COMMUNICATIONS IN NUMERICAL METHODS IN ENGINEERING
Commun. Numer. Meth. Engng 2004; **20**:511–519 (DOI: 10.1002/cnm.689)

A hybrid discontinuous Galerkin/interface method for the computational modelling of failure

J. Mergheim, E. Kuhl and P. Steinmann*,†

Department of Mechanical Engineering, Technical University of Kaiserslautern, P.O. Box 3049, D-67653 Kaiserslautern, Germany

SUMMARY

The present contribution is concerned with the computational modelling of failure along well-defined surfaces, which occur for example in the case of light-weight composite materials. A hybrid method will be introduced which makes use of the discontinuous Galerkin method in combination with a finite element interface approach. As a natural choice interface elements are introduced along the known failure surface. The discontinuous Galerkin method is applied in the pre-failure regime to avoid the unphysical use of penalty terms and instead to enforce the continuity of the solution along the interface weakly. Once a particular failure criterion is fulfilled, the behaviour of the interface is determined constitutively, depending on the displacement jump. The applicability of the proposed method is illustrated by means of two computational model problems. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: discontinuous Galerkin method; interface approach; failure; FE-technology

1. INTRODUCTION

The application of light-weight composite materials has become increasingly popular in recent years. The load carrying capacity of such composite structures is typically characterized through the failure of the weakest link, i.e. through the debonding of the adhesive layer in between two components or through the failure of the boundary layer very close to the adhesive. The accurate description of the delamination process can thus be considered the most essential ingredient in the design of composite structures, see References [1, 2].

When failure takes place along well-defined failure surfaces, the use of interface elements represents the most natural choice. In the case of pasted structures, for example, interfaces are placed in the adhesive layer. As soon as a particular failure criterion is met, the behaviour of the interface is determined constitutively through a traction-separation law whereby the

*Correspondence to: P. Steinmann, Department of Mechanical Engineering, Technical University of Kaiserslautern, P.O. Box 3049, D-67653 Kaiserslautern, Germany.
†Email: ps@rhrk.uni-kl.de

Contract/grant sponsor: German Science Foundation; contract/grant number:814

Received 18 July 2003
Accepted 27 November 2003

interface traction is typically introduced as a non-linear function of the displacement jump. The characterization of the post-failure regime is thus straightforward and well-accepted in the related literature, see References [3–5].

However, the classical treatment of the pre-failure regime is rather adhoc and somewhat inconsistent. Since the displacement field is double-valued at the interface, two finite element nodes have to be introduced at one material point. Traditionally, prior to failure, these two nodes are held together artificially with the help of a penalty method, whereby the choice of an appropriate penalty parameter is rather questionable.

The present contribution aims at deriving a consistent interface formulation by refraining from the use of penalty methods in the pre-failure regime. Rather, we suggest the weak enforcement of continuity at the interface by making use of Nitsche's method [6], which can be seen as the origin of the discontinuous Galerkin methods. Nitsche introduced a method to enforce the Dirichlet boundary conditions in a weak sense. Later Douglas and Dupont [7], Arnold [8] and Wheeler [9] extended Nitsche's approach to the weak enforcement of the continuity of the solution at the interior boundaries. In the last few years the discontinuous Galerkin method was extended and applied to various problems, see Reference [10] for an overview. Only recently the discontinuous Galerkin method gained an increased interest in the structural mechanics community through the works of Engel *et al.* [11], Hansbo and Hansbo [12] and Hansbo and Larsson [13].

This weak enforcement of continuity, which has also been applied successfully in combination with domain decomposition techniques, see Reference [14], represents a consistent strategy to tie together pairs of finite element nodes at the interface prior to failure. Like in classical discontinuous Galerkin methods, the average interface traction is enforced to vanish in an integral sense. Double-valued fields are thus treated consistently in the present approach and the use of otherwise unphysical penalty parameters is only necessary to stabilize the method.

The paper is organized as follows: firstly, we will review the strong form of the boundary value problem of a geometrically linear solid. Then the discontinuous Galerkin method is formulated for linear elasticity and afterwards the weak formulation of the interface approach is derived and the constitutive traction-separation law is given. Based on the previous results the hybrid method is formulated. Finally, some aspects of the spatial discretization are given and two numerical examples demonstrate the performance of the proposed method.

