

2 Introduction to mechanics

2.1 Motivation

Thermodynamic bodies are being characterized by two competing opposite phenomena, energy and entropy which some researchers in thermodynamics would classify as 'cause' and 'chance' or 'determinancy' and 'random walk', see Müller and Weiss for a nice discussion of both [9]. While at low temperatures energy driven phenomena dominate the body's behavior in terms of mechanical strain energy, stress and strain, the influence of entropy increases significantly with increasing temperature. Think about biopolymers and biomembranes. When heated up, polymers will no longer rest at a constant position in space. Rather their individual molecules will move around giving rise to thermal fluctuation. Not only the temperature but also the polymer density in the solution determines the state of a biological substrate. At higher densities, the substrate takes a condensed state of matter and energy dominates its behavior. At lower densities, however, the solution phase is favored in which molecules can move around freely. This state is rather dominated by entropic behavior. To account for both phenomena, it is common to express the overall free energy ψ as a sum of strain energy W and the entropy S whereby the latter is weighted by the negative absolute temperature T .

$$\psi = W - T S \tag{2.1.1}$$

At a condensed state, the behavior is almost solid like, the strain energy W dominates the response at relatively high density or zero temperature T . At a solute state, the behavior is rather fluid like, the entropy contribution $T S$ dominates at relatively low density or finite temperature T . A transition between phases obviously takes place at $\psi = 0$, or, equivalently, at $W = T S$.

Remember what Chris said in class about the zero-temperature limit. This does not literally mean that the cell is frozen to zero degrees. Rather it is a common way of expressing that for the particular process we are interested in, the thermal or entropic contribution $-T S$ is negligible as compared to the mechanic contribution W .

2.2 Energy

2.2.1 Motivation

Mechanics is actually super simple and nothing to be afraid of. In fact, it all boils down to three basic equations. Unfortunately, for technical reasons, or what some peo-

ple would claim simplifications, these three equations are often phrased in different terms and different symbols are used for clarification, or sometimes rather for confusion. Within these notes, we have decided to use the same notation throughout and point out explicitly when a different notation has found common acceptance in the literature. Now, think of your strength of materials class, or, if you hadn't taken one before, just think of your high school classes in physics! What are the three most important equations that you would remember when thinking of mechanics?

Your three most important equations of mechanics Here are the results of our class survey. What are the most important equations that you would remember when thinking of mechanics? Surprisingly, everybody remembers the **constitutive equations**, the stress strain relations, which are sometimes also referred to as material model. Seven of you listed its one dimensional version in the form of Hooke's law two listed Hooke's law for a linear spring $F = kx$, one listed the its version for torsion $T = L\Theta$ and two listed even listed the three dimensional version $\epsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y - \nu\sigma_z]$. Related to the above equation, one listed the definition of Poisson's ratio $\nu = -\epsilon^{\text{trans}}/\epsilon^{\text{long}}$ and one mentioned the strain energy of a Hooke'an material $W = 1/2\sigma\epsilon$. Four of you stated the **equilibrium equations**, which are sometimes also referred to as momentum balance in the related continuum mechanics literature. Two of them used the one dimensional form for rigid bodies $F = ma$, two mentioned its form for a deformable continuum $\text{div}\sigma + \rho b = \rho\ddot{u}$. Three remembered the **kinematic equations**, two expressed them for a one-dimensional bar $\epsilon = \Delta l / l$, one for a continuum $\epsilon = \partial u / \partial x$. Finally, two of you remembered the relation between stress and stress resultants $\sigma = N / A$. Two even stated the differential equation for beam bending $y'' = M / [EI]$ which actually is a result of the combination of the three sets of equations discussed above. The cool thing is, all the equations you remembered will be addressed in this class!

We will repeatedly use the three equations that most of you mentioned. In this section, we will relate them to a simple one dimensional truss. Later, we will discuss the same set of equations for three dimensional continuous bodies, for beams, for membranes and for shells. But for now, let's just stick to the one dimensional truss.

