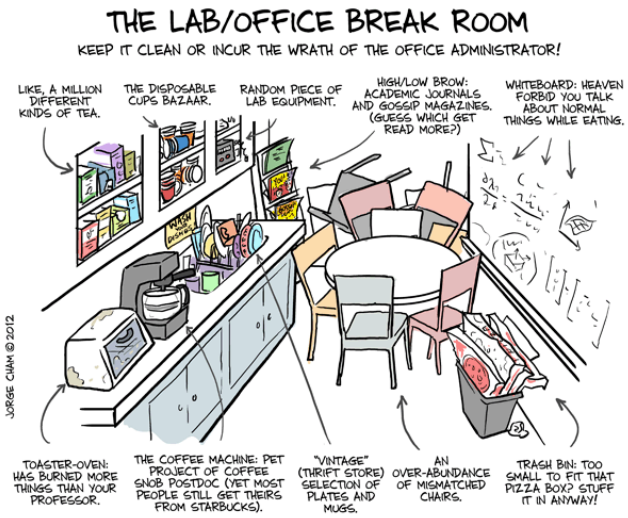
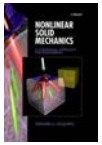
17 - midterm summary



me338 - syllabus

day	date		topic	chapters	pages
tue	sep	24	why continuum mechanics?		
thu	sep	26	introduction to vectors and tensors	1.1-1.5	1-32
tue	oct	01	introduction to vectors and tensors	1.6-1.9	32-55
thu	oct	03	kinematics	2.1-2.4	55-76
tue	oct	08	kinematics	2.5-2.8	76-109
thu	oct	10	concept of stress	3.1-3.4	109-131
tue	oct	15	balance principles	4.1-4.4	131-161
thu	oct	17	balance principles	4.5-4.7	161-179
tue	oct	22	aspects of objectivity	5.1-5.4	179-205
thu	oct	24	hyperelastic materials	6.1-6.2	205-222
tue	oct	29	hyperelastic materials	6.3-6.5	222-252
thu	oct	31	hyperelastic materials	6.6-6.8	252-278
tue	nov	05	hyperelastic materials	6.9-6.10	285-295
thu	nov	07	isotropic damage	6.11	295-304
tue	nov	12	thermodynamics of materials	7.1-7.6	305-337
thu	nov	14	thermodynamics of materials	7.7-7.9	337-371
tue	nov	19	final prep		
thu	nov	21	final		
tue	dec	03	selected topics		
thu	dec	05	selected topics		
thu	dec	05	written part of final projects due		



the potato equations

- **kinematic equations - what's strain?** $\epsilon = \frac{\Delta l}{l}$
  general equations that characterize the deformation of a physical body without studying its physical cause
- **balance equations - what's stress?** $\sigma = \frac{F}{A}$
  general equations that characterize the cause of motion of any body
- **constitutive equations - how are they related?** $\sigma = E\,\epsilon$
  material specific equations that complement the set of governing equations

the potato equations

- **kinematic equations - why not** $\epsilon = \frac{\Delta l}{l}$ **?**
  inhomogeneous deformation » non-constant
  finite deformation » non-linear $\boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$
  inelastic deformation » growth tensor $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{g}}$
- **balance equations - why not** $\sigma = \frac{F}{A}$**?** $\mathrm{Div}(\boldsymbol{P}) + \rho\,\boldsymbol{b}_0 = \boldsymbol{0}$
  equilibrium in deformed configuration » multiple stresses
- **constitutive equations - why not** $\sigma = E\,\epsilon$ **?**
  finite deformation » non-linear $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F})$
  inelastic deformation » internal variables $\boldsymbol{P} = \boldsymbol{P}(\rho, \boldsymbol{F}, \boldsymbol{F}_{\mathrm{g}})$

tensor vs index notation

tensor notation: compact • precise • common in the modern scientific literature

index notation: lengthy • easy to understand • easy to translate into algorithms • a bit outdated

$(\boldsymbol{u}\otimes\boldsymbol{v})\cdot\boldsymbol{w}=(\boldsymbol{v}\cdot\boldsymbol{w})\,\boldsymbol{u}$	$(u_i\,v_j)\,w_j=(v_j\,w_j)\,u_i$
$(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v})\otimes\boldsymbol{w}=\alpha\,(\boldsymbol{u}\otimes\boldsymbol{w})+\beta\,(\boldsymbol{v}\otimes\boldsymbol{w})$	$(\alpha\,u_i+\beta\,v_i)w_j=\alpha\,(u_i\,w_j)+\beta\,(v_i\,w_j)$
$\boldsymbol{u}\otimes(\alpha\,\boldsymbol{v}+\beta\,\boldsymbol{w})=\alpha\,(\boldsymbol{u}\otimes\boldsymbol{v})+\beta\,(\boldsymbol{u}\otimes\boldsymbol{w})$	$u_i(\alpha\,v_j+\beta\,w_j)=\alpha\,(u_i\,v_j)+\beta\,(u_i\,w_j)$
$(\boldsymbol{u}\otimes\boldsymbol{v})\cdot(\boldsymbol{w}\otimes\boldsymbol{x})=(\boldsymbol{v}\cdot\boldsymbol{w})\,(\boldsymbol{u}\otimes\boldsymbol{x})$	$(u_i\,v_j)(w_j\,x_k)=(v_j\,w_j)\,(u_i x_k)$
$\boldsymbol{A}\cdot(\boldsymbol{u}\otimes\boldsymbol{v})=(\boldsymbol{A}\cdot\boldsymbol{u})\otimes\boldsymbol{v}$	$A_{ij}(u_j\,v_k)=(A_{ij}\,u_i)v_k$
$(\boldsymbol{u}\otimes\boldsymbol{v})\cdot\boldsymbol{A}=\boldsymbol{u}\otimes(\boldsymbol{A}^{\mathrm{t}}\cdot\boldsymbol{v})$	$(u_i\,v_j)A_{jk}=u_i(A_{kj}\,v_j)$

tensor algebra - invariants

- (principal) invariants of second order tensor

$$\begin{aligned} I_A &= \mathrm{tr}\,(\boldsymbol{A}) \\ II_A &= \tfrac{1}{2}\left[\mathrm{tr}^2\,(\boldsymbol{A})-\mathrm{tr}\,(\boldsymbol{A}^2)\right] \\ III_A &= \det(\boldsymbol{A}) \end{aligned}$$

- derivatives of invariants wrt second order tensor

$$\begin{aligned} \partial_{\boldsymbol{A}}\ I_A &= \boldsymbol{I} \\ \partial_{\boldsymbol{A}}\ II_A &= I_A\,\boldsymbol{I}-\boldsymbol{A} \\ \partial_{\boldsymbol{A}}\ III_A &= III_A\,\boldsymbol{A}^{-\mathrm{t}} \end{aligned}$$

tensor algebra - trace

- trace of second order tensor $\qquad \mathrm{tr}\,(\boldsymbol{u}\otimes\boldsymbol{v})=\boldsymbol{u}\cdot\boldsymbol{v}$

$$\begin{aligned} \mathrm{tr}(\boldsymbol{A}) &= \mathrm{tr}(A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j) \\ &= A_{ij}\,\mathrm{tr}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)=A_{ij}\,\boldsymbol{e}_i\cdot\boldsymbol{e}_j \\ &= A_{ij}\,\delta_{ij}=A_{ii}=A_{11}+A_{22}+A_{33} \end{aligned}$$

