A novel strategy to identify the critical conditions for growth-induced instabilities

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Geometric instabilities in living structures can be critical for healthy biological function, and abnormal buckling, folding, or wrinkling patterns are often important indicators of disease. Mathematical models typically attribute these instabilities to differential growth, and characterize them using the concept of fictitious configurations. This kinematic approach toward growth-induced instabilities is based on the multiplicative decomposition of the total deformation gradient into a reversible elastic part and an irreversible growth part. While this generic concept is generally accepted and well established today, the critical conditions for the formation of growth-induced instabilities remain elusive and poorly understood. Here we propose a novel strategy for the stability analysis of growing structures motivated by the idea of replacing growth by prestress. Conceptually speaking, we kinematically map the stress-free grown configuration onto a prestressed initial configuration. This allows us to adopt a classical infinitesimal stability analysis to identify critical material parameter ranges beyond which growth-induced instabilities may occur. We illustrate the proposed concept by a series of numerical examples using the finite element method. Understanding the critical conditions for growth-induced instabilities may have immediate applications in plastic and reconstructive surgery, asthma, obstructive sleep apnoea, and brain development.

1. Introduction

Structural instabilities in the form of creases, folds, or wrinkles are inherent to living matter. In many living systems, the formation of structural instabilities is critical to biological function, e.g., to increase the surface-to-volume ratio of the system (Wyczalkowski et al., 2012). Typical examples are wrinkling of skin (Buganza Tepole et al., 2011), villi formation in the intestine (Balbi and Ciarletta, 2013), and folding of the developing brain (Xu et al., 2010). In other biological systems, however, the formation of structural instabilities can be a critical hallmark of disease, e.g., when associated with a narrowing lumen. The most prominent example of this latter category is the folding of the mucous membrane in asthmatic airways (Wiggs et al., 1997). It is thus not surprising that the mathematical modeling of folding in tubular organs (Ciarletta and Ben Amar, 2012), in particular the modeling of the folding mucous membrane (Moulton and Goriely, 2011; Li et al., 2011; Xie et al., 2013), has drawn increasing scientific attention within the past decade.

Continuum approaches toward the formation of geometric instabilities in living systems typically adopt the concept of...
finite growth (Rodriguez et al., 1994). This kinematic approach toward growth is based on the introduction of a fictitious growth configuration (Garikipati, 2009) and on the multiplicative decomposition of the deformation gradient into a reversible elastic and an irreversible growth part (Taber, 1995). Mathematically speaking, in this approach, growth is represented through a second order, isotropic (Himpel et al., 2005), transversely isotropic (Zöllner et al., 2012), orthotropic (Göktepe et al., 2010), or generally anisotropic growth tensor. As discussed in a recent review article on growth (Ambrosi et al., 2011), the evolution of this growth tensor is typically either morphogenetically driven (Li et al., 2012) or mechanically driven (Menzel and Kuhl, 2012).

Here we focus on morphogenetically driven growth and on its role in the formation of structural instabilities. The first rigorous mathematical analyses of growth-induced morphogenetic instabilities studied the failure models of a shrinking spherical shell (Goriely and BenAmar, 2005) and of a growing spherical shell under external pressure (Ben Amar and Goriely, 2005). Motivated by the clinical problem of mucosal folding during chronic airway wall remodeling, recent studies explored the buckling of single-layered (Moulton and Goriely, 2011) and double-layered (Cao et al., 2012; Jia et al., 2012) hollow cylindrical tubes. In realistic airway wall geometries, the thickness of the folding inner layer is typically orders of magnitude smaller than the cylindrical airway structure itself. Accordingly, a recent study suggested to model mucosal folding using the concept of surface growth (Papastavrou et al., 2013), an approach for which the mucosal surface itself is equipped with its own potential energy (Steinmann, 2008). A similar approach was recently proposed for the longitudinal growth in double-layered cylinders (Vandiver and Goriely, 2009) to simulate the effects of surface growth in plants (Holland et al., in press).

Inhomogeneous growth induces a state of prestrain or residual stress (Fung, 1991). Residual stress, which must not be confused with prestress in this paper, is the stress in a body in the unloaded configuration (Menzel, 2005; Rausch et al., in press; Rausch and Kuhl, 2013). In the following we distinguish between two geometrically identical reference configurations, one stress-free and one pre-stressed. We assume that the reverse (elastic) deformation from the stress-free growth configuration to the pre-stressed reference configuration results in pre-stress. The theory of elasticity for a body under initial stress (prestress here) was first established by Biot (1939). More recently, Johnson and Hoger (1993) studied the dependence of the elasticity tensor on residual stress where the residual stress was produced by an elastic deformation (Hoger, 1986; Marlow, 1992; Hoger, 1993). The problem of dead loading in the mathematical theory of linear elasticity with initial stress (Man and Carlson, 1994) bears certain similarities to our analysis here. Ogden (1992) details on stability and uniqueness of solution of incremental boundary value problem. In Ogden (2003), he employed his generic method to soft tissues and tube extension/inflection problems with residual stresses arising from a uniform circumferential stress. For further mathematical details of symmetry, bifurcation, and instabilities in the context of elasticity with a prestressed reference configuration we refer to Bharatha and Levinson (1978), Capriz and Guidugli (1979), and Wan and Marsden (1983).

In the theory of linear elasticity, a condition for the existence of solutions is that incremental deformations require positive energy. This physically motivated condition translates into the mathematically motivated condition of pointwise stability. In the classical theory of elasticity the pointwise stability condition corresponds to the positive definiteness of the constitutive tensor. However, in the context of prestress, we cannot simply adopt this classical pointwise stability condition for two reasons: (i) The condition that the elasticity tensor is positive definite may no longer be feasible; and (ii) The pointwise stability condition does not directly render the positive definiteness of the constitutive tensor, since there are several elasticity tensors, all of which are functions of the prestress that describe prestressed materials (see similar discussions in Hoger, 1995, for the case of residual stress). In this paper, we re-establish the governing equations for prestressed continua and properly impose the pointwise stability condition.