Let us first agree on how we could characterize the state of deformation of the truss! A typical measure would be the elongation or change in length Δl . Sometimes we are not only interested in the total elongation at the end of the truss but rather in the elongation everywhere in the truss. We thus introduce the notion displacement $u(x)$ with the understanding that the total elongation is equivalent to the end displacement $u(x = l) = \Delta l$ provided the other end of the truss is fixed $u(x = 0) = 0$. More important than the absolute displacement $u(x)$ is the relative displacement with respect to the original length $\epsilon = \Delta l / l$. This ratio would be referred to as strain and is typically associated with the symbol ϵ provided we restrict ourselves to small deformations. It takes the following general definition.

$$\circ \text{ kinematics} \quad \epsilon = \lim_{x \rightarrow 0} \frac{u}{x} = \frac{du}{dx} = u_{,x} \quad \text{homogeneous} \quad \epsilon = \frac{\Delta l}{l} \quad (2.2.1)$$

Hey, I hope you don't mind if I continue using the abbreviation $d(\circ) / dx = (\circ)_{,x}$ which someone in class referred to as the European notation for the derivative. Sometimes, for one dimensional problems, the notion stretch is used instead of strain, stretches are often denoted by λ but take the same definition. For homogeneous one dimensional structures $u_{,x}$ is constant along the length l and thus, indeed $\varepsilon = \Delta l / l$. The second important equation is the relation between strain and stress which essentially characterizes the material behavior. In the simplest form, the strain ε can be related to the stress σ in a linear way through Hooke's law.

$$\circ \text{ constitutive equation} \quad \sigma = \sigma(\varepsilon) \quad \text{linear elastic} \quad \sigma = E \varepsilon \quad (2.2.2)$$

The constant E is referred to as Young's modulus, sometimes it therefore also denoted as Y . Recall that stresses have the unit of force per length squared or force per area and strains are unitless. Young's modulus thus has the same unit as stress, i.e. force per length squared.

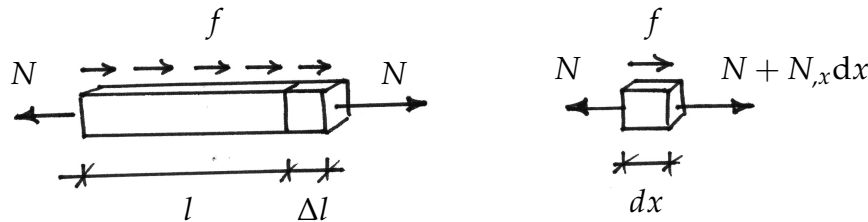


Figure 2.1: One dimensional truss of initial length l subject to axial force f which stretches it by the amount Δl (left) and infinitesimal truss element of length dx with resultant forces N on the negative and $N + N_{,x} dx$ on the positive side

The third and last equation you should remember is the equilibrium equation. Unfortunately, there is no such thing as stress equilibrium. Equilibrium is a relation between forces. So now, how do we relate the stress inside the truss to the force acting on a particular cross section of the truss? Do you remember the definition of stress $\sigma = N / A$ as force N per area A ? In general, the force N sometimes also referred to as stress resultant can be obtained by integrating the stress over the cross section area $dydz = A$.

$$\circ \text{ stress resultant} \quad N = \iint \sigma \, dydz \quad \text{homogeneous} \quad \sigma = \frac{N}{A} \quad (2.2.3)$$

For the particular case of a homogeneous cross section with constant EA , we have $N = EA \varepsilon$. Now, recall force equilibrium as it was introduced in your statics class. Take a look at a small, an infinitesimal, truss element. Equilibrium states that the sum of all forces acting on element must vanish, $\sum f \doteq 0$. Summing up all forces along the truss axis as displayed in figure 5.3, we get the following expression $[-N] + [N + N_{,x} dx] + [f dx] = 0$. This equation can be further simplified, as the $-N$ term and $+N$ term cancel. The remaining equation is divided by dx to render the force equilibrium equation for a one dimensional truss.