2. STRONG FORM OF THE BOUNDARY VALUE PROBLEM

Let $\Omega \subset \mathbb{R}^{n_{\dim}}$ denote the configuration occupied by an initially linear elastic body with placements in $\mathbb{R}^{n_{\dim}}$ denoted by $\mathbf{x}$. The boundary $\partial\Omega$ of Ω with the outward normal $\mathbf{n}$ is subdivided into disjoint parts $\partial\Omega = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}}$ with $\Gamma_{\mathrm{N}} \cap \Gamma_{\mathrm{D}} = \emptyset$, where either Neumann or Dirichlet boundary conditions are prescribed.

Then the unknown displacement $\mathbf{u}$ must satisfy:

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega \\ \mathbf{u} &= \mathbf{u}_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \\ \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} &= \mathbf{g} && \text{on } \Gamma_{\mathrm{N}} \end{aligned} \tag{1}$$

Here $\boldsymbol{\sigma}$ is the symmetric stress tensor with

$$\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon} \tag{2}$$

and $\boldsymbol{\varepsilon}$ is the strain tensor

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \tfrac{1}{2}[\nabla \mathbf{u} + \nabla^{t} \mathbf{u}] \tag{3}$$

$\mathbf{D}$ is the fourth-order constitutive tensor, $\mathbf{f}$ is the body force, $\mathbf{g}$ is the traction vector on the Neumann boundary, and $\mathbf{u}_{D}$ is the prescribed displacement on the Dirichlet boundary.

3. DISCONTINUOUS GALERKIN METHOD

In this section the modelling of the pre-failure state with the discontinuous Galerkin method is described. Therefore we assume that the potential failure zone is known and introduce an internal surface Γ_{I} along this zone. Γ_{I} divides the domain Ω into the parts Ω^1 and Ω^2. We associate a unit normal vector $\mathbf{n}$ and a tangential vector $\mathbf{m}$ to Γ_{I}. Thereby $\mathbf{n}$ points from Ω^2 to Ω^1, so that $\mathbf{n} = -\mathbf{n}^1 = \mathbf{n}^2$.

The discontinuous Galerkin method allows for discontinuities in the displacement field along Γ_{I}. Therefore we introduce a jump term and an average term

$$[\![\mathbf{u}]\!] := \mathbf{u}^1 - \mathbf{u}^2, \quad \{\mathbf{u}\} := 0.5[\mathbf{u}^1 + \mathbf{u}^2] \tag{4}$$

The average stress along Γ_{I} is defined by

$$\{\boldsymbol{\sigma}(\mathbf{u})\} := 0.5[\boldsymbol{\sigma}(\mathbf{u}^1) + \boldsymbol{\sigma}(\mathbf{u}^2)] \tag{5}$$

To obtain the weak formulation of the boundary value problem, we multiply the strong form of the boundary value problem (1) with a test function $\delta\mathbf{u}$ and integrate by parts over Ω^1 and Ω^2:

$$\begin{aligned} &\int_{\Omega^1} \delta\boldsymbol{\varepsilon}^1 : \boldsymbol{\sigma}(\mathbf{u}^1)\,\mathrm{d}V + \int_{\Omega^2} \delta\boldsymbol{\varepsilon}^2 : \boldsymbol{\sigma}(\mathbf{u}^2)\,\mathrm{d}V - \int_{\Gamma_I^1} \delta\mathbf{u}^1 \cdot \boldsymbol{\sigma}(\mathbf{u}^1)\cdot\mathbf{n}^1\,\mathrm{d}A - \int_{\Gamma_I^2} \delta\mathbf{u}^2 \cdot \boldsymbol{\sigma}(\mathbf{u}^2)\cdot\mathbf{n}^2\,\mathrm{d}A \\ &= \int_{\Omega^1\cup\Omega^2} \delta\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}V + \int_{\Gamma_N} \delta\mathbf{u}\cdot\mathbf{g}\,\mathrm{d}A \end{aligned} \tag{6}$$

We recall the definition of the normal vector $\mathbf{n}$ and consider that

$$\int_{\Gamma_I^1} \delta\mathbf{u}^1 \cdot \boldsymbol{\sigma}(\mathbf{u}^1)\cdot\mathbf{n}\,\mathrm{d}A - \int_{\Gamma_I^2} \delta\mathbf{u}^2 \cdot \boldsymbol{\sigma}(\mathbf{u}^2)\cdot\mathbf{n}\,\mathrm{d}A = \int_{\Gamma_I} [\![\delta\mathbf{u}\cdot\boldsymbol{\sigma}(\mathbf{u})]\!]\cdot\mathbf{n}\,\mathrm{d}A \tag{7}$$