The necessary and sufficient conditions for the loss of well-posedness of the boundary value problem for linear elastic, homogeneous continua are the loss of strong ellipticity of the governing equations and the boundary complementing condition (see e.g. Simpson and Spector, 1985; Benallal et al., 1993). A sufficient condition for stability of elastic continua is the pointwise stability criterion (Hill, 1957). A general theory of uniqueness and stability for elasto-plastic solids was given by Hill (1958). The propagation of surface waves in bodies has been investigated in Dowaiik and Ogden (1990). Bifurcation in the form of surface instabilities has been investigated for arbitrary nonlinear elastic materials under conditions of an equibiaxial prestress (Reddy, 1982), and for plane strain (Reddy, 1983). A comprehensive study on uniqueness, loss of ellipticity, and localization for the time-discrete, rate-dependent boundary value problems with softening can be found in Benallal et al. (2010).

In view of these considerations, the goal of this contribution is to explore the critical conditions for growth-induced instabilities in living structures. In Section 2, we illustrate the kinematics of finite growth based on the multiplicative decomposition of the total deformation gradient into an elastic and a growth part. In Section 3, we introduce the key idea of this work, the conceptual replacement of this growth part by prestress. In Section 4, we discuss the condition for strong ellipticity, the condition for pointwise stability, and the boundary complementing condition in the context of prestress. In Section 5, we illustrate these three conditions for a simple homogeneous model problem, and for the inhomogeneous problems of growth and shrinkage of a hollow cylinder and of a solid sphere. We conclude with a critical discussion and an outlook in Section 6.

### 1.1. Notation and definitions

The three-dimensional Euclidean space is denoted $\mathbb{E}^3$. Direct notation is adopted throughout. Occasional use is made of index notation, the summation convention for repeated indices being implied. The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., the single contraction, is denoted $\mathbf{a} \cdot \mathbf{b} = [\mathbf{a}]^T [\mathbf{b}]$. The scalar product of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$, i.e., the double contraction, is denoted $\mathbf{A} : \mathbf{B} = [\mathbf{A}^T] [\mathbf{B}]$. The action of a second-order tensor $\mathbf{A}$ on a vector $\mathbf{a}$ is understood as $[\mathbf{A} \cdot \mathbf{a}] = [\mathbf{A}]^T [\mathbf{a}]$, and $[\mathbf{a} \cdot \mathbf{A}] = [\mathbf{a}]^T [\mathbf{A}]$. The double contraction of a third-order tensor $\mathbf{C}$ and a second-order tensor $\mathbf{B}$ renders a vector according to $[\mathbf{C} : \mathbf{B}] = [\mathbf{C}]_{\mathbf{B}} [\mathbf{B}]$. The action of a third-order tensor $\mathbf{C}$ on a vector $\mathbf{a}$, denoted $\mathbf{C} : \mathbf{a}$, is a second-order tensor with components $[\mathbf{C} : \mathbf{a}]_{ij} = [\mathbf{C}]_{ijm} [\mathbf{a}]_m$. 
The composition of two second-order tensors A and B, denoted $A \cdot B$, is a second-order tensor with components $[A \cdot B]_{ij} = [A]_{ij} [B]_{ij}$. The tensor product of two vectors $a$ and $b$ is a second-order tensor $D = a \otimes b$ with $[D]_{ij} = [a]_i [b]_j$. The tensor product of two second-order tensors $A$ and $B$ is a fourth-order tensor $D = A \otimes B$ with $[D]_{ijkl} = [A]_{ij} [B]_{kl}$. The two standard double contractions of a fourth-order tensor $D$ and a second-order tensor $A$ render the second-order tensor with components $[D \cdot A]_{ij} = [D]_{ij} [A]_{ij}$ and $[A \cdot D]_{ij} = [A]_{ij} [D]_{ij}$. The two non-standard tensor products of two second-order tensors $A$ and $B$ are the fourth-order tensors $[A \boxtimes B]_{ijkl} = [A]_{ij} [B]_{kl}$ and $[A \bowtie B]_{ijkl} = [A]_{ij} [B]_{kl}$.

2. Kinematic equations for growing continua

We consider a continuum body in the material configuration $B_0 \subset \mathbb{E}^3$ at time $t=0$ and in the spatial configuration $B_1$ at time $t>0$, where $t \subset T = [0, t_{\text{end}}] \subset \mathbb{R}_+$. denotes the time domain of interest. As depicted in Fig. 1, we denote the surface outward unit normals in the material and spatial configurations by $N$ and $n$. We denote a motion of the reference placement by the orientation-preserving map $\varphi : B_0 \times T \to \mathbb{E}^3$ for any time $t \in T$. In particular, we denote the current placement of the bulk associated with the motion $\varphi$ by $B_t = \varphi(R_t(X), t)$ and designate its positions as $x = \varphi(x, t) \in B_t$. The deformation gradient $F : T B_0 \to T B_1$, $F(X, t) = \nabla \varphi(X, t)$ with $\nabla \{\ast\} = \{\ast\} \otimes \partial x$, is the invertible linear tangent map of a material line element $dX \in T B_0$ to a spatial line element $dx \in T B_1$, where $T B_0$ and $T B_1$ are the corresponding tangent spaces, and $J = \det F$ denotes the corresponding Jacobian. The classical approach to model volumetric growth (Rodriguez et al., 1994; Göktepe et al., 2010) is to multiplicatively decompose the deformation gradient $F$ into an elastic part $F_e$ and a growth part $F_g$, see Fig. 2 (left):

$$F = F_e \cdot F_g \Rightarrow F_e = F \cdot F_g^{-1},$$

such that the free energy is a function of the elastic part $F_e$ alone, and unconstrained growth $F_g$ does not induce stress. The goal of this paper is to explore whether an infinitesimal, incremental perturbation of a stable state triggers a kinematic instability. Although this analysis requires the deformation gradient $F$ to be close to identity, the underlying framework does allow for finite growth. Note that the growth is neither restricted to be compatible nor infinitesimal in this work. We interpret the classical multiplicative decomposition as shown in Fig. 2 (right). In particular, we rewrite the elastic part of the deformation gradient $F_e = F \cdot F_g^{-1}$, which maps the stress-free grown configuration to the stressed current configuration $B_1$, as a deformation part $F$ and an inverse growth part $F_g^{-1}$, thus

$$F = F_e \cdot F_g = F \cdot F_g^{-1} \cdot F_g = F \cdot I.$$

In this approach, the configuration $B_1$ is only geometrically and not necessarily energetically identical to $B_0$. This implies that mapping the stress-free grown configuration $B_3$ reversely onto $B_1$ does, in general, induce stresses in the body. Conceptually speaking, we can understand the configuration $B_1$ as the prestressed reference configuration, to which the deformation $\varphi$ and the associated deformation gradient $F$ are applied. $F$ may be thought of mapping from the configuration $B_1$, that is obtained by applying a prestress to $B_3$ so that the growth

Fig. 1 – The material and spatial configurations of a continuum body, and the associated motions and deformation gradients.