- properties of traces of second order tensors

$$\begin{aligned} &\mathrm{tr}\,(\boldsymbol{I})=3 \\ &\mathrm{tr}\,(\boldsymbol{A}^{\mathrm{t}})=\mathrm{tr}\,(\boldsymbol{A}) \\ &\mathrm{tr}\,(\boldsymbol{A}\cdot\boldsymbol{B})=\mathrm{tr}\,(\boldsymbol{B}\cdot\boldsymbol{A}) \\ &\mathrm{tr}\,(\alpha\,\boldsymbol{A}+\beta\,\boldsymbol{B})=\alpha\,\mathrm{tr}(\boldsymbol{A})+\beta\,\mathrm{tr}(\boldsymbol{B}) \\ &\mathrm{tr}\,(\boldsymbol{A}\cdot\boldsymbol{B}^{\mathrm{t}})=\boldsymbol{A}:\boldsymbol{B} \\ &\mathrm{tr}\,(\boldsymbol{A})=\mathrm{tr}\,(\boldsymbol{A}\cdot\boldsymbol{I})=\boldsymbol{A}:\boldsymbol{I} \end{aligned}$$

tensor algebra - determinant

- determinant of second order tensor $\qquad III_A=\det(\boldsymbol{A})$

$$\begin{aligned} \det(\boldsymbol{A}) &= \det(A_{ij})=\tfrac{1}{6}\,e_{ijk}\,e_{abc}\,A_{ia}\,A_{jb}\,A_{kc} \\ &= A_{11}A_{22}A_{33}+A_{21}A_{32}A_{13}+A_{31}A_{12}A_{23} \\ &- A_{11}A_{23}A_{32}-A_{22}A_{31}A_{13}-A_{33}A_{12}A_{21} \end{aligned}$$

- properties of determinants of second order tensors

$$\begin{aligned} &\det\,(\boldsymbol{I})=1 \\ &\det\,(\boldsymbol{A}^{\mathrm{t}})=\det\,(\boldsymbol{A}) \\ &\det\,(\alpha\,\boldsymbol{A})=\alpha^3\det(\boldsymbol{A}) \\ &\det\,(\boldsymbol{A}\cdot\boldsymbol{B})=\det(\boldsymbol{A})\det(\boldsymbol{B}) \\ &\det(\boldsymbol{u}\otimes\boldsymbol{v})=0 \end{aligned}$$

tensor algebra - spectral decomposition

- eigenvalue problem of second order tensor
  $\boldsymbol{A} \cdot \boldsymbol{n}_A = \lambda_A \boldsymbol{n}_A \qquad [\boldsymbol{A} - \lambda_A \boldsymbol{I}] \cdot \boldsymbol{n}_A = \boldsymbol{0}$
- solution $\det(\boldsymbol{A} - \lambda_A \boldsymbol{I}) = 0$ in terms of scalar triple product
  $[\boldsymbol{A} \cdot \boldsymbol{u} - \lambda_A \boldsymbol{u}, \boldsymbol{A} \cdot \boldsymbol{v} - \lambda_A \boldsymbol{v}, \boldsymbol{A} \cdot \boldsymbol{w} - \lambda_A \boldsymbol{w}] = 0$
- characteristic equation
  $\lambda_A^3 - I_A \lambda_A^2 + II_A \lambda_A - III_A = 0$
  $I_A = \operatorname{tr}(\boldsymbol{A})$
  $II_A = \frac{1}{2}[\operatorname{tr}^2(\boldsymbol{A}) - \operatorname{tr}(\boldsymbol{A}^2)]$
  $III_A = \det(\boldsymbol{A})$
- spectral decomposition
  $\boldsymbol{A} = \sum_{i=1}^{3} \lambda_{Ai} \boldsymbol{n}_{Ai} \otimes \boldsymbol{n}_{Ai}$
- cayleigh hamilton theorem
  $\boldsymbol{A}^3 - I_A \boldsymbol{A}^2 + II_A \boldsymbol{A} - III_A \boldsymbol{I} = \boldsymbol{0}$

tensor analysis - gradient

- consider scalar- and vector field in domain $\mathcal{B} \in \mathcal{R}^3$
  $f: \mathcal{B} \to \mathcal{R} \qquad f: \boldsymbol{x} \to f(\boldsymbol{x})$
  $\boldsymbol{f}: \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f}: \boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x})$
- gradient of scalar- and vector field
  $\nabla f(\boldsymbol{x}) = \frac{\partial f(\boldsymbol{x})}{\partial x_i} = f_{,i}(\boldsymbol{x})\, \boldsymbol{e}_i \qquad \nabla f(\boldsymbol{x}) = \begin{bmatrix} f_{,1} \\ f_{,2} \\ f_{,3} \end{bmatrix}$
  $\nabla \boldsymbol{f}(\boldsymbol{x}) = \frac{\partial f_i(\boldsymbol{x})}{\partial x_j} = f_{i,j}(\boldsymbol{x})\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \nabla \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix}$

  renders vector- and 2nd order tensor field

tensor analysis - divergence

- consider vector- and 2nd order tensor field in domain $\mathcal{B}$
  $\boldsymbol{f}: \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f}: \boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x})$
  $\boldsymbol{F}: \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 \qquad \boldsymbol{F}: \boldsymbol{x} \to \boldsymbol{F}(\boldsymbol{x})$
- divergence of vector- and 2nd order tensor field
  $\operatorname{div}(\boldsymbol{f}(\boldsymbol{x})) = \operatorname{tr}(\nabla \boldsymbol{f}(\boldsymbol{x})) = \nabla \boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{I}$
  $\operatorname{div}(\boldsymbol{f}(\boldsymbol{x})) = f_{i,i}(\boldsymbol{x}) = f_{1,1} + f_{2,2} + f_{3,3}$

  $\operatorname{div}(\boldsymbol{F}(\boldsymbol{x})) = \operatorname{tr}(\nabla \boldsymbol{F}(\boldsymbol{x})) = \nabla \boldsymbol{F}(\boldsymbol{x}) : \boldsymbol{I}$
  $\operatorname{div}(\boldsymbol{F}(\boldsymbol{x})) = F_{ij,j}(\boldsymbol{x}) = \begin{bmatrix} F_{11,1} + F_{12,2} + F_{13,3} \\ F_{21,1} + F_{22,2} + F_{23,3} \\ F_{31,1} + F_{32,2} + F_{33,3} \end{bmatrix}$