With the relation

$$[\![\delta\mathbf{u}\cdot\boldsymbol{\sigma}]\!] = [\![\delta\mathbf{u}]\!]\cdot\{\boldsymbol{\sigma}\} + \{\delta\mathbf{u}\}\cdot[\![\boldsymbol{\sigma}]\!] \tag{8}$$

and provided that $\boldsymbol{\sigma}\cdot\mathbf{n}$ is continuous along Γ_I, so that $[\![\boldsymbol{\sigma}]\!]\cdot\mathbf{n}=0$, we get

$$\int_{\Omega^1\cup\Omega^2}\delta\boldsymbol{\varepsilon}:\boldsymbol{\sigma}(\mathbf{u})\,\mathrm{d}V+\int_{\Gamma_\mathrm{I}}[\![\delta\mathbf{u}]\!]\cdot\{\boldsymbol{\sigma}(\mathbf{u})\}\cdot\mathbf{n}\,\mathrm{d}A=\int_{\Omega^1\cup\Omega^2}\delta\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}V+\int_{\Gamma_\mathrm{N}}\delta\mathbf{u}\cdot\mathbf{g}\,\mathrm{d}A \tag{9}$$

Since the resulting equation is neither symmetric nor stable so far, the term $\int_{\Gamma_\mathrm{I}}\mathbf{n}\cdot\{\boldsymbol{\sigma}(\delta\mathbf{u})\}\cdot[\![\mathbf{u}]\!]\,\mathrm{d}A$ is added to symmetrize the method. And furthermore, in terms of Nitsche's method a penalty term $\int_{\Gamma_\mathrm{I}}(\gamma/h)[\![\delta\mathbf{u}]\!]\cdot[\![\mathbf{u}]\!]\,\mathrm{d}A$, γ being a penalty factor and h being the mesh size, is added to obtain a stabilized symmetric method. As we consider a continuous displacement field $\mathbf{u}$, the jump in the displacements $[\![\mathbf{u}]\!]$ will vanish along Γ_I and the equation is not changed due to the additional terms:

$$\begin{aligned}&\int_{\Omega^1\cup\Omega^2}\delta\boldsymbol{\varepsilon}:\boldsymbol{\sigma}(\mathbf{u})\,\mathrm{d}V+\int_{\Gamma_\mathrm{I}}\left[[\![\delta\mathbf{u}]\!]\cdot\{\boldsymbol{\sigma}(\mathbf{u})\}\cdot\mathbf{n}+\mathbf{n}\cdot\{\boldsymbol{\sigma}(\delta\mathbf{u})\}\cdot[\![\mathbf{u}]\!]\right]\mathrm{d}A\\&\quad+\int_{\Gamma_\mathrm{I}}\frac{\gamma}{h}[\![\delta\mathbf{u}]\!]\cdot[\![\mathbf{u}]\!]\,\mathrm{d}A=\int_{\Omega^1\cup\Omega^2}\delta\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}V+\int_{\Gamma_\mathrm{N}}\delta\mathbf{u}\cdot\mathbf{g}\,\mathrm{d}A\end{aligned} \tag{10}$$

Formulation (10) assures the weak enforcement of the continuity of the solution at Γ_I, which is required in the pre-critical state.

4. INTERFACE APPROACH

A finite element interface formulation is applied to model the post-critical state, after a failure criterion has been met. The interface formulation accounts for strong discontinuities in the displacement field along the discontinuity surface Γ_I. The post-critical material behaviour, namely the development of the discontinuities in the displacements, is governed by a constitutive traction-separation law of the interface.