Fig. 2 – Illustration of kinematics of growth. The classical approach (left) is based on the multiplicative decomposition of the deformation gradient $F$ into an elastic part $F_e$ and a growth part $F_g$. Instability analysis based on this approach is rather complex and it accounts for geometrically induced instabilities which are not the purpose of this paper. The current approach (right) modifies the classical decomposition through the inverse mapping $F_g^{-1}$. Instability analysis based on the current approach is straightforward and focuses on materially induced instabilities. The admissible range for material parameters and growth can be precisely predicted for the general case.
deformation is reversed, to the configuration $B_s$. Here we shall compute the necessary prestress from an assumed constitutive law. For the simplest case of isotropic volume growth, we can express the growth tensor as

$$F_0 = g I$$

where $g$ is a single scalar-valued growth multiplier. Growth induces a prestress in the body in configuration $B_s$, and the amount of prestress depends on the magnitude of growth $g$. In the simplest case, the prestress $\gamma$ is isotropic and proportional to the magnitude of growth $g$, thus the assumed constitutive behavior upon prestress renders

$$\gamma = -\kappa (\det F_0 - 1) I = -\kappa (g^2 - 1) I,$$

where $\kappa$ denotes the bulk modulus and $\gamma$ is the scalar-valued isotropic prestress. The negative sign arises from the fact that the prestress $\gamma$ is the necessary stress to transform the grown configuration $B_s$ into $B_t$. Throughout this paper, we will use the term growth for the multiplier $g$ with the understanding that $g$ can represent both shrinkage and growth. In view of Eq. (5), we distinguish the following three scenarios:

$$\begin{cases}
  g < 1 \Rightarrow \gamma > 0 & \text{shrinkage}, \\
  g = 1 \Rightarrow \gamma = 0 & \text{homeostatic equilibrium}, \\
  g > 1 \Rightarrow \gamma < 0 & \text{growth}.
\end{cases}$$

3. Governing equations for prestressed continua

This section focuses on characterizing the mapping from the prestressed configuration $B_t$ to the current configuration $B_s$, as illustrated in Fig. 2 (right) and on identifying the corresponding equilibrium and constitutive equations. Here we restrict ourselves to infinitesimal deformations, for which the deformation gradient $F$ is close to the identity. We reparameterize $F$ in terms of the displacement vector $u$ via the classical relation $F = I + u V$. In the spirit of Section 2, we re-interpret the configurations $B_t$ and $B_s$ as the material and spatial configurations. In the absence of body forces, the balance of linear momentum reads

$$\text{Div} \mathbf{P} = 0 \quad \text{in} \quad B_t,$$

where $\mathbf{P}$ is the Piola stress and $\text{Div}(\bullet) = \nabla (\bullet) \cdot I$ denotes the material divergence operator. We supplement the equilibrium equation (7) by the boundary condition

$$\mathbf{P} \cdot \mathbf{n} = T^b \quad \text{on} \quad \partial B_t,$$

in which $T^b$ denotes the externally prescribed tractions on the boundary $\partial B_t$. To constitutively define the Piola stress $\mathbf{P}$, we introduce the following isotropic neo-Hookean Helmholtz energy per unit volume of the prestressed reference configuration $B_t$:

$$\psi = \frac{1}{2} \kappa (J^2 - 1)^2 + \frac{1}{2} \mu (J^{-2/3} F : F^{-1} - 3) + \gamma$$

with $\kappa = \lambda + \frac{2}{3} \mu$.

Here $\lambda$ and $\mu$ are the standard Lamé parameters and $\gamma$ is the scalar-valued prestress introduced in Eq. (5). The Piola stress follows from thermodynamic considerations in terms of the Helmholtz energy function (9) as thermodynamically conjugate to the total deformation gradient $F$:

$$\mathbf{P} = \frac{\partial \psi}{\partial F} = \kappa \left[ J^2 F^{-1} + \mu J^{-2/3} F : F^{-1} \right] + \gamma F^{-1}.$$  

To reformulate the governing equations in the context of the infinitesimal deformation theory, we linearize the Piola stress at the reference configuration $B_t$ where $F = I$:

$$\text{Lin} \mathbf{P}_t = \gamma I + \frac{\partial \psi}{\partial F} \bigg|_{F=I} : \left[ F - I \right].$$

The corresponding fourth-order tangent reads

$$\frac{\partial \mathbf{P}}{\partial F} = \kappa \left[ J^2 F^{-1} : F^{-1} - J^2 \right] + \gamma J^3 F^{-1} = \frac{\partial \psi}{\partial F}$$

and its evaluation at $B_t$ results in

$$\frac{\partial \mathbf{P}}{\partial F} \bigg|_{F=I} = \gamma I + \left[ J^2 \right] \frac{\partial \psi}{\partial F} : \left[ I + \gamma I^{-1} \right] \frac{\partial \psi}{\partial F} : \left[ I + \gamma I^{-1} \right],$$

in terms of the fourth order tensors, $[\psi] = \frac{\partial \psi}{\partial F} : \left[ I + \gamma I^{-1} \right]$ and $[\psi] = \frac{\partial \psi}{\partial F} : \left[ I + \gamma I^{-1} \right]$. With the evaluation of the Piola stress $\mathbf{P}$ at $F = I$ as $P = \gamma I$ and with the minor symmetries of $[\psi]$ and $[\psi]$, Eq. (11) reads

$$\text{Lin} \mathbf{P}_t = \gamma I + \left[ J^2 \right] \frac{\partial \psi}{\partial F} : \left[ I + \gamma I^{-1} \right] : \left[ I + \gamma I^{-1} \right].$$