$$\circ \text{ equilibrium} \quad \sum f \doteq 0 \quad \text{in axial direction} \quad N_{,x} + f = 0 \quad (2.2.4)$$

This equation tells us something about forces acting on the truss but, unfortunately, it does not tell us anything about what's going on inside the truss! To look inside the truss, we have to 'convert' the stress resultant N to the stress measure $N = A \sigma$, make use of the constitutive equation $\sigma = E \varepsilon$ and the truss kinematics $\varepsilon = u_{,x}$. The equilibrium equation $N_{,x} + f = A \sigma_{,x} + f = EA \varepsilon_{,x} + f = EA u_{,xx} + f = 0$ then translates to the classical Laplace equation for trusses

$$\circ \text{ differential equation} \quad EA u_{,xx} + f = 0 \quad \text{or} \quad EA \Delta u + f = 0 \quad (2.2.5)$$

In the mathematical literature, this second order differential equation for the unknown displacement u would typically be expressed in terms of the Laplace differential operator $\Delta = \nabla^2 = d^2(\circ) / dx^2 = (\circ)_{,xx}$. For homogeneous problems the solution would simply be given as $u = NL / [EA]$ with EA characterizing the axial stiffness of the truss. Now, let's try to generalize this concept to three dimensions!

2.2.2 Kinematics: The strain displacement relation

What is strain? The equations that relate strain and displacement are the kinematic equations. These are general equations that characterize the deformation of a physical body without studying its physical cause. The strain components can be represented in a matrix. Mathematically speaking, the strain components in the matrix can be transformed into any other coordinate system, therefore the strain is considered to be a second order tensor, a $n_{\text{dim}} \times n_{\text{dim}}$ matrix with related base vectors where $n_{\text{dim}} = 1, 2, 3$ is the spatial dimension of the problem. The matrix of strain components is symmetric such that $\varepsilon_{xy} = \varepsilon_{yx}$, $\varepsilon_{yz} = \varepsilon_{zy}$ and $\varepsilon_{zx} = \varepsilon_{xz}$. Let us look at small strains first and try to generalize the strain expression we had introduced for the one dimensional truss. For the general three dimensional setting there are three normal strains ε_{xx} , ε_{yy} and ε_{zz} , one for each direction in space.

$$\begin{aligned} \varepsilon_{xx} &= \lim_{x \rightarrow 0} \frac{u}{x} = \frac{du}{dx} & \varepsilon_{xx} &= u_{,x} \\ \varepsilon_{yy} &= \lim_{y \rightarrow 0} \frac{v}{y} = \frac{dv}{dy} & \varepsilon_{yy} &= v_{,y} \\ \varepsilon_{zz} &= \lim_{z \rightarrow 0} \frac{w}{z} = \frac{dw}{dz} & \varepsilon_{zz} &= w_{,z} \end{aligned} \quad (2.2.6)$$

Normal strains indicate a stretch of the body, they are related to volumetric changes. There are deformation modes, however, for which the body does not undergo any volumetric changes at all. These isochoric deformations are related to changes in angles which are represented through the shear strains ε_{xy} , ε_{yz} and ε_{zx} .

$$\begin{aligned} \varepsilon_{xy} &= \frac{1}{2} \left[\frac{du}{dy} + \frac{dv}{dx} \right] & \varepsilon_{xy} &= \frac{1}{2} [u_{,y} + v_{,x}] = \varepsilon_{yx} \\ \varepsilon_{yz} &= \frac{1}{2} \left[\frac{dv}{dz} + \frac{dw}{dy} \right] & \varepsilon_{yz} &= \frac{1}{2} [v_{,z} + w_{,y}] = \varepsilon_{zy} \\ \varepsilon_{zx} &= \frac{1}{2} \left[\frac{dw}{dx} + \frac{du}{dz} \right] & \varepsilon_{zx} &= \frac{1}{2} [w_{,x} + u_{,z}] = \varepsilon_{xz} \end{aligned} \quad (2.2.7)$$

You could imagine those as sliding modes of deformation. To clearly distinguish normal and shear strains, sometimes the symbol γ is used for the components of the shear strain. The notation γ introduces the engineering strain, it differs from the continuum strain ε by a factor two, i.e. $\gamma_{xy} = 2\varepsilon_{xy}$, $\gamma_{yz} = 2\varepsilon_{yz}$ and $\gamma_{zx} = 2\varepsilon_{zx}$.