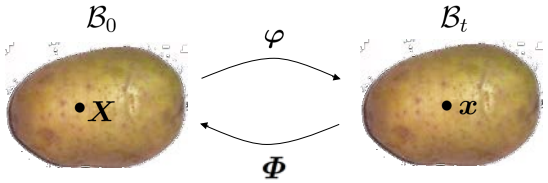
  renders scalar- and vector field

tensor analysis - laplace operator

- consider scalar- and vector field in domain $\mathcal{B} \in \mathcal{R}^3$
  $f: \mathcal{B} \to \mathcal{R} \qquad f: \boldsymbol{x} \to f(\boldsymbol{x})$
  $\boldsymbol{f}: \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f}: \boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x})$
- laplace operator acting on scalar- and vector field
  $\Delta f(\boldsymbol{x}) = \operatorname{div}(\nabla(f(\boldsymbol{x}))) \qquad \Delta f(\boldsymbol{x}) = f_{,ii} = f_{,11} + f_{,22} + f_{,33}$
  $\Delta \boldsymbol{f}(\boldsymbol{x}) = \operatorname{div}(\nabla(\boldsymbol{f}(\boldsymbol{x}))) \qquad \Delta \boldsymbol{f}(\boldsymbol{x}) = f_{i,jj} = \begin{bmatrix} f_{1,11} + f_{1,22} + f_{1,33} \\ f_{2,11} + f_{2,22} + f_{2,33} \\ f_{3,11} + f_{3,22} + f_{3,33} \end{bmatrix}$

  renders scalar- and vector field

motion



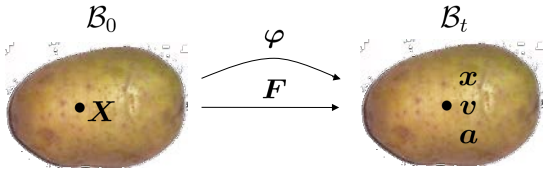
- spatial deformation map

$$\boldsymbol{x} = \varphi(\boldsymbol{X}, t) \quad \text{with} \quad \varphi : \mathcal{B}_0 \times \mathbb{R} \to \mathcal{B}_t$$

- material deformation map

$$\boldsymbol{X} = \boldsymbol{\Phi}(\boldsymbol{x}, t) \quad \text{with} \quad \boldsymbol{\Phi} : \mathcal{B}_t \times \mathbb{R} \to \mathcal{B}_0$$

temporal derivatives



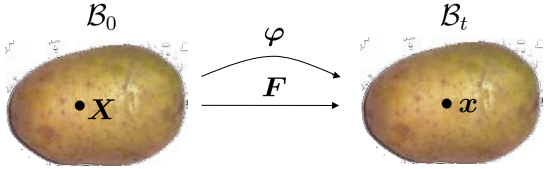
- temporal derivative of φ - velocity (material time deriv.)

$$\boldsymbol{v} = \mathrm{D}_t\varphi = \left.\frac{\partial \varphi}{\partial t}\right|_{X \text{ fixed}} \quad \text{with} \quad \boldsymbol{v} : \mathcal{B}_0 \times \mathbb{R} \to \mathbb{R}^3$$

- temporal derivative of $\boldsymbol{v}$ - acceleration (material time der.)

$$\boldsymbol{a} = \mathrm{D}_t\boldsymbol{v} = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{X \text{ fixed}} = \left.\frac{\partial^2 \varphi}{\partial t^2}\right|_{X \text{ fixed}} \quad \text{with} \quad \boldsymbol{a} : \mathcal{B}_0 \times \mathbb{R} \to \mathbb{R}^3$$

spatial derivatives



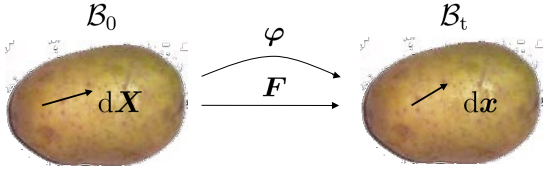
- nonlinear deformation map φ

$$\boldsymbol{x} = \varphi(\boldsymbol{X}, t) \quad \text{with} \quad \varphi : \mathcal{B}_0 \times \mathbb{R} \to \mathcal{B}_t$$

- spatial derivative of φ - deformation gradient $\boldsymbol{F}$

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} \quad \text{with} \quad \boldsymbol{F} : T\mathcal{B}_0 \to T\mathcal{B}_t \qquad \boldsymbol{F} = \left.\frac{\partial \varphi}{\partial \boldsymbol{X}}\right|_{t \text{ fixed}}$$

deformation gradient



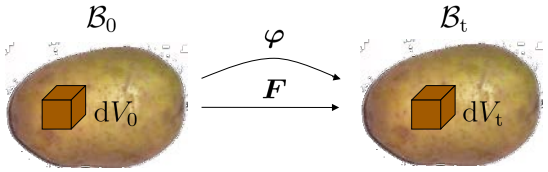
- transformation of line elements deformation gradient F_{ij}

$$\mathrm{d}x_i = F_{ij}\,\mathrm{d}X_j \quad \text{with} \quad F_{ij} : T\mathcal{B}_0 \to T\mathcal{B}_t \qquad F_{ij} = \left.\frac{\partial \varphi_i}{\partial X_j}\right|_{t \text{ fixed}}$$

- uniaxial tension (incompr), simple shear, rotation

$$F_{ij}^{\text{uni}} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha^{-\frac{1}{2}} \end{bmatrix} \quad F_{ij}^{\text{shr}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F_{ij}^{\text{rot}} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

jacobian



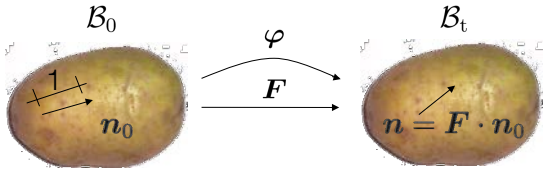
- transformation of volume elements - determinant of $\boldsymbol{F}$

$$\mathrm{d}V_0 = \mathrm{d}\boldsymbol{X}_1 \cdot [\mathrm{d}\boldsymbol{X}_2 \times \mathrm{d}\boldsymbol{X}_3] \qquad \mathrm{d}V_t = \mathrm{d}\boldsymbol{x}_1 \cdot [\mathrm{d}\boldsymbol{x}_2 \times \mathrm{d}\boldsymbol{x}_3]$$
$$= \det([\mathrm{d}\boldsymbol{x}_1, \mathrm{d}\boldsymbol{x}_2, \mathrm{d}\boldsymbol{x}_3])$$
$$= \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3]) \qquad = \det(\boldsymbol{F})\ \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3])$$

- changes in volume - determinant of deformation gradient J

$$\mathrm{d}V_t = J\,\mathrm{d}V_0 \qquad J = \det(\boldsymbol{F})$$

stretch & deformation tensor



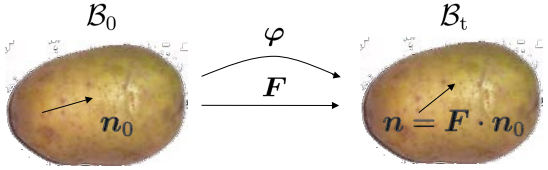
- stretch $\lambda = |\boldsymbol{n}|$ with $\boldsymbol{n} = \boldsymbol{F} \cdot \boldsymbol{n}_0$ and $|\boldsymbol{n}_0| = 1$

$$\lambda^2 = \boldsymbol{n} \cdot \boldsymbol{n} = [\boldsymbol{F} \cdot \boldsymbol{n}_0] \cdot [\boldsymbol{F} \cdot \boldsymbol{n}_0]$$
$$= \boldsymbol{n}_0 \cdot \boldsymbol{F}^t \cdot \boldsymbol{F} \cdot \boldsymbol{n}_0 = \boldsymbol{n}_0 \cdot \boldsymbol{C} \cdot \boldsymbol{n}_0$$