Remark: Alternative stress measures. The linearization of the different stress measures, which is frequently used to interpret instabilities in morphogenesis (Goriely and Ben Amar, 2005), provides additional insight in the interpretation of prestress. Although the non-symmetric Piola stress $\mathbf{P}$ is the only stress measure relevant to evaluate the balance Eq. (7), it is possible to compute the corresponding alternative finite deformation stress measures in the context of the infinitesimal deformations. Accordingly, the fully material symmetric Piola–Kirchhoff stress $\mathbf{S}$ and the fully spatial
Here, since our constitutive tensor \( c \) can be linearized at \( B_2 \):
\[
\begin{align*}
\text{Lin } S_{F, f} &= S_{F, f} + \frac{\partial S}{\partial F} \bigg|_{F=I} : [F - I] = [3\lambda + 2\mu\text{ vol} + 2(\mu\text{ sym})] : \varepsilon \\
\text{Lin }\sigma_{F, f} &= \sigma_{F, f} + \frac{\partial \sigma}{\partial F} \bigg|_{F=I} : [F - I] = [3\lambda] + [2\mu\text{ vol} + 2\mu\text{ sym}] : \varepsilon.
\end{align*}
\]

In the absence of deformation, i.e., for \( \nabla u = 0 \), all stress measures assume the same prestress \( \delta \). In the absence of the prestress, i.e., for \( \gamma = 0 \), all linearized stress measures are identical. This is why, with a few exceptions, e.g., Bazant (1971) and Hoger (1986), the classical linear elasticity theory typically does not distinguish between the different stress measures. □

4. Stability analysis for prestressed continua

To ensure that the boundary value problem is well-posed, it has to satisfy specific conditions. In the following, we expand the classical condition of strong ellipticity, the classical condition of pointwise stability, and the classical boundary complementing condition originally established for continua without prestress (Mansden and Hughes, 1994) to prestressed continua.

4.1. Strong ellipticity

First, we explore the condition of strong ellipticity in the context of prestress. The elasticity tensor \( c \) is strongly elliptic, associated with a real wave speed, if, for all vectors \( a, b \neq 0 \),
\[
[a \otimes b] : [C_{\text{eff}} + \gamma I] : [a \otimes b] > 0.
\]

For our particular constitutive model with prestress (9), for which \( c = C_{\text{eff}} + \gamma I \), the condition of strong ellipticity translates into the following expression:
\[
[a \otimes b] : [C_{\text{eff}} + \gamma I] : [a \otimes b] > 0.
\]

For the effective tangent moduli \( C_{\text{eff}} = 3\lambda_{\text{eff}}\text{ vol} + 2\mu_{\text{eff}}\text{ sym} \) according to Eq. (15), this implies that strong ellipticity: \( \mu > 0 \) and \( \lambda + 2\mu > 0 \).

Alternatively, we can state the condition of strong ellipticity through the positive-definiteness of the acoustic tensor \( q \) defined in terms of the (arbitrary) unit vector \( b \) with \( |b| = 1 \):
\[
q : \text{positive definite with } q = [I \otimes b] : [C_{\text{eff}} + \gamma I] : b.
\]

4.2. Pointwise stability

Second, we study the condition of pointwise stability in the context of prestress. The elasticity tensor \( c \) is said to be pointwise stable if, for all symmetric second order tensors \( e = e_i \),
\[
\delta e : c : \delta e > 0.
\]

Here, since our constitutive tensor \( c = C_{\text{eff}} + \gamma I \) does not possess minor and major symmetries similar to the classical elasticity tensor \( c \), we cannot simply adapt the classical pointwise stability condition above. Pointwise stability requires the incremental work \( \delta e : \delta e \) to increase for increasing incremental strains \( \delta e \), thus \( \delta e : \delta e > 0 \). In our case, pointwise stability translates into the following condition, \( \nabla u : [C_{\text{eff}} + \gamma I] : \nabla u - \nabla u : \gamma I : \nabla u > 0 \), or alternatively, for all displacement gradients \( \nabla u \), not necessarily symmetric,
\[
\nabla u : C_{\text{eff}} : \nabla u > 0.
\]

Here we have assumed that the prestress is subtracted from the total stress \( \Sigma \) since it is a priori in balance with externally prescribed tractions \( T^o \). Furthermore, we have subtracted the geometrical stiffness term \( \nabla u : \gamma I : \nabla u \) to properly impose pointwise stability (Hoger, 1995). From \( C_{\text{eff}} = 3\lambda_{\text{eff}}\text{ vol} + 2\mu_{\text{eff}}\text{ sym} \), we conclude that the pointwise stability condition only affects the symmetric part \( e = \text{vol} \nabla u \) of the displacement gradient \( \nabla u \), thus
\[
\epsilon : [3\lambda_{\text{eff}}\text{ vol} + 2\mu_{\text{eff}}\text{ sym}] > 0.
\]

Alternatively, we can reformulate the above equation in terms of volumetric and deviatoric parts:
\[
\epsilon : \left[ 3\lambda_{\text{eff}} + 2\mu_{\text{eff}} \right]_{\text{vol}} + 2\mu_{\text{eff}}\text{ sym} : \epsilon > 0.
\]

where \( \epsilon_{\text{sym}} = \text{vol} - 1 \). With \( 3\lambda_{\text{eff}} + 2\mu_{\text{eff}} = 3\lambda + 2\mu + \gamma \) and \( 2\mu_{\text{eff}} = 2\mu - 2\gamma \) this implies that pointwise stability holds within the following prestress limits \( \gamma \):

pointwise stability: \( -3\lambda - 2\mu \leq \gamma < -\mu. \)

For Lamé parameters \( \lambda \geq 0 \) and \( \mu \geq 0 \) that are strictly non-negative, the above condition implies that pointwise stability can either be violated by negative growth or shrinkage associated with a tensile prestress \( \gamma > \mu \) or by positive growth associated with a compressive prestress \( \gamma < -3\lambda - 2\mu \).