The normal and shear strains we have introduced so far characterize the kinematics of body at small deformations. Actually, there are some non-linear terms in the strain expression which we have neglected thus far. Just imagine cell squeezed around in its physiological environment! Its deformations can be huge as compared to its size! For bodies which undergo large deformations, it is essential to introduce finite kinematics with a truly nonlinear strain displacement relation. The key kinematic quantity in large deformation problems is the deformation gradient F , the gradient of the new position $[x + u, y + v, z + w]$ with respect to the old position $[x, y, z]$. In contrast to the strains, the deformation gradient is a non-symmetric second order tensor that consists of a matrix with the following entries and the corresponding base vectors.

$$\begin{aligned} F_{xx} &= 1 + u_{,x} & F_{xy} &= u_{,y} & F_{xz} &= u_{,z} \\ F_{yx} &= v_{,x} & F_{yy} &= 1 + v_{,y} & F_{yz} &= v_{,z} \\ F_{zx} &= w_{,x} & F_{zy} &= w_{,y} & F_{zz} &= 1 + w_{,z} \end{aligned} \quad (2.2.8)$$

The strains at large deformation that will be relevant in the following chapters are the Green Lagrange strains. In the continuum mechanics literature, they are typically introduced as $E_{ij} = \frac{1}{2}[F_{ki}F_{kj} - \delta_{ij}]$ which is a short term notation for $E_{ij} = \frac{1}{2}[F_{xi}F_{xj} + F_{yi}F_{yj} + F_{zi}F_{zj} - \delta_{ij}]$. This abstract definition introduces the three normal strain components E_{xx} , E_{yy} and E_{zz} and the three shear strain components E_{xy} , E_{yz} and E_{zx} .

$$\begin{aligned} E_{xx} &= u_{,x} + \frac{1}{2} [u_{,x}^2 + v_{,x}^2 + w_{,x}^2] \\ E_{yy} &= v_{,y} + \frac{1}{2} [u_{,y}^2 + v_{,y}^2 + w_{,y}^2] \\ E_{zz} &= w_{,z} + \frac{1}{2} [u_{,z}^2 + v_{,z}^2 + w_{,z}^2] \\ E_{xy} &= \frac{1}{2} [u_{,y} + v_{,x}] + \frac{1}{2} [u_{,x}u_{,y} + v_{,x}v_{,y} + w_{,x}w_{,y}] = E_{yx} \\ E_{yz} &= \frac{1}{2} [v_{,z} + w_{,y}] + \frac{1}{2} [u_{,y}u_{,z} + v_{,y}v_{,z} + w_{,y}w_{,z}] = E_{zy} \\ E_{zx} &= \frac{1}{2} [w_{,x} + u_{,z}] + \frac{1}{2} [u_{,z}u_{,x} + v_{,z}v_{,x} + w_{,z}w_{,x}] = E_{xz} \end{aligned} \quad (2.2.9)$$

By its very definition, the matrix of the Green Lagrange strain components is again symmetric. It is also easy to show that in the limit of small deformation, when products of derivatives are really small as compared to the derivatives themselves and thus negligible, the Green Lagrange strains E correspond to the small strains ε of equations (1.2.6) and (1.2.7). For example, at small strains $E_{xy} = 1/2 [u_{,y} + v_{,x}] + 1/2 [u_{,x}u_{,y} + v_{,x}v_{,y} + w_{,x}w_{,y}] \approx 1/2 [u_{,y} + v_{,x}] = \varepsilon_{xy}$.