- right Cauchy-Green deformation tensor

$$\boldsymbol{C} = \boldsymbol{F}^t \cdot \boldsymbol{F} \quad \text{with} \quad C_{AB} = F_{aA} F_{aB}$$
$$\det(\boldsymbol{C}) = \det^2(\boldsymbol{F}) = J^2 > 0$$

green lagrange strain tensor



- stretch

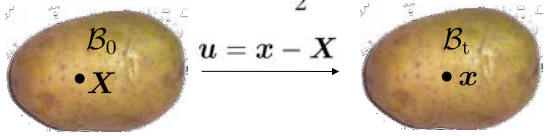
$$\tfrac{1}{2}[\lambda^2 - \lambda_0^2] = \tfrac{1}{2}[\boldsymbol{n}\cdot\boldsymbol{n} - \boldsymbol{n}_0\cdot\boldsymbol{n}_0]$$
$$= \tfrac{1}{2}[\boldsymbol{n}_0\cdot\boldsymbol{F}^t\cdot\boldsymbol{F}\cdot\boldsymbol{n}_0 - \boldsymbol{n}_0\cdot\boldsymbol{I}\cdot\boldsymbol{n}_0]$$
$$= \boldsymbol{n}_0\cdot\tfrac{1}{2}[\boldsymbol{F}^t\cdot\boldsymbol{F} - \boldsymbol{I}]\cdot\boldsymbol{n}_0 = \boldsymbol{n}_0\cdot\boldsymbol{E}\cdot\boldsymbol{n}_0$$

- green-lagrange strain tensor

$$\boldsymbol{E} = \tfrac{1}{2}[\boldsymbol{F}^t \cdot \boldsymbol{F} - \boldsymbol{I}] \quad \text{with} \quad E_{AB} = \tfrac{1}{2}[F_{aA}F_{aB} - \delta_{AB}]$$

green lagrange strain tensor

- small strain $\boldsymbol{\varepsilon} = \nabla^{\mathrm{sym}}\boldsymbol{u} = \frac{1}{2}[\nabla\boldsymbol{u} + \nabla^t\boldsymbol{u}]$



- deformation gradient

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{X}} = \frac{\mathrm{d}[\boldsymbol{X} + \boldsymbol{u}]}{\mathrm{d}\boldsymbol{X}} = \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\boldsymbol{X}} + \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\boldsymbol{X}} = \boldsymbol{I} + \nabla_{\boldsymbol{X}}\boldsymbol{u}$$

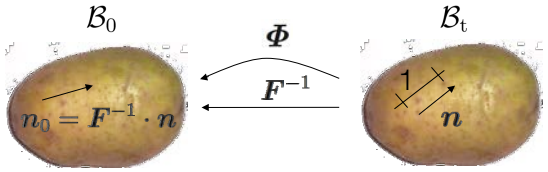
- green-lagrange strain tensor

$$\boldsymbol{E} = \tfrac{1}{2}[\boldsymbol{F}^t \cdot \boldsymbol{F} - \boldsymbol{I}] = \tfrac{1}{2}[[\boldsymbol{I}^t + \nabla^t\boldsymbol{u}] \cdot [\boldsymbol{I} + \nabla\boldsymbol{u}] - \boldsymbol{I}]$$

nonlinear term

$$\boldsymbol{E} = \tfrac{1}{2}[\boldsymbol{I} + \nabla^t\boldsymbol{u} + \nabla\boldsymbol{u} + \nabla^t\boldsymbol{u} \cdot \nabla\boldsymbol{u} - \boldsymbol{I}] = \boldsymbol{\varepsilon} + \tfrac{1}{2}[\nabla^t\boldsymbol{u} \cdot \nabla\boldsymbol{u}]$$

stretch & deformation tensor



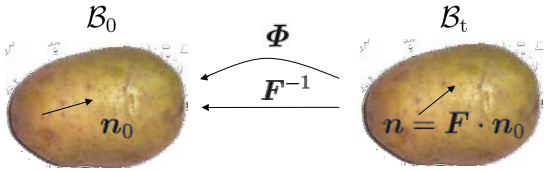
- stretch $\quad \Lambda = |\boldsymbol{n}_0| \quad$ with $\quad \boldsymbol{n}_0 = \boldsymbol{F}^{-1} \cdot \boldsymbol{n} \quad$ and $\quad |\boldsymbol{n}| = 1$

$$\Lambda^2 = \boldsymbol{n}_0 \cdot \boldsymbol{n}_0 = [\boldsymbol{F}^{-1} \cdot \boldsymbol{n}] \cdot [\boldsymbol{F}^{-1} \cdot \boldsymbol{n}]$$
$$= \boldsymbol{n} \cdot \boldsymbol{F}^{-\mathrm{t}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{b}^{-1} \cdot \boldsymbol{n}$$

- left Cauchy-Green deformation tensor / finger tensor

$$\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{t}} \quad \text{with} \quad b_{ab} = F_{aA} F_{bA}$$
$$\det(\boldsymbol{b}) = \det^2(\boldsymbol{F}) = J^2 > 0$$

euler almansi strain tensor

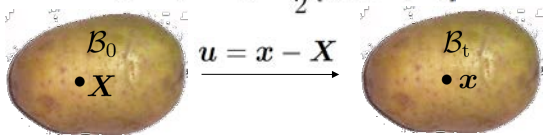


- euler-almansi strain tensor and co-variant push forward

$$\boldsymbol{e} = \tfrac{1}{2}\left[\boldsymbol{I} - \boldsymbol{F}^{-\mathrm{t}} \cdot \boldsymbol{F}^{-1}\right] \quad \text{with} \quad e_{ab} = \tfrac{1}{2}\left[\delta_{ab} - F^{-1}_{Aa} F^{-1}_{Ab}\right]$$
$$= \boldsymbol{F}^{-\mathrm{t}} \cdot \tfrac{1}{2}\left[\boldsymbol{F}^{\mathrm{t}} \cdot \left[\boldsymbol{I} - \boldsymbol{F}^{-\mathrm{t}} \cdot \boldsymbol{F}^{-1}\right] \cdot \boldsymbol{F}\right] \cdot \boldsymbol{F}^{-1}$$
$$= \boldsymbol{F}^{-\mathrm{t}} \cdot \tfrac{1}{2}\left[\boldsymbol{F}^{\mathrm{t}} \cdot \boldsymbol{F} - \boldsymbol{I}\right] \cdot \boldsymbol{F}^{-1} = \boldsymbol{F}^{-\mathrm{t}} \cdot \boldsymbol{E} \cdot \boldsymbol{F}^{-1}$$
$$\boldsymbol{E} = \tfrac{1}{2}\left[\boldsymbol{F}^{\mathrm{t}} \cdot \boldsymbol{F} - \boldsymbol{I}\right] \quad \text{with} \quad E_{AB} = \tfrac{1}{2}\left[F_{aA} F_{aB} - \delta_{AB}\right]$$

euler almansi strain tensor

- small strain $\quad \boldsymbol{\varepsilon} = \nabla^{\mathrm{sym}} \boldsymbol{u} = \dfrac{1}{2}\left[\nabla \boldsymbol{u} + \nabla^{\mathrm{t}} \boldsymbol{u}\right]$