4.3. Boundary complementing condition

Third, we study the boundary complementing condition associated with the infinitesimal, quasi-static, incremental boundary value problem. We consider a semi-infinite half-space of bulk material subjected to an incremental deformation \( \delta u \) such that the total deformation is \( u + \delta u \). The corresponding stress \( \Sigma + \delta \Sigma \) has to satisfy the balance equation and its boundary condition (16), i.e.,
\[
\text{Div}(\Sigma + \delta \Sigma) = 0 \quad \text{in } B_t \quad \text{subject to } (\Sigma + \delta \Sigma) \cdot N = T^o \text{ on } \partial B_t.
\]

Since, \( \text{Div } \Sigma = 0 \) and \( \Sigma \cdot N = T^o \), the incremental balance of linear momentum in the bulk in the absence of a body force field reduces to the following expression along with its boundary condition:
\[
\text{Div } \delta \Sigma = 0 \quad \text{in } B_t \quad \text{subject to } \delta \Sigma \cdot N = 0 \text{ on } \partial B_t.
\]

Here, for the case of isotropic volume growth, we assume that the externally prescribed tractions are constant and responsible to map the grown configuration to the stressed configuration, hence \( T^o = y N \). The incremental stress tensor \( \delta \Sigma \) is related to the incremental deformation tensor \( \delta \nabla u \) via the constitutive relation
\[
\delta \Sigma = [C_{\text{eff}} + \gamma I] : \delta \nabla u.
\]

To explore the boundary complementing condition, we consider a hypothetical stationary wave on the surface of the bulk half-space as illustrated in Fig. 3 (Benallal et al., 1993; Utzinger et al., 2008; Javili et al., 2012). The wave lies in the plane spanned by the surface base vectors \( t \) and \( b \) and decays toward the bulk along the inward-pointing surface normal \( d \). In what follows, we denote relevant vectors and tensors by
small letters and restrict the displacement increment $\Delta u$ to be infinitesimal. We label the position vector of a point within the bulk with respect to the origin of the local surface orthonormal base system $t, b, d$ as $\mathbf{x}$, and denote its projections onto the $d$ and $t$ directions as $\xi$ and $\eta$:

$$\xi = \mathbf{x} \cdot d \quad \text{and} \quad \eta = \mathbf{x} \cdot t.$$  \hspace{1cm} (29)

To represent a stationary wave as shown in Fig. 3, we prescribe a wave-type ansatz for the incremental displacement $\Delta u$ as the product of a decay function $u(\xi)$ and a waviness term $\exp(ik\eta)$:

$$\Delta u = u(\xi) \exp(ik\eta).$$  \hspace{1cm} (30)

where $k>0$ is the wave number and $i$ is the imaginary unit. The first and the second gradients of $\Delta u$ are denoted as $V \Delta u$ and $V_2 \Delta u$, respectively. We reformulate the incremental balance of linear momentum in the bulk in terms of the incremental displacement $\Delta u$ as

$$\text{Div}\ \sigma = \text{Div}[[C_{\text{eff}} + f]\mathbf{I}] : \delta \mathbf{u} = [[C_{\text{eff}} + f] : V_2 \Delta \mathbf{u}] : \mathbf{I}.$$  \hspace{1cm} (31)

To study the boundary complementing condition on the surface, we consider the incremental balance of linear momentum on the surface (27,2), here simply the incremental boundary condition restated as follows:

$$\overline{\sigma} \cdot \mathbf{n} = \overline{\sigma}_\mathbf{d} \cdot [-\mathbf{d}] = 0 \quad \text{with} \quad \overline{\sigma}_\mathbf{d} = \overline{\sigma} \mathbf{d} = 0.$$  \hspace{1cm} (32)

Here, we have introduced the hat symbol to indicate the evaluation of the corresponding quantity, here the incremental stress $\overline{\sigma}$, on the material surface, i.e., at $\xi = 0$. Using the incremental constitutive equation (28), together with proper wave-type ansatz governed by incremental balance of momentum in the bulk (31), along with the mathematical transformations detailed in the Appendix, we obtain the following expression:

$$\mathbf{b} \cdot \mathbf{a} = 0, \quad \text{with} \quad \mathbf{b} = \begin{bmatrix} -2\mu k + \gamma k & \mu & 0 \\ -2\mu k + \gamma k & \lambda + 2\mu & 0 \\ 0 & 0 & -\mu k \end{bmatrix} \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$  \hspace{1cm} (33)

Non-trivial solutions of Eq. (33) with $k > 0$ correspond to possible stationary surface wave-type solutions of the incremental boundary-value problem. In summary, we can state the condition for a possible bifurcation in the following form:

$$\text{boundary complementing condition :} \quad \det \mathbf{B} = \mu k^2 [\lambda + \mu(2\mu - \gamma)] = 0.$$  \hspace{1cm} (34)

Remark: Classical case of non-prestressed solids recovered. For the classical case of a solid without prestress with $\gamma = 0$,

Eq. (34) reduces to the classical condition for the loss of the boundary complementing condition:

$$\det \mathbf{B} = 2\mu^2 k^2 [\lambda + \mu] = 0 \Rightarrow \mu = 0 \text{ or } \lambda + \mu = 0,$$  \hspace{1cm} (35)

which is eliminated a priori because of the pointwise stability of the bulk. \hspace{1cm} □

Remark: Pointwise stability and strong ellipticity eliminate surface bifurcations. For the case of a solid subjected to prestress with $\gamma > 0$, the conditions for the possible bifurcation with a positive wave number $k > 0$ are

$$\mu = 0 \text{ or } \lambda + \mu = 0 \text{ or } 2\mu - \gamma = 0,$$  \hspace{1cm} (36)

which are necessarily eliminated a priori because of pointwise stability and strong ellipticity. From Eq. (34), we conclude that if a bifurcation does occur, it can have any arbitrary wave number $k > 0$ and any wavelength. \hspace{1cm} □

5. Examples of growth-induced instabilities

The following examples elucidate the theory presented in the previous sections. First, to investigate the effect of prestress, we illustrate the eigenvalue analysis in the homogeneous setting of a single trilinear finite element. Then, to explore biologically relevant growth, we simulate growth-induced instabilities in the inhomogeneous settings of a hollow cylinder mimicking the airway wall (Papastavrou et al., 2013) and of a solid half-sphere mimicking the developing brain (Bayly et al., 2013). Since our main objective is to provide a generic framework for the critical conditions of growth-induced instabilities, we choose arbitrary values for the Lamé parameters of $\mu = 8.0$ and $\lambda = 12.0$ in all four problems.

5.1. Homogeneous problem of eigenvalue analysis for a single trilinear element

To investigate the interplay between negative eigenvalues and bulk prestress $\gamma$, we perform a stability analysis for a single trilinear finite element. For simplicity, we assume the dimensions of the element to be unity. The derivation of the underlying weak form and its finite element discretization is standard
(Hughes, 1987), except for the local constitutive tensor and the global stiffness matrix, which follow from the boundary value problem (16).