2.2.3 Constitutive equations: The stress-strain relation

How are strain and stress related? The equations that relate stress and strain are the constitutive equations. Unlike the kinematic equations and the equilibrium equations, the constitutive equations are not general. They are they are material specific equations that complement the set of governing equations and are therefore sometimes just referred to as material model. The three dimensional generalization of the $\sigma = E \varepsilon$ law introduced in the motivation section is Hooke's law for isotropic, linear elastic solids.

$$\begin{aligned}
 \sigma_{xx} &= \frac{E}{1-\nu^2} [\varepsilon_{xx} + \nu \varepsilon_{yy} + \nu \varepsilon_{zz}] & \sigma_{xy} &= \frac{E}{1+\nu} \varepsilon_{xy} = \sigma_{yx} \\
 \sigma_{yy} &= \frac{E}{1-\nu^2} [\varepsilon_{yy} + \nu \varepsilon_{xx} + \nu \varepsilon_{zz}] & \sigma_{yz} &= \frac{E}{1+\nu} \varepsilon_{yz} = \sigma_{zy} \\
 \sigma_{zz} &= \frac{E}{1-\nu^2} [\varepsilon_{zz} + \nu \varepsilon_{xx} + \nu \varepsilon_{yy}] & \sigma_{zx} &= \frac{E}{1+\nu} \varepsilon_{zx} = \sigma_{xz}
 \end{aligned} \tag{2.2.10}$$

Inversely, Hooke's law could be rephrased to gain an expression for the strains in terms of given stresses.

$$\begin{aligned}
 \varepsilon_{xx} &= \frac{1}{E} [\sigma_{xx} - \nu \sigma_{yy} - \nu \sigma_{zz}] & \varepsilon_{xy} &= \frac{1+\nu}{E} \sigma_{yx} = \varepsilon_{xy} \\
 \varepsilon_{yy} &= \frac{1}{E} [\sigma_{yy} - \nu \sigma_{xx} - \nu \sigma_{zz}] & \varepsilon_{yz} &= \frac{1+\nu}{E} \sigma_{yz} = \varepsilon_{yz} \\
 \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu \sigma_{xx} - \nu \sigma_{yy}] & \varepsilon_{zx} &= \frac{1+\nu}{E} \sigma_{zx} = \varepsilon_{xz}
 \end{aligned} \tag{2.2.11}$$

In contrast to the one dimensional truss model, we have now introduced two constants, the material parameters E and ν , Young's modulus and Poisson's ratio. We have already seen the interpretation of Young's modulus, it basically tells us the stress that is generated by stretching a material at a particular strain. Its unit is similar to the one of stresses, i.e., force per length squared. As you can guess from the above equations, Poisson's ratio is unitless. It is a measure of how much a material contracts in the lateral direction when stretched along one axis. Unlike Young's modulus, Poisson's ratio is limited to the regime $-1.0 \leq \nu \leq 0.5$.

2.2.4 Equilibrium: The stress vs force relation

What is stress? The equations that relate external or applied forces to internal forces are the equilibrium equations. Like the kinematic equations, they are general equations that are valid for any solid, independent of the material it is made of. Unfortunately, equilibrium cannot be expressed in terms of stresses right away, it has to be phrased in terms of forces. Therefore, it is important to first determine the forces that act on a particular cross section. Just like in the one dimensional example, but now a bit more cumbersome, these forces can be obtained by intergrating the stresses over the total area which they are acting on. The resulting forces are therefore often referred to as stress resultants. Each of the nine stress resultants is then related to one of the stress

components introduced in equation (1.2.10).

$$\begin{aligned}
 N_{xx} &= \iint \sigma_{xx} \, dydz & N_{xy} &= \iint \sigma_{xy} \, dydz & N_{xz} &= \iint \sigma_{xz} \, dydz \\
 N_{yx} &= \iint \sigma_{yx} \, dx dz & N_{yy} &= \iint \sigma_{yy} \, dx dz & N_{yz} &= \iint \sigma_{yz} \, dx dz \\
 N_{zx} &= \iint \sigma_{zx} \, dx dy & N_{zy} &= \iint \sigma_{zy} \, dx dy & N_{zz} &= \iint \sigma_{zz} \, dx dy
 \end{aligned} \quad (2.2.12)$$