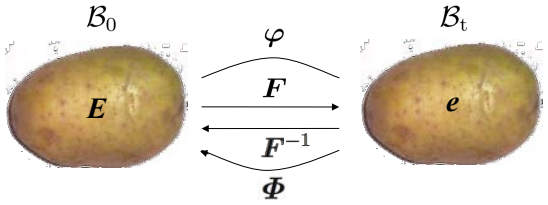
- inverse deformation gradient

$$\boldsymbol{F}^{-1} = \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\boldsymbol{x}} = \frac{\mathrm{d}[\boldsymbol{x} - \boldsymbol{u}]}{\mathrm{d}\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{x}} - \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\boldsymbol{x}} = \boldsymbol{I} - \nabla_{\boldsymbol{x}} \boldsymbol{u}$$

- euler-almansi strain tensor

$$\boldsymbol{e} = \tfrac{1}{2}\left[\boldsymbol{I} - \boldsymbol{F}^{-\mathrm{t}} \cdot \boldsymbol{F}^{-1}\right] = \tfrac{1}{2}\left[\boldsymbol{I} - \left[\boldsymbol{I}^{\mathrm{t}} - \nabla^{\mathrm{t}} \boldsymbol{u}\right] \cdot \left[\boldsymbol{I} - \nabla \boldsymbol{u}\right]\right] \quad \text{nonlinear term}$$
$$\boldsymbol{e} = \tfrac{1}{2}\left[\boldsymbol{I} - \boldsymbol{I} + \nabla^{\mathrm{t}} \boldsymbol{u} + \nabla \boldsymbol{u} - \nabla^{\mathrm{t}} \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right] = \boldsymbol{\varepsilon} - \tfrac{1}{2}\left[\nabla^{\mathrm{t}} \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right]$$

strain tensors



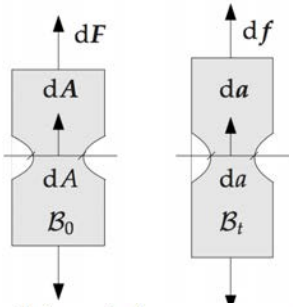
- green lagrange strain tensor

$$\boldsymbol{E} = \tfrac{1}{2}\left[\boldsymbol{F}^{\mathrm{t}} \cdot \boldsymbol{F} - \boldsymbol{I}\right] \quad \text{with} \quad E_{AB} = \tfrac{1}{2}\left[F_{aA} F_{aB} - \delta_{AB}\right]$$

- euler-almansi strain tensor

$$\boldsymbol{e} = \tfrac{1}{2}\left[\boldsymbol{I} - \boldsymbol{F}^{-\mathrm{t}} \cdot \boldsymbol{F}^{-1}\right] \quad \text{with} \quad e_{ab} = \tfrac{1}{2}\left[\delta_{ab} - F^{-1}_{Aa} F^{-1}_{Ab}\right]$$

stress tensors



cauchy / true stress
relates spatial force to spatial area

$$df = t\,da = \sigma \cdot n\,da = \sigma \cdot da$$

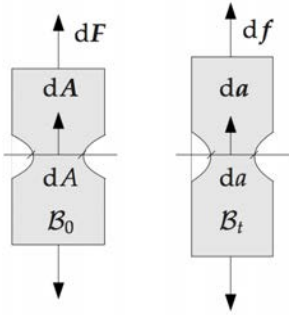
first piola kirchhoff / nominal stress
relates spatial force to material area

$$df = t\,da = \sigma \cdot n\,da = \sigma \cdot da = J\,\sigma \cdot F^{-t} \cdot dA = P \cdot dA$$

second piola kirchhoff stress
relates material force to material area

$$dF = F^{-1} \cdot df = F^{-1} \cdot P \cdot dA = J\,F^{-1} \cdot \sigma \cdot F^{-t} \cdot dA = S \cdot dA$$

stress tensors



cauchy / true stress
relates spatial force to spatial area

$$t = \sigma \cdot n \qquad t_i = \sigma_{ij}\, n_j$$

first piola kirchhoff / nominal stress
relates spatial force to material area

$$P = J\,\sigma \cdot F^{-t} \qquad P_{iJ} = J\,\sigma_{ik}\, F^{-t}_{kJ}$$

second piola kirchhoff stress
relates material force to material area

$$S = F^{-1} \cdot P = J\,F^{-1} \cdot \sigma \cdot F^{-t}$$

$$S_{IJ} = F^{-1}_{Ik}\; P_{kJ} = J\,F^{-1}_{Ik}\; \sigma_{kl}\; F^{-t}_{lJ}$$

stress tensors

gustav robert kirchhoff
[1824-1887]

first piola kirchhoff

$$P = F \cdot S$$

$$P = J\,\sigma \cdot F^{-t}$$

augustin louis caucy
[1789-1857]

second piola kirchhoff

$$S = F^{-1} \cdot P$$

$$S = J\,F^{-1} \cdot \sigma \cdot F^{-t}$$

cauchy

$$\sigma = \frac{1}{J}\, P \cdot F^{t}$$

$$\sigma = \frac{1}{J}\, F \cdot S \cdot F^{t}$$

transport mechanisms

covariant / strains

$$E = F^{t} \cdot e \cdot F \qquad \longleftarrow \text{ pull back}$$

$$E_{IJ} = F^{t}_{Ik}\; e_{kl}\; F_{lJ}$$

push forward $\longrightarrow$

$$e = F^{-t} \cdot E \cdot F^{-1}$$

$$e_{ij} = F^{-t}_{iK}\; E_{KL}\; F^{-1}_{Lj}$$

contravariant / stresses

$$S = J\,F^{-1} \cdot \sigma \cdot F^{-t} \qquad \longleftarrow \text{ pull back}$$

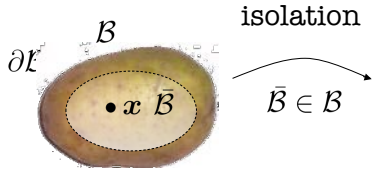
$$S_{IJ} = J\,F^{-1}_{Ik}\; \sigma_{kl}\; F^{-t}_{lJ}$$

push forward $\longrightarrow$

$$\sigma = \frac{1}{J}\, F \cdot S \cdot F^{t}$$

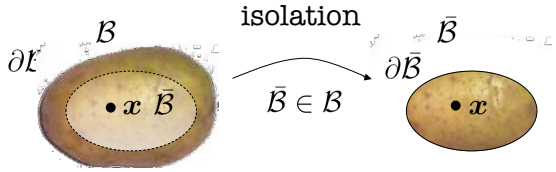
$$\sigma_{ij} = \frac{1}{J}\, F_{iK}\; S_{KL}\; F^{t}_{Lj}$$

potato balance equations



[1] isolate subset $\bar{\mathcal{B}}$ from $\mathcal{B}$

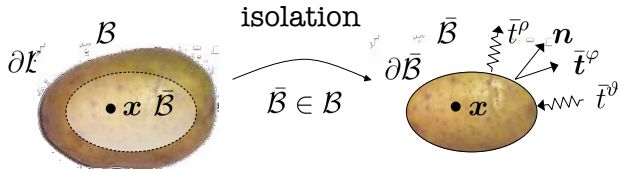
potato balance equations



[1] isolate subset $\bar{\mathcal{B}}$ from $\mathcal{B}$

[2] **characterize** influence of remaining body through phenomenological quantities - contact fluxes $\bar{t}^{\rho}$, $\bar{\boldsymbol{t}}^{\varphi}$& $\bar{t}^{\vartheta}$

potato balance equations

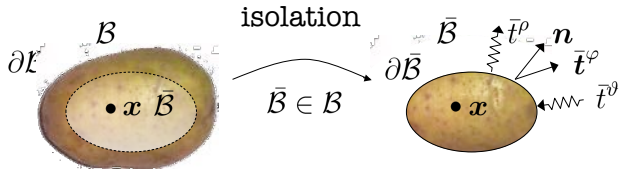


[1] isolate subset $\bar{\mathcal{B}}$ from $\mathcal{B}$

[2] characterize influence of remaining body through phenomenological quantities - contact fluxes $\bar{t}^{\rho}$, $\bar{\boldsymbol{t}}^{\varphi}$& $\bar{t}^{\vartheta}$