Fig. 4 illustrates the eigenvalues of the global stiffness matrix plotted against the prestress. For any prestress level $γ > 0$, we obtain six zero eigenvalues, which correspond to three rigid body translations and the three rigid body rotations. According to conditions (19) and (25), at $γ = -3\lambda + 2μ = -52$ and at $γ = μ = 8$, the analysis predicts at least one eigenmode associated with a zero eigenvalue, as illustrated in Fig. 4. Each of the predicted zero energy modes is associated with either a purely volumetric deformation or a purely deviatoric deformation mode. In addition, mixed mode conditions can also result in zero energy modes. In summary, we observe various zero energy modes in addition to the plain modes indicated in conditions (19) and (25).

5.2. Inhomogeneous problem of shrinkage-induced instabilities in a hollow cylinder

The first inhomogeneous example is a shrinking infinitely long hollow cylinder, modeled as a two-dimensional ring under plane strain conditions. The ring has an outer diameter of unit length and an inner radius of 0.3. We discretize the geometry using 1920 bilinear quadrilateral finite elements. While we allow the inner boundary to move freely upon growth, we subject the outer boundary to homogeneous Dirichlet boundary conditions. To mimic the effect of negative growth or shrinkage, we apply a tensile prestress of $γ = 9$, a value beyond the critical prestress level of $γ = μ = 8$, as predicted by the pointwise stability condition in Eq. (25).

Fig. 5 illustrates selected instability patterns along with their associated eigenvalues. Tensile prestress beyond the critical limit is associated with shrinkage-induced instabilities. In contrast to many studies in the literature, here, the associated unstable modes of Fig. 5 result exclusively from the eigenvalue analysis of the global stiffness matrix and are not associated with randomly applied perturbations. If we apply a perturbation, we observe mixed instability modes. Our analysis is capable of capturing a tetragonal, pentagonal, hexagonal instability patterns. Without perturbation, it is even possible to see no instabilities at all.

5.3. Inhomogeneous problem of growth-induced instabilities in a hollow cylinder

The second inhomogeneous example is a growing thin hollow cylinder, modeled as a three-dimensional ring under plane stress conditions. The ring has an outer diameter of unit length, an inner radius of 0.3, and a thickness of 0.001. We discretize the geometry using 1280 trilinear hexahedral finite elements. Similar to the previous problem, we allow the inner boundary to move freely upon growth, and subject the outer boundary to homogeneous Dirichlet boundary conditions. To mimic the effect of positive growth, we apply a compressive prestress of $γ = -55$, a value beyond the critical prestress level of $γ = -3\lambda - 2μ = -52$, as predicted by the pointwise stability condition in Eq. (25).

Fig. 6 illustrates selected instability patterns along with their associated eigenvalues. Compressive prestress beyond the critical limit is associated with growth-induced instabilities. Similar to the previous example, our analysis is capable of capturing a hexagonal, heptagonal, and octagonal instability patterns. We observed that, to obtain more unstable modes, we need to either apply a finer discretization or a smaller prestress $γ$, larger in magnitude and further away from the critical threshold. Fig. 7 illustrates several instability patterns together with their associated eigenvalues for the prestress of $γ = -70$.

5.4. Inhomogeneous problem of growth-induced instabilities in a solid half-sphere

The final example is a growing solid half-sphere in a fully three-dimensional setting. The sphere has a radius of 0.3. We discretize its geometry with 3456 trilinear hexahedral finite elements. We allow the outer boundary to move freely upon growth, fix the center against rigid body translations and rotations, and apply homogeneous Dirichlet boundary conditions in $z$-direction at the bottom to account for symmetry conditions. To mimic the effect of positive growth, we apply a compressive prestress of $γ = -100$, a value beyond the critical prestress level of $γ = -3\lambda - 2μ = -52$, as predicted by the pointwise stability condition in Eq. (25).

![Fig. 4](image)

Fig. 4 – Homogeneous problem of eigenvalue analysis for a single trilinear element. Eigenvalues are plotted versus varying prestress levels $γ$. Negative eigenvalues are associated with unstable eigenmodes. Circles indicate when an eigenmode reaches its critical condition and becomes unstable.
Fig. 8 illustrates selected instability patterns along with their associated eigenvalues. Compressive prestress beyond the critical limit is associated with growth-induced instabilities. Similar to the previous examples, our analysis is capable of capturing different instability patterns associated with the different eigenmodes.

6. Discussion

Many living structures undergo biological growth during development or under diseased conditions. Growth is often the origin for geometric instabilities in the form of folding or wrinkling. Typical examples are folding patterns in the developing brain and wrinkling patterns in chronically obstructed airways. To better understand malformations during development or disease progression, it is important to know when and how these instabilities form. Here we have presented our first attempts to systematically characterize the critical conditions for growth-induced instabilities in living structures.

The classical approach to model biological growth is to kinematically introduce an incompatible growth configuration and to multiplicatively decompose the deformation gradient into an elastic part and a growth part. The key idea of this paper is to conceptually replace growth with prestress and establish a generic framework for growth- or rather prestress-induced instabilities. In terms of configurations, we map the stress-free grown configuration onto a stressed configuration, which is geometrically identical to the initial configuration before growth. However, this configuration is not stress-free and therefore not energetically identical to the initial configuration. This concept allows us to adopt a classical instability analysis, and highlight critical conditions of strong ellipticity and of pointwise stability and the boundary complementing condition.

Our linear stability analysis allows us to predict critical prestress levels for negative growth or shrinkage, associated with a tensile prestress, and for positive growth, associated with compressive prestress. In particular, the pointwise stability condition reveals the critical prestress limits under tension and compression as functions of the material parameters $\lambda$ and $\mu$ as $-3\lambda-2\mu<\gamma<\mu$. The accompanying eigenvalue analysis of the finite-element based global stiffness matrix provides a first insight into the corresponding unstable modes. While the concepts of incremental...
deformation and infinitesimal perturbation have been widely used to study growth-induced instabilities (Ben Amar and Goriely, 2005; Goriely and BenAmar, 2005), we have to keep in mind that linearized theories bear several important limitations: First, infinitesimal perturbations only represent a subset of a wider class of possible perturbations; they neglect the symmetry of finite deformation (Moulton and Goriely, 2011). Second, and most importantly, linearized theories only allow us to identify critical conditions at the onset of instabilities; they are unable to predict the progression of the unstable mode and the fully developed unstable state (Balbi and Ciarletta, 2013). To study the structural response beyond the onset of instabilities, a natural extension of the proposed work would be to perform a post-buckling analysis based on a fully geometrically non-linear approach.