Again, the first index refers to the direction of the cross section normal, the second index refers to the direction of the stress resultant. Similar to the one dimensional

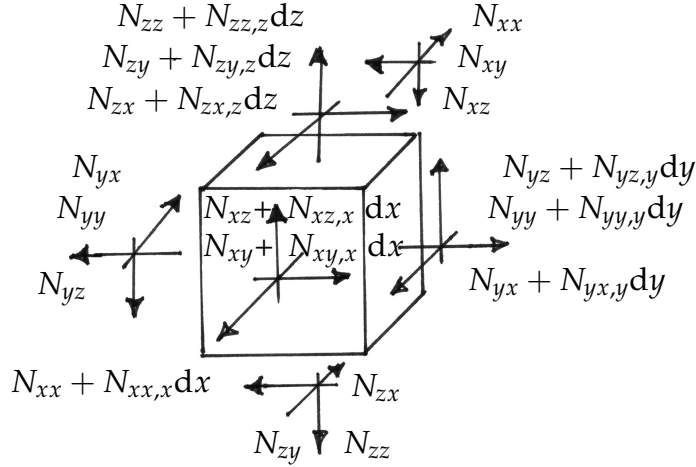


Figure 2.2: Infinitesimal three dimensional element with corresponding stress resultants N

problem, we now look at an infinitesimal element as displayed in figure 5.3. We sum all the forces in x, y and z direction to obtain the three equilibrium equations of a three dimensional solid.

$$\begin{aligned}
 \sum f_x &\doteq 0 & [-N_{xx} + N_{xx} + N_{xx,x} dx] \\
 & & + [-N_{yx} + N_{yx} + N_{yx,y} dy] \\
 & & + [-N_{zx} + N_{zx} + N_{zx,z} dz] + f_x = 0 \\
 \sum f_y &\doteq 0 & [-N_{yy} + N_{yy} + N_{yy,y} dy] \\
 & & + [-N_{xy} + N_{xy} + N_{xy,x} dx] \\
 & & + [-N_{zy} + N_{zy} + N_{zy,z} dz] + f_y = 0 \\
 \sum f_z &\doteq 0 & [-N_{zz} + N_{zz} + N_{zz,z} dz] \\
 & & + [-N_{xz} + N_{xz} + N_{xz,x} dx] \\
 & & + [-N_{yz} + N_{yz} + N_{yz,y} dy] + f_z = 0
 \end{aligned} \quad (2.2.13)$$

These equations can be slightly rearranged. Obviously, the first two terms of each row cancel. The remaining terms can be expressed in terms of the stress components σ and divided by the volume $dx \, dy \, dz$. The equilibrium equations then take the following

remarkably simple format.

$$\begin{aligned}
 \sum f_x \doteq 0 & \quad \sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z} + f_x = 0 \\
 \sum f_y \doteq 0 & \quad \sigma_{yy,y} + \sigma_{xy,x} + \sigma_{zy,z} + f_y = 0 \\
 \sum f_z \doteq 0 & \quad \sigma_{zz,z} + \sigma_{xz,x} + \sigma_{yz,y} + f_z = 0
 \end{aligned} \tag{2.2.14}$$

In continuum mechanics, these three equations are typically expressed in one single compact vector equation by making use of the index notation $\sigma_{ij,j} + f_i = 0_i$ or by introducing the divergence operator $\text{div}(\circ) = (\circ)_{ij,j}$ such that in compact tensor notation equilibrium simply reads $\text{div}(\boldsymbol{\sigma}^t) + \mathbf{f} = \mathbf{0}$.