Recall that jumps of field quantities $(\bullet)$ across Γ_I are denoted by $[\![(\bullet)]\!]=(\bullet)^1-(\bullet)^2$. We then conclude that the test function or virtual displacement function $\delta\mathbf{u}$ exhibits a discontinuity $[\![\delta\mathbf{u}]\!]$ along Γ_I. Taking into account the condition that $\mathbf{t}$ is continuous along Γ_I with $\mathbf{t}=-\mathbf{t}^1=\mathbf{t}^2$, we get the weak formulation of the interface approach with an additional contribution due to the discontinuity:

$$\int_{\Omega^1\cup\Omega^2}\delta\boldsymbol{\varepsilon}:\boldsymbol{\sigma}(\mathbf{u})\,\mathrm{d}V+\int_{\Gamma_\mathrm{I}}[\![\delta\mathbf{u}]\!]\cdot\mathbf{t}([\![\mathbf{u}]\!])\,\mathrm{d}A=\int_{\Omega^1\cup\Omega^2}\delta\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}V+\int_{\Gamma_\mathrm{N}}\delta\mathbf{u}\cdot\mathbf{g}\,\mathrm{d}A \tag{11}$$

The relation of the traction vector $\mathbf{t}$ and the jump of the displacements $[\![\mathbf{u}]\!]$ is given constitutively. Here the constitutive law of the interface is chosen independently of the constitutive setting of the surrounding domain. An exponential softening of the material is assumed in the post-critical state and can be formulated as

$$\begin{aligned}t_n([\![\mathbf{u}\cdot\mathbf{n}]\!])&=\bar{t}_n\exp(-c\,[\![\mathbf{u}\cdot\mathbf{n}]\!])\\t_m([\![\mathbf{u}\cdot\mathbf{m}]\!])&=\bar{t}_m\exp(-c\,[\![\mathbf{u}\cdot\mathbf{m}]\!])\end{aligned} \tag{12}$$

where c affects the gradient of the curve. Obviously, the normal and tangential components t_n and t_m has to be considered separately, whereby $\mathbf{t}=t_n\mathbf{n}+t_m\mathbf{m}$.

5. HYBRID METHOD

Based on the approaches introduced in the last two sections, we are now able to formulate the hybrid method. The basic idea of the hybrid method is to combine the discontinuous Galerkin method with the interface approach in a way, that the discontinuous Galerkin method assures the weak enforcement of the continuity of the solution along Γ_{I} in the pre-failure regime and that the interface approach controls the jump in the displacements in the post-critical state. Therefore, we combine the weak formulations (10) and (11) with a switching factor α and obtain the weak formulation of the hybrid method:

$$\int_{\Omega^1\cup\Omega^2} \delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma}(\mathbf{u})\,\mathrm{d}V + [1-\alpha]\int_{\Gamma_{\mathrm{I}}} \left[[\![\delta\mathbf{u}]\!]\cdot\{\boldsymbol{\sigma}(\mathbf{u})\}\cdot\mathbf{n} + \mathbf{n}\cdot\{\boldsymbol{\sigma}(\delta\mathbf{u})\}\cdot[\![\mathbf{u}]\!] \right]\mathrm{d}A$$
$$+ \int_{\Gamma_{\mathrm{I}}} [\![\delta\mathbf{u}]\!]\cdot\left[[1-\alpha]\frac{\gamma}{h}[\![\mathbf{u}]\!] + \alpha\mathbf{t}([\![\mathbf{u}]\!])\right]\mathrm{d}A = \int_{\Omega^1\cup\Omega^2} \delta\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}V + \int_{\Gamma_{\mathrm{N}}} \delta\mathbf{u}\cdot\mathbf{g}\,\mathrm{d}A \tag{13}$$

Thereby the factor α controls the switch from the discontinuous Galerkin method to the interface approach. We set $\alpha = 0$ in the pre-critical state and once a certain failure criterion is met, $\alpha = 1$ and remains constant thereafter. As mentioned, the decisive factor for the change of the method is given by the failure criterion, which is chosen depending on the traction vector of the discontinuity surface Γ_{I}:

$$\{\boldsymbol{\sigma}(\mathbf{u})\}:[\mathbf{n}\otimes\mathbf{n}] + \beta|\{\boldsymbol{\sigma}(\mathbf{u})\}:[\mathbf{n}\otimes\mathbf{m}]| - t^{\mathrm{crit}} \leqslant 0, \quad \rightarrow \alpha = 0 \tag{14}$$

To ensure a continuous transition from the discontinuous Galerkin method to the interface approach, the values, which are reached for $\{\boldsymbol{\sigma}(\mathbf{u})\}:[\mathbf{n}\otimes\mathbf{n}]$ and $\{\boldsymbol{\sigma}(\mathbf{u})\}:[\mathbf{n}\otimes\mathbf{m}]$ in the moment of failure provide the normal and tangential components of the traction vector $\mathbf{t}$ for $[\![\mathbf{u}]\!] = \mathbf{0}$, namely $\bar{t}_n$ and $\bar{t}_m$. Note that this continuous transition becomes more difficult in the case of non-matching meshes.