Under plain strain conditions, our critical condition for growth-induced instabilities is $2\lambda + 2\mu = 0$, which is clearly independent of the prestress level $\gamma$. At first glance, this condition seems to contradict the literature on growth-induced instabilities in the airway wall (Li et al., 2011). However, there are two major differences between the models in the literature and our model: First, most previously reported geometric models are based on double-layered cylindrical tubes (Cao et al., 2012; Jin et al., 2011), while we have limited our analysis to a single-layered tube (Moulton and Goriely, 2011). Second, and more importantly, most previously reported growth models constrain growth in the axial direction (Moulton and Goriely, 2011; Li et al., 2011), while we have assumed that a fully three-dimensional growth results in a fully isotropic prestress state. A natural extension of our model would be to incorporate a second layer and in-plane only growth to analyze to which extent our approach can reproduce similar unstable modes under similar conditions.

For conceptual simplicity, in this study we have limited our analysis to isotropic growth phenomena. While isotropy is a reasonable assumption for several types of tissues such as...
as tumors (Ambrosi and Mollica, 2002), most soft biological tissues are truly anisotropic (Menzel and Kuhl, 2012). To generalize the proposed concept to model anisotropic growth, we would need to replace the isotropic prestress \( \gamma \) by a generalized tensorial prestress \( \gamma \). A particularly interesting type of anisotropic growth is surface growth, a phenomenon, which is characteristic for thin films such as the mucus membrane (Moulton and Goriely, 2011), whose thickness is orders of magnitude smaller than the structural dimensions of the overall system (Papastavrou et al., 2013). An exciting extension of the proposed work would be to model surface wrinkling and folding (Zöllner et al., 2012) using the concept of surface tension and surface elasticity (Gurtin and Murdoch, 1975; Javili and Steinmann, 2009). This would advance our understanding of growing thin films (Holland et al., in press), which could be equipped with their own energies independent of the bulk (Javili and Steinmann, 2010a,b).

To reliably predict the onset of growth-induced instabilities in living systems, it is critical to experimentally calibrate and validate the underlying growth law (Ambrosi et al., 2011). Recent first attempts have demonstrated the in vivo characterization of growth in the heart wall (Tsamis et al., 2012) and in heart valves (Rausch et al., 2012). However, longitudinal studies of controlled growth at multiple points in time are rare, and calibrating the time-evolution of growth remains challenging.

In summary, this paper presents our approach to address the growth instabilities in living systems. The performed analysis so far does not require the bulk to be isotropic, nevertheless, in the remainder of this work we apply the analysis to the special case of isotropy. For the case of isotropic, linear elasticity the bulk localization tensors are simply:

\[
q_0 = \left[ \lambda_0 + \mu_0 \right] t \otimes t + \mu_0 \left[ \lambda_0 + 2 \mu_0 \right] I \quad \text{and} \quad q_1 = \left[ \lambda_0 + \mu_0 \right] d \otimes d + \mu_0 I.
\]

In particular the bulk acoustic (localization) tensor, defined by \( q_0 = q_{ac} \), can be reformulated as

\[
q_{ac} = \left[ \lambda + 2 \mu \right] d \otimes d + \mu I.
\]

Note that any contribution from the prestress \( \gamma \) eventually vanishes in the definitions of the acoustic tensor and this is expected since the acoustic tensors are representing the material behavior and not its loading conditions. It is clear from the strong ellipticity of the bulk, detailed in Section 4.1, that the coefficients \( \lambda + 2 \mu \) and \( \mu \) will be strictly positive.

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### Appendix

It is enlightening to provide the intermediate steps to study the boundary complementing condition briefly introduced in Section 4.3. The position vector of a point within the bulk with respect to the origin of the local surface orthonormal base system \( t, h, d \) is denoted \( \mathbf{x} \). The projections of the position vector \( \mathbf{x} \) in the \( t \) and \( d \) directions are denoted as

\[
\xi = \mathbf{x} \cdot d \quad \text{and} \quad \eta = \mathbf{x} \cdot t.
\]

A stationary wave-type ansatz for the incremental displacement \( \delta u \) is prescribed as the product of a decay function \( u(\xi) \) and a waviness term:

\[
\delta u = u(\xi) \exp(ik\eta),
\]

where \( k > 0 \) is the wave number and \( i \) is the imaginary unit. The first and the second gradients of \( \delta u \) are thus given by

\[
\begin{align*}
\nabla \delta u &= [ \delta u \otimes d + ik \delta u \otimes t ] \exp(ik\eta), \\

\nabla^2 \delta u &= [ \delta u \otimes d \otimes d + ik \delta u \otimes t \otimes d + \delta u \otimes d \otimes \eta ] \exp(ik\eta) \\
&\quad - [k^2 \delta u \otimes t \otimes \eta ] \exp(ik\eta).
\end{align*}
\]

The incremental balance of momentum in the bulk can be related to the incremental displacement \( \delta u \) as

\[
\text{Div} \delta \mathbf{\Sigma} = \text{Div}[C_{\text{eff}} + \gamma I] \cdot \delta \mathbf{\nu} = [C_{\text{eff}} + \gamma I] \cdot \nabla \delta \mathbf{u} : \mathbf{I}.
\]

Eq. (39) yields the ordinary differential equation

\[
q_2 \cdot \mathbf{w} + ik q_1 \cdot \mathbf{w} - k^2 q_0 \cdot \mathbf{w} = 0,
\]

for the decay function \( u(\xi) \) where the localization tensors \( q_0, q_1 \), and \( q_2 \) are defined by

\[
\begin{align*}
q_0 &= [ \lambda + \mu ] [ \mathbf{t} \otimes \mathbf{t} ] \\
q_1 &= [ \lambda + \mu ] [ \mathbf{d} \otimes \mathbf{d} ] + [ \mathbf{t} \otimes \mathbf{t} ] \\
q_2 &= [ \lambda + \mu ] [ \mathbf{d} \otimes \mathbf{d} ].
\end{align*}
\]