To review what we just did, let us summarize the process that is common in continuum mechanics modeling. If the main goal was the characterization of the mechanical behavior of the cell, we would start with the two general equations, kinematics (1.2.6,1.2.7) or, in the context of large deformations (1.2.9) and equilibrium (1.2.14). Then, we would postulate a particular class of material behavior, here we have assumed the solid to be linear elastic. We identify an appropriate constitutive equation (1.2.10), here we have chosen Hooke's law. It introduces two material parameters, Young's modulus E and Poisson's ratio ν . Through experimental investigation, we can then determine the values of these parameters, identify ranges of validity, verify and validate the model and hopefully make useful predictions.

2.2.5 Structural elements

The derived set of equations is rather general and, as you might agree, somewhat complicated. In some cases, it can be reduced significantly by making particular assumptions concerning the dimensions of the solid of interest. Additional simplifications can be made once we know the relevant loading situation. Based on the following assumptions for the width w , the height h and the length l of the structure, we will distinguish six different structural elements with a reduced set of governing equations.

	dimension	geometry	loading	deformation	gov eqn
truss	1d straight	$w, h \ll l$	axial	tension	2 nd order
beam	1d straight	$w, h \ll l$	transverse	bending	4 th order
wall	2d flat	$h \ll w, l$	in plane	tension/shear	2 nd order
plate	2d flat	$h \ll w, l$	transverse	bending	4 th order
membrane	3d curved	$h \ll w, l$	in plane	tension/shear	2 nd order
shell	3d curved	$h \ll w, l$	transverse	bending	4 th order

Table 2.1: Classification of structural elements based on dimension, geometry and loading

Typical examples of trusses or beams would be biopolymers such as microtubules, actin and intermediate filaments. A typical example of a membrane or shell is the lipid

bilayer or the cell membrane. As you can see from the table, based on different assumptions of geometry and loading, the structure will either be tension/shear dominated or bending dominated, or, possibly a combination of both. We will see later that axial or in plane loading, stretch, tension and shear result in a second order partial differential equation relating the in plane deformation to the applied force. Transverse or out of plane loading and bending deformation result in a fourth order partial differential equation relating the out of plane deformation to the applied force.

For structural elements, it proves convenient to combine the information about the material properties, e.g., Young's modulus and Poisson's ratio, with information about the cross section geometry, e.g. the cross section area or the moment of inertia. Typical examples would be the axial stiffness, the bending stiffness, or the area compression modulus. We will address these individually in the sections on the mechanics of biopolymers and biomembranes.

2.3 Summary

2.4 Problems

Problem 2.1 - Green Lagrange strains

Try to recapitulate, at least for one normal and one shear term, how the individual components of the Green Lagrange strains (1.2.9) can be derived with the help of the definition $E_{ij} = \frac{1}{2}[F_{ki}F_{kj} - \delta_{ij}]$ or rather $E_{ij} = \frac{1}{2}[F_{xi}F_{xj} + F_{yi}F_{yj} + F_{zi}F_{zj} - \delta_{ij}]$. Write out the six equations for $i, j = x, y, z$ and make use of the definition of the deformation gradient (1.2.8).

$$\begin{aligned}
 E_{xx} &= \frac{1}{2} [F_{xx}F_{xx} + F_{yx}F_{yx} + F_{zx}F_{zx} - 1] \\
 E_{yy} &= \frac{1}{2} [F_{xy}F_{xy} + F_{yy}F_{yy} + F_{zy}F_{zy} - 1] \\
 E_{zz} &= \frac{1}{2} [F_{xz}F_{xz} + F_{yz}F_{yz} + F_{zz}F_{zz} - 1] \\
 E_{xy} &= \frac{1}{2} [F_{xx}F_{xy} + F_{yx}F_{yy} + F_{zx}F_{zy} - 1] = E_{yx} \\
 E_{yz} &= \frac{1}{2} [F_{xy}F_{xz} + F_{yy}F_{yz} + F_{zy}F_{zz} - 1] = E_{zy} \\
 E_{zx} &= \frac{1}{2} [F_{xz}F_{xx} + F_{yz}F_{yx} + F_{zz}F_{zx} - 1] = E_{xz}
 \end{aligned}$$

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