6. DISCRETIZATION

The weak form associated with the domains Ω^1 and Ω^2 is discretized with standard isoparametric elements. The geometry $\mathbf{x}$ is expanded elementwise by shape functions N^k

$$\Omega = \bigcup_e \Omega_e, \quad \mathbf{x}|_{\Omega_e} = \sum_k N^k \mathbf{x}_k \tag{15}$$

and in terms of the isoparametric concept, the displacement field $\mathbf{u}$ and its variation $\delta\mathbf{u}$ are expanded by the same shape functions

$$\mathbf{u}|_{\Omega_e} = \sum_k N^k \mathbf{u}_k, \quad \delta\mathbf{u}|_{\Omega_e} = \sum_k N^k \delta\mathbf{u}_k \tag{16}$$

Based on the above discretizations the corresponding gradients $\boldsymbol{\varepsilon}$ and $\delta\boldsymbol{\varepsilon}$ take the format

$$\boldsymbol{\varepsilon}|_{\Omega_e} = \sum_k [\mathbf{u}_k \otimes \nabla N^k]^{\mathrm{sym}}, \quad \delta\boldsymbol{\varepsilon}|_{\Omega_e} = \sum_k [\delta\mathbf{u}_k \otimes \nabla N^k]^{\mathrm{sym}} \tag{17}$$

We denote two elements, which border on Γ_{I} with Ω_e^1 and Ω_e^2. $\check{N}_k$ indicates the set of shape functions N_k evaluated at the relevant element boundary. The discretization of the corresponding jump and average terms reads

$$\begin{aligned} [\![\mathbf{u}]\!]|_{\Gamma_{\mathrm{I}}} &= \sum_k [\check{N}^{k1}\mathbf{u}_k|_{\Omega_e^1} - \check{N}^{k2}\mathbf{u}_k|_{\Omega_e^2}] \\ \{\mathbf{u}\}|_{\Gamma_{\mathrm{I}}} &= 0.5\sum_k [\check{N}^{k1}\mathbf{u}_k|_{\Omega_e^1} + \check{N}^{k2}\mathbf{u}_k|_{\Omega_e^2}] \end{aligned} \tag{18}$$

In the pre-critical region the discontinuous Galerkin method provides a linear and symmetric system of equations, which can be solved directly. Due to the non-linear constitutive law of the interface approach a non-linear system of equations is generated in the post-critical state, which is solved iteratively by a Newton–Raphson scheme.

7. NUMERICAL EXAMPLES

The following two numerical examples demonstrate the applicability of the proposed method. For the sake of simplicity we restrict ourselves to the special case of matching meshes. In the first example a purely mode I failure is considered to study the influence of different discretizations and to check the transition from the discontinuous Galerkin method to the interface approach. The geometry and the loading conditions of the model problem are pictured in Figure 1(a), the potential failure zone is introduced in the middle of the bar and the quasi-static force is applied to both sides of the bar by displacement control. It is shown in Figure 1(b) that the load-displacement curve is independent of the discretization and that the transition from the discontinuous Galerkin method to the interface approach is smooth. Furthermore the effect of the choice of the factor c on the softening behaviour is shown. The larger the coefficient c, the more brittle, the material response. Figure 2 displays the

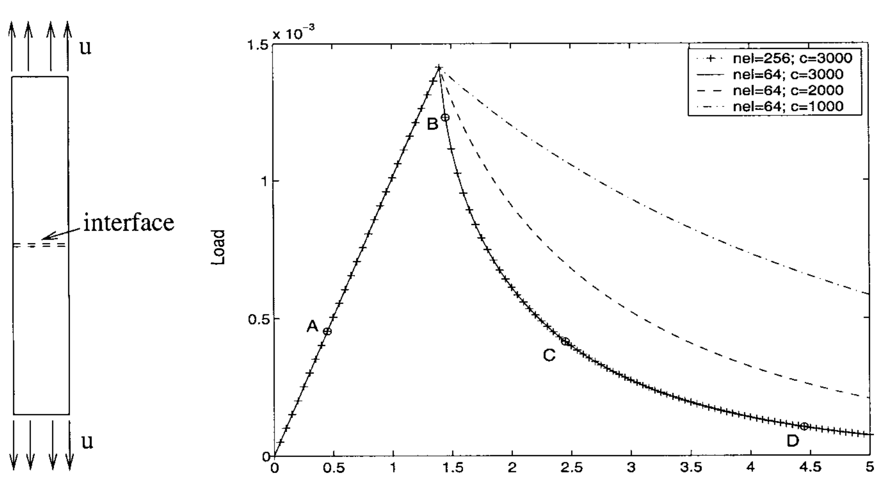

Figure 1. Example 1—geometry of the structure and load–displacement relation.