[3] **define** basic physical quantities - mass, linear and angular momentum, energy

potato balance equations



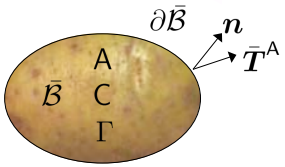
[1] isolate subset $\bar{\mathcal{B}}$ from $\mathcal{B}$

[2] characterize influence of remaining body through phenomenological quantities - contact fluxes $\bar{t}^{\rho}$, $\bar{\boldsymbol{t}}^{\varphi}$& $\bar{t}^{\vartheta}$

[3] define basic physical quantities - mass, linear and angular momentum, energy

[4] **postulate** balance of these quantities

generic balance equation

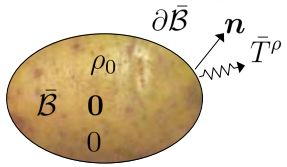


general format

A ... balance quantity
B ... flux $\quad \mathbf{B} \cdot \boldsymbol{n} = \bar{\boldsymbol{T}}^{\mathsf{A}}$
C ... source
Γ ... production

$$\mathrm{D}_t \mathsf{A} = \mathrm{Div}(\mathbf{B}) + \mathsf{C} + \Gamma$$

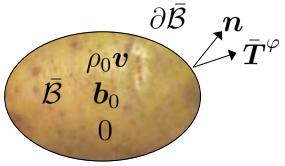
balance of mass



ρ_0 ... density
$\mathbf{0}$ ... no mass flux $\qquad \bar{T}^{\rho} = 0$
0 ... no mass source
0 ... no mass production

continuity equation $\quad \mathrm{D}_t \rho_0 = 0$

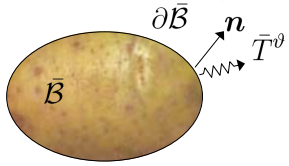
balance of (linear) momentum



$\rho_0 \boldsymbol{v}$ ... linear momentum density
$\boldsymbol{P}$ ... momentum flux - stress $\qquad \boldsymbol{P} \cdot \boldsymbol{n} = \bar{\boldsymbol{T}}^{\varphi}$
$\boldsymbol{b}_0$ ... momentum source - force
0 ... no momentum production

equilibrium equation $\quad \rho_0 \dot{\boldsymbol{v}} = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0$

balance of (internal) energy

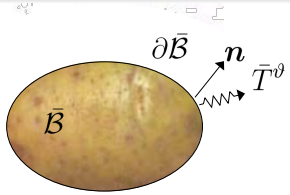


E_0 ... internal energy density
$\boldsymbol{Q}$ ... heat flux $\qquad -\boldsymbol{Q} \cdot \boldsymbol{n} = \bar{T}^{\vartheta}$
R_0 ... heat source
0 ... no heat production

energy equation $\quad \dot{E}_0 = \boldsymbol{P} : \dot{\boldsymbol{F}} - \mathrm{Div}(\boldsymbol{Q}) + R_0$

internal mechanical power $\qquad$ external thermal power

entropy inequality



local entropy inequality

$$\mathcal{D}_0^{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi}_0 - \eta_0 \dot{\theta} - \frac{1}{\theta} \boldsymbol{Q} \cdot \nabla_X \theta \geq 0$$

thermal part > heat flux / mechanical part > stress

$$-\frac{1}{\theta} \boldsymbol{Q} \cdot \nabla_X \theta \geq 0 \qquad \mathcal{D}_0^{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi}_0 - \eta_0 \dot{\theta} \geq 0$$

09 - balance principles

example: neo hooke'ian elasticity

- free energy $\psi_0^{\text{neo}} = \frac{1}{2} \lambda_0 \ln^2(\det(\boldsymbol{F})) + \frac{1}{2} \mu_0 [\, \boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2 \ln(\det(\boldsymbol{F})) \,]$
- definition of stress

$$\begin{aligned} \boldsymbol{P}^{\text{neo}} &= \mathrm{D}_F \psi_0^{\text{neo}} \\ &= \tfrac{1}{2} \lambda_0 \, 2 \ln(\det \boldsymbol{F}) \boldsymbol{F}^{-\text{t}} + \tfrac{1}{2} \mu_0 \, 2 \, \boldsymbol{F} - \mu_0 \boldsymbol{F}^{-\text{t}} \\ &= \mu_0 \, \boldsymbol{F} + [\, \lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0 \,] \boldsymbol{F}^{-\text{t}} \end{aligned}$$

- definition of tangent operator

$$\begin{aligned} \mathsf{A}^{\text{neo}} &= \mathrm{D}_{FF} \psi_0^{\text{neo}} = \mathrm{D}_F \boldsymbol{P}^{\text{neo}} \\ &= \lambda_0 \boldsymbol{F}^{-\text{t}} \otimes \boldsymbol{F}^{-\text{t}} + \mu_0 \boldsymbol{I} \overline{\otimes} \boldsymbol{I} \\ &+ [\mu_0 - \lambda_0 \ln(\det(\boldsymbol{F}))] \, \boldsymbol{F}^{-\text{t}} \underline{\otimes} \boldsymbol{F}^{-1} \end{aligned}$$

coleman noll evaluation

entropy $\mathcal{D}^{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi} - \eta \dot{\theta} \geq 0$

assume $\psi = \psi(\boldsymbol{F})$ $\dot{\psi} = \dfrac{\partial \psi(\boldsymbol{F})}{\partial t} = \dfrac{\partial \psi(\boldsymbol{F})}{\partial \boldsymbol{F}} : \dfrac{\partial \boldsymbol{F}}{\partial t} = \dfrac{\partial \psi(\boldsymbol{F})}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}}$

$$\mathcal{D}^{\text{int}} = \left[\boldsymbol{P} - \frac{\partial \psi(\boldsymbol{F})}{\partial \boldsymbol{F}} \right] : \dot{\boldsymbol{F}} \geq 0$$

with $\mathcal{D}^{\text{int}} = 0$ and $\forall \, \dot{\boldsymbol{F}}$

first piola kirchhoff stress

$$\boldsymbol{P} = \frac{\partial \psi}{\partial \boldsymbol{F}} \qquad \text{with} \qquad P_{aA} = \frac{\partial \psi}{\partial F_{aA}}$$