The decay function \( u(\xi) \) can be expressed as

\[
u(\xi) = m \exp(\eta),
\]

where \( m \) is the amplitude of the wave and \( \eta < 0 \) is the decay exponent. Thus Eq. (40) can be written as

\[
[\mathbf{I} - i k \mathbf{q}_2 \cdot -k^2 \mathbf{q}_0] \cdot \mathbf{m} = \mathbf{q}_1, \quad \mathbf{m} = \mathbf{0},
\]

or alternatively in a form without the localization tensors as

\[
[C_{\text{eff}} + \gamma I] \cdot [\mathbf{I} - i k \mathbf{t} \otimes \mathbf{t} - i k \mathbf{d} \otimes \mathbf{d}] = \mathbf{q}_1 \cdot \mathbf{m} = \mathbf{0},
\]

where \( \mathbf{q}_1 \) denotes the surface localization tensor. The decay exponent \( r \) corresponding to non-trivial solutions for the amplitude \( \mathbf{m} \) are given by

\[
r = \arg_{\eta}(\det \mathbf{q}_1) = 0.
\]

Eq. (42) states the necessary conditions for the possibility of stationary wave-type solutions on the surface. Thus, Eq. (41) can be expressed in the following general form for distinct \( r \) and \( m \) with the coefficients \( a_i \) dependent on the boundary conditions:

\[
u(\xi) = \sum_{i=0}^{\infty} a_i m_i \exp(\eta_i).
\]

The performed analysis so far does not require the bulk to be isotropic, nevertheless, in the remainder of this work we apply the analysis to the special case of isotropy. For the case of isotropic, linear elasticity the bulk localization tensors \( q_0, q_1 \), and \( q_2 \) simplify to

\[
q_0 = [\lambda + \mu] t \otimes t + \mu [\lambda + 2 \mu] I \quad \text{and} \quad q_1 = [\lambda + \mu] d \otimes d + \mu I.
\]

In particular the bulk acoustic (localization) tensor, defined by \( q_0 = q_{ac} \), can be reformulated as

\[
q_{ac} = [\lambda + 2 \mu] d \otimes d + \mu I.
\]

Note that any contribution from the prestress \( \gamma \) eventually vanishes in the definitions of the acoustic tensor and this is expected since the acoustic tensors are representing the material behavior and not its loading conditions. It is clear from the strong ellipticity of the bulk, detailed in Section 4.1, that the coefficients \( \lambda + 2 \mu \) and \( \mu \) will be strictly positive.
The determinant of the surface localization tensor, which vanishes for non-trivial solutions, is given by

$$\det q_b = |k^2 - r^2|^2 [i + 2\mu] = 0 \quad \det q_b \neq 0$$

(44)

which corresponds to the triple roots $r = \pm \sqrt{k}$. In order to have bounded solutions, only negative values of the decay exponent $r$ are admissible, and hence $r = -\sqrt{k}$. Thus by manipulating Eq. (43) the decay function reads

$$w_\xi(t) = \left[ t + id \right] [a_0 + a_1 \xi + b_2] \exp(-k\xi)$$

(45)

The decay function is then inserted into stationary wave-type ansatz (38) which yields the following expression for the incremental displacement and its gradient:

$$\delta u = w_\xi \exp(ik\theta) = \left[ t + id \right] [a_0 + a_1 \xi + b_2] \exp(k[(\eta - \xi)])$$

$$\nabla \delta u = \left[ t + id \right] [a_0 - k(a_1 \xi + \eta) + b_2] \exp(k[(\eta - \xi)]) \otimes \delta$$

In order to study the boundary complementing condition on the surface, consider the incremental balance of linear momentum at the surface (27), here simply the incremental boundary condition, restated as

$$\vec{\delta} \cdot \vec{N} = \vec{\delta} \cdot [-\delta] = 0 \quad \text{with} \quad \vec{\delta} \vec{\delta} = \vec{\delta} \vec{\delta} |_{\xi = 0} = 0$$

(46)

where the evaluation of the incremental stress in the bulk at the surface, i.e., $\xi = 0$, is denoted by a hat above the stress. Employing the aforementioned constitutive assumptions and incremental constitutive law (28) in the bulk yields

$$\vec{\delta} = [C_{\text{eff}} + \gamma I] : \vec{\delta} w_\xi \exp(ik\theta)$$

(47)

The evaluation of the incremental displacement and its first gradient at the surface is given by

$$\nabla \delta u = \nabla \delta u |_{\xi = 0} = w_\xi(0) \exp(ik\theta) = \left[ t + id \right] [a_0 + a_1 \xi + b_2] \exp(ik\theta)$$

$$\nabla \delta u = \nabla(\delta u |_{\xi = 0}) = \left[ t + id \right] [a_0 - k(a_1 \xi + \eta) + b_2] \exp(k[(\eta - \xi)]) \otimes \delta \exp(ik\theta)$$

(48)

Note that the evaluation of the displacement gradient at the surface is not, in general, equal to the gradient of the displacement evaluated at the surface, i.e.,

$$\nabla \delta u = \nabla(\delta u |_{\xi = 0}) \neq \nabla \delta u = \nabla(\delta u |_{\xi = 0})$$

With these definitions at hand, and using Eq. (47), the left-hand side of the incremental balance of linear momentum at the surface (46) is computed next and is eventually set to zero. Inserting Eq. (47) into (48) yields the following expression for the incremental stress at the surface:

$$\vec{\delta} = [C_{\text{eff}} + \gamma I] : \vec{\delta} w_\xi = [3\delta_{\text{eff}}^{\text{vol}} + 2\mu_{\text{eff}}^{\text{symm}} + \gamma I] : [\left[ t + id \right] [a_0 - ka_2] - b_2] \otimes \delta \exp(ik\theta)$$

Next, the quantity $\vec{\delta} \cdot \delta$ is computed and set to zero.

$$\vec{\delta} \cdot \delta = [\delta_{\text{eff}}^{\text{vol}} + \mu_{\text{eff}}^{\text{symm}}] \left[ t + 2(a_1 - ka_2) \exp(ik\theta) + \gamma I \right] \otimes \delta = 0$$

(49)

or alternatively in matrix format as

$$B \cdot \delta = 0 \quad \text{where} \quad B = \begin{bmatrix} -2\mu k + \gamma k & \mu & 0 \\ -2\mu k + \gamma k & \lambda + 2\mu & 0 \\ 0 & 0 & -\gamma k \end{bmatrix} \quad \text{and} \quad \delta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

References