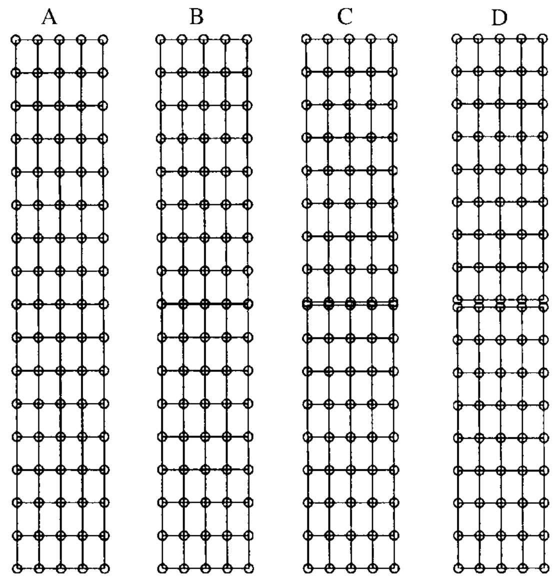

Figure 2. Example 1—deformation of the structure.

deformation of the structure at the different time steps A–D, as indicated in Figure 1(b). The opening of the interface can be seen clearly.

The second example is concerned with mixed mode failure. Figure 3 depicts the geometry of the structure, the loading conditions and the resulting load-displacement relation. The load-displacement curve is not as smooth as in the first example, because the failure criterion is not met at the same time for all element boundaries at Γ_I, but successively. The deformation of the structure and the relaxation of stresses and strains in the two continuous parts as well as the lateral sliding as a consequence of the development of the discontinuity in the displacement field are shown in Figure 4. In this example an additional penalty term, which enforces that $[\![\mathbf{u} \cdot \mathbf{n}]\!] \geqslant 0$, is added to the weak formulation of the hybrid form (13), to prevent the penetration of the two parts of the structure after the failure of the interface.

8. CONCLUSION

A consistent hybrid formulation for the computational modelling of failure along a known interface has been proposed. Prior to failure the discontinuous Galerkin method is applied to enforce the continuity of the solution weakly and to refrain from the use of unphysical penalty parameters. As soon as the failure criterion is met, a switch from the discontinuous Galerkin method to the interface approach takes place. The material behaviour in the post-critical regime is described by the constitutive traction-separation law of the interface, which is chosen independently of the constitutive setting of the surrounding domain. The applicability of the method was shown by means of two numerical examples and the expected results were achieved. Future work will consider the case of non-matching meshes, which can lead

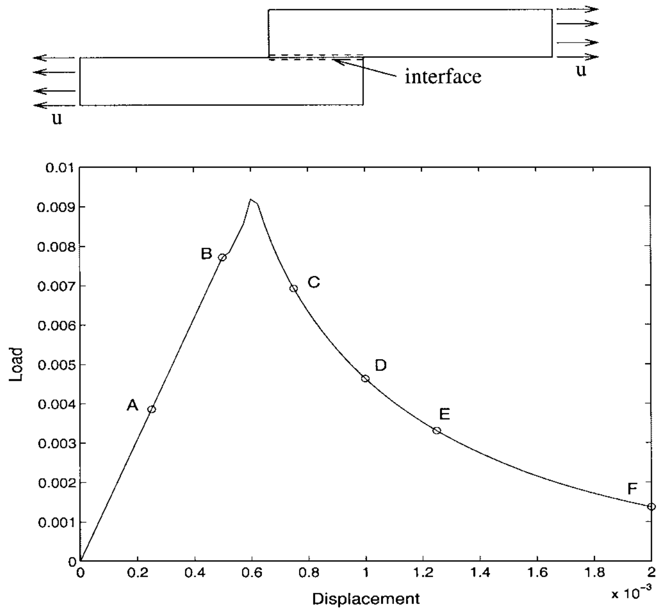

Figure 3. Example 2—geometry of the structure and load–displacement relation.