09 - balance principles

example: neo hooke'ian elasticity

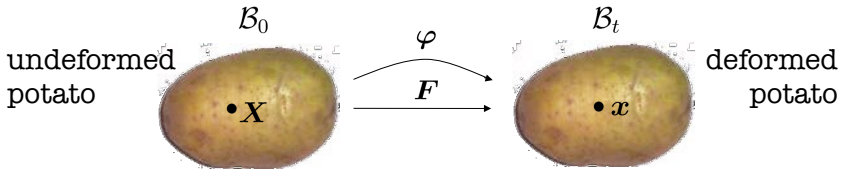
- free energy $\psi_0^{\text{neo}} = \frac{1}{2} \lambda_0 \ln^2(\det(F_{ij})) + \frac{1}{2} \mu_0 [\, F_{ij} F_{ij} - n^{\text{dim}} - 2 \ln(\det(F_{ij})) \,]$
- definition of stress

$$\begin{aligned} P_{ij}^{\text{neo}} &= \mathrm{D}_{F_{ij}} \psi_0^{\text{neo}} \\ &= \tfrac{1}{2} \lambda_0 \, 2 \ln(\det F_{ij}) F_{ji}^{-1} + \tfrac{1}{2} \mu_0 \, 2 \, F_{ij} - \mu_0 F_{ji}^{-1} \\ &= \mu_0 \, F_{ij} + [\, \lambda_0 \ln(\det(F_{ij})) - \mu_0 \,] F_{ji}^{-1} \end{aligned}$$

- definition of tangent operator

$$\begin{aligned} \mathsf{A}_{ijkl}^{\text{neo}} &= \mathrm{D}_{F_{ij} F_{kl}} \psi_0^{\text{neo}} = \mathrm{D}_{F_{kl}} P_{ij}^{\text{neo}} \\ &= \lambda_0 F_{ji}^{-1} \, F_{lk}^{-1} + \mu_0 I_{ik} \, I_{jl} \\ &+ [\mu_0 - \lambda_0 \ln(\det(F_{ij}))] \, F_{li}^{-1} \, F_{jk}^{-1} \end{aligned}$$

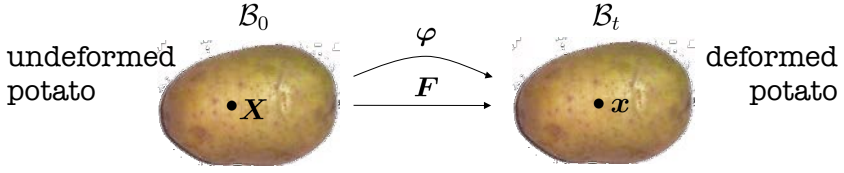
example: neo hooke‘ian elasticity



- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\lambda_0 \ln^2(\det(\boldsymbol{F})) + \frac{1}{2}\mu_0[\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\ln(\det(\boldsymbol{F}))]$
- definition of stress - neo hookean elasticity

$$\boldsymbol{P}^{\text{neo}} = \mathrm{D}_F \psi_0^{\text{neo}} = \mu_0 \boldsymbol{F} + [\lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0]\boldsymbol{F}^{-\text{t}}$$

example: neo hooke‘ian elasticity



- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\lambda_0 \ln^2(\det(\boldsymbol{F})) + \frac{1}{2}\mu_0[\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\ln(\det(\boldsymbol{F}))]$
- large strain - lamé parameters and bulk modulus

$$\lambda = \frac{E\,\nu}{[1+\nu][1-2\,\nu]} \qquad \mu = \frac{E}{2\,[1+\nu]} \qquad \kappa = \frac{E}{3\,[1-2\,\nu]}$$

- small strain - young ‘s modulus and poisson ‘s ratio

$$E = 3\,\kappa\,[1-2\,\nu] \qquad \nu = \frac{3\kappa - 2\mu}{2\,[3\kappa+\mu]}$$

example: neo hooke‘ian elasticity

- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\lambda_0 \ln^2(\det(\boldsymbol{F})) + \frac{1}{2}\mu_0[\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\ln(\det(\boldsymbol{F}))]$
- definition of stress - neo hookean elasticity

$$\boldsymbol{P}^{\text{neo}} = \mathrm{D}_F \psi_0^{\text{neo}} = \mu_0 \boldsymbol{F} + [\lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0]\boldsymbol{F}^{-\text{t}}$$

- remember! mashing potatoes is not an elastic process!

example: invariant-based materials

$$\psi = \psi(\boldsymbol{C}) = \psi(I_1^C, I_2^C, I_3^C)$$

second piola kirchhoff stress

$$\boldsymbol{S} = 2\frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\left[\frac{\partial \psi}{\partial I_1^C}\frac{\partial I_1^C}{\partial \boldsymbol{C}} + \frac{\partial \psi}{\partial I_2^C}\frac{\partial I_2^C}{\partial \boldsymbol{C}} + \frac{\partial \psi}{\partial I_3^C}\frac{\partial I_3^C}{\partial \boldsymbol{C}}\right]$$

invariants and their derivatives

$$I_1^C = \mathrm{tr}(\boldsymbol{C}) \qquad \partial I_1^C/\partial \boldsymbol{C} = \boldsymbol{I}$$
$$I_2^C = \tfrac{1}{2}[\mathrm{tr}^2(\boldsymbol{C}) - \mathrm{tr}(\boldsymbol{C}^2)] \qquad \partial I_2^C/\partial \boldsymbol{C} = I_1^C\,\boldsymbol{I} - \boldsymbol{C}$$
$$I_3^C = \det(\boldsymbol{C}) \qquad \partial I_3^C/\partial \boldsymbol{C} = I_3^C\,\boldsymbol{C}^{-1}$$

second piola kirchhoff stress

$$\boldsymbol{S} = 2\left[\left[\frac{\partial \psi}{\partial I_1^C} + I_1^C\frac{\partial \psi}{\partial I_2^C}\right]\boldsymbol{I} - \frac{\partial \psi}{\partial I_2^C}\boldsymbol{C} + I_3^C\frac{\partial \psi}{\partial I_3^C}\boldsymbol{C}^{-1}\right]$$

example: invariant-based materials

$$\psi = \psi(\boldsymbol{b}) = \psi(I_1^b, I_2^b, I_3^b)$$

cauchy stress

$$\boldsymbol{\sigma} = \frac{2}{J}\frac{\partial \psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} = \frac{2}{J}\boldsymbol{b} \cdot \frac{\partial \psi(\boldsymbol{b})}{\partial \boldsymbol{b}}$$

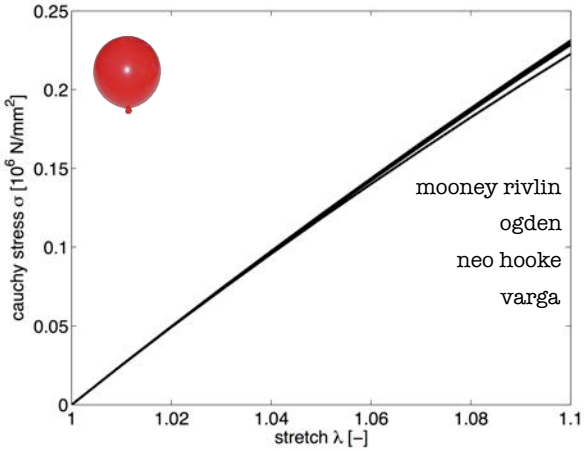
invariants in terms of principal stretches

$$\begin{aligned} I_1^b &= \operatorname{tr}(\boldsymbol{b}) &&= \lambda_I + \lambda_{II} + \lambda_{III} \\ I_2^b &= \tfrac{1}{2}[\operatorname{tr}^2(\boldsymbol{b}) - \operatorname{tr}(\boldsymbol{b}^2)] &&= \lambda_I \cdot \lambda_{II} + \lambda_{II} \cdot \lambda_{III} + \lambda_{III} \cdot \lambda_I \\ I_3^b &= \det(\boldsymbol{b}) &&= \lambda_I \cdot \lambda_{II} \cdot \lambda_{III} \end{aligned}$$