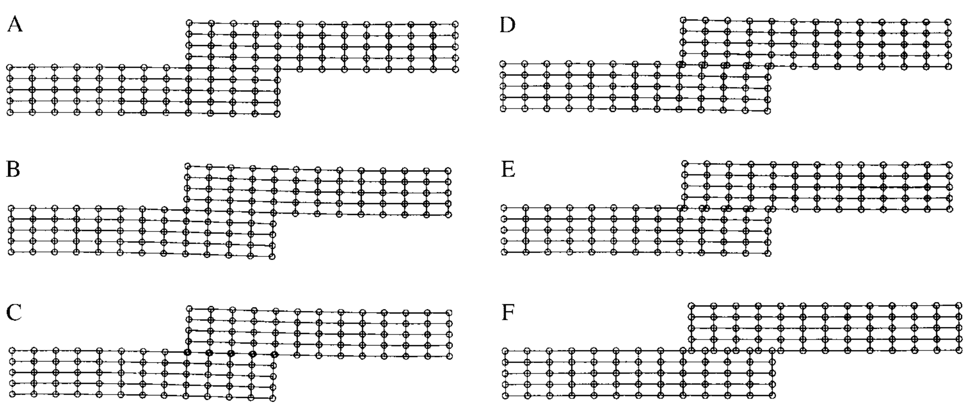

Figure 4. Example 2—deformation of the structure.

to stability problems and will require a more careful transition from the dG method to the interface approach.

ACKNOWLEDGEMENT

We gratefully acknowledge the financial support by the German Science Foundation: DFG Graduiertenkolleg 814, Engineering materials on different scales: experiment, modelling and simulation.

REFERENCES

1. Hashagen F. Numerical analysis of failure mechanisms in fibre metal laminates. *Ph.D. Thesis*, Technische Universiteit Delft, Delft, Netherlands, 1998.
2. Schellekens JCJ. Computational strategies for composite structures. *Ph.D. Thesis*, Technische Universiteit Delft, Delft, Netherlands, 1998.
3. Miehe C, Schröder J. Post critical discontinuous localization analysis of small-strain softening elastoplastic solids. *Archive of Applied Mechanics* 1984; **64**:267–285.
4. Larsson R, Runesson K, Ottosen NS. Discontinuous displacement approximation for capturing plastic localization. *International Journal for Numerical Methods in Engineering* 1993; **36**:2087–2105.
5. Schellekens JCJ, De Borst R. On the numerical integration of interface elements. *International Journal for Numerical Methods in Engineering* 1993; **36**:43–66.
6. Nitsche J. Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abhandiungen sus dem Mathematischen der Universitat Hamburg* 1970; **36**:9–15.
7. Douglas J, Dupont T. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing Methods in Applied Science*, Lecture Notes in Physics, vol. 58. Springer: Berlin, 1976; 207–276.
8. Arnold DN. An interior penalty finite element method with discontinuous elements. *SIAM Journal on Numerical Analysis* 1982; **19**(4):742–760
9. Wheeler MF. An elliptic collocation-finite element method with interior penalties. *SIAM Journal on Numerical Analysis* 1978; **15**(1):152–162.
10. Cockburn B, Karniadakis GE, Shu CW. *Discontinuous Galerkin Methods*: *Theory*, *Computation and Applications*. Springer: Berlin, 2000.
11. Engel G, Garikipati K, Hughes TJR, Larson MG, Mazzei L, Taylor RL. Continuous/discontinuous finite element approximation of fourth order elliptic problems in structural and continuum mechanics with application to thin beams and plates and strain gradient plasticity. *Computer Methods in Applied Mechanics and Engineering* 2002; **191**:3669–3750.
12. Hansbo A, Hansbo P. A Finite Element method for the simulation of strong and weak discontinuities in elasticity. *Chalmers Finite Element Center*, *Preprint*, 2003; 09.
13. Hansbo P, Larson MG. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Computer Methods in Applied Mechanics and Engineering* 2002; **191**:1895–1908.
14. Becker R, Hansbo P, Stenberg R. A finite element method for domain decomposition with non-matching grids. *Mathematical Modelling and Numerical Analysis* 2003; **37**:209–225.