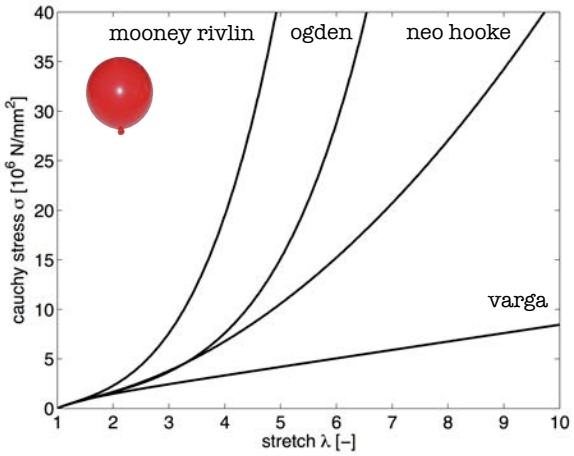
principal cauchy stresses

$$\sigma_I = -p + \lambda_I \frac{\partial \psi}{\partial \lambda_I} \qquad I = 1, 2, 3$$

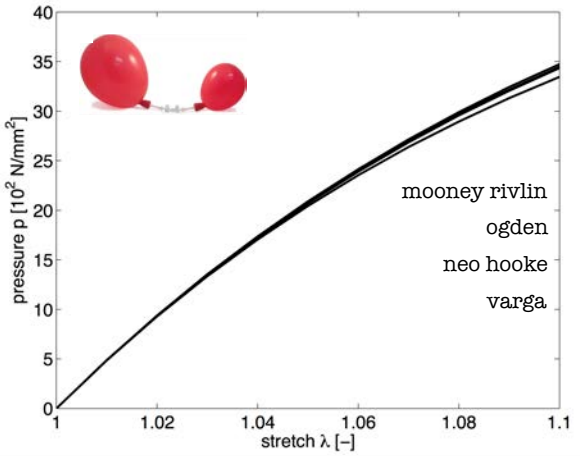
inflation of a spherical rubber balloon



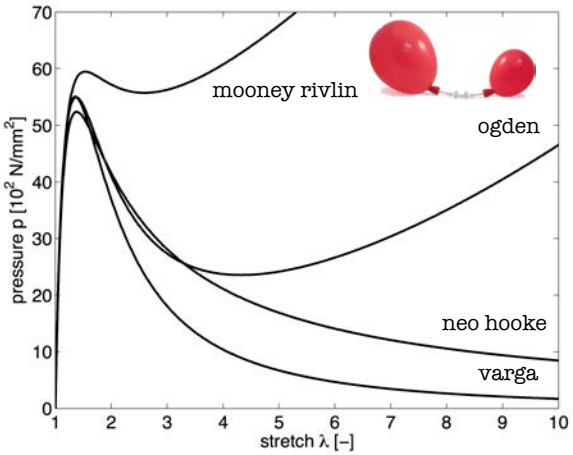
inflation of a spherical rubber balloon



inflation of a spherical rubber balloon



inflation of a spherical rubber balloon

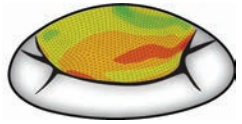


transversely isotropic incompressible

incompressible material

$$\boldsymbol{F} = \boldsymbol{F}^{\text{vol}} \cdot \bar{\boldsymbol{F}}$$

volumetric part $\boldsymbol{F}^{\text{vol}} = J^{1/3}\boldsymbol{I}$



isochoric part $\bar{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F}$

transversely isotropic material

fiber orientation $||\boldsymbol{n}_0|| = 1$

structural tensor $\boldsymbol{N} = \boldsymbol{n}_0 \otimes \boldsymbol{n}_0$

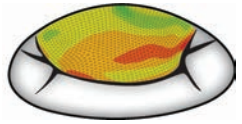
transversely isotropic incompressible

$$\psi = \psi^{\text{vol}}(J) + \bar{\psi}(\bar{I}_1, \bar{I}_4)$$

$$\begin{aligned} J &= \det(\boldsymbol{F}) \\ \bar{I}_1 &= \bar{\boldsymbol{C}} : \boldsymbol{I} \\ \bar{I}_4 &= \bar{\boldsymbol{C}} : \boldsymbol{N} \end{aligned} \quad \text{with} \quad \begin{aligned} \partial_{\boldsymbol{C}} J &= \tfrac{1}{2} J \boldsymbol{C}^{-1} \\ \partial_{\bar{\boldsymbol{C}}} \bar{I}_1 &= \boldsymbol{I} \\ \partial_{\bar{\boldsymbol{C}}} I_4 &= \boldsymbol{N} \,. \end{aligned}$$

transversely isotropic incompressible

$$\boldsymbol{S} = 2\,\frac{\partial \psi}{\partial \boldsymbol{C}} = \boldsymbol{S}^{\text{vol}} + \boldsymbol{S}^{\text{iso}}$$



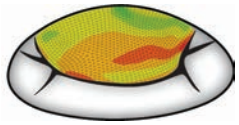
volumetric part $\boldsymbol{S}^{\text{vol}} = 2\,\dfrac{\partial \psi^{\text{vol}}}{\partial \boldsymbol{C}} = J\,p\,\boldsymbol{C}^{-1}$

isochoric part $\boldsymbol{S}^{\text{iso}} = 2\,\dfrac{\partial \bar{\psi}}{\partial \boldsymbol{C}} = J^{-2/3}\,\mathbb{P} : \bar{\boldsymbol{S}}$

$$\bar{\boldsymbol{S}} = 2\,\frac{\partial \bar{\psi}}{\partial \bar{\boldsymbol{C}}} = 2\,\bar{\psi}_1\,\boldsymbol{I} + 2\,\bar{\psi}_4\,\boldsymbol{N}$$

example 01 - neo hooke model

$$\bar{\psi} = c_0\,[\,\bar{I}_1 - 3\,]$$

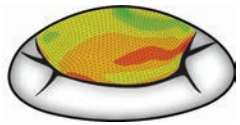


$$\bar{\boldsymbol{S}} = 2\,\frac{\partial\bar{\psi}}{\partial\bar{\boldsymbol{C}}} = 2\,\bar{\psi}_1\,\boldsymbol{I} + 2\,\bar{\psi}_4\,\boldsymbol{N} \quad \text{with} \quad \bar{\psi}_i = \partial\bar{\psi}/\partial\bar{I}_i$$

$$\bar{\psi}_1 = c_0 \quad \text{and} \quad \bar{\psi}_4 = 0$$

example 02 - may newman model

$$\bar{\psi} = c_0\,[\exp(c_1[\,\bar{I}_1 - 3\,]^2 + c_2[\,\bar{I}_4 - 1\,]^2) - 1\,]$$

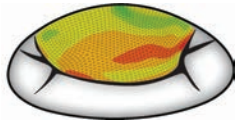


$$\bar{\boldsymbol{S}} = 2\,\frac{\partial\bar{\psi}}{\partial\bar{\boldsymbol{C}}} = 2\,\bar{\psi}_1\,\boldsymbol{I} + 2\,\bar{\psi}_4\,\boldsymbol{N} \quad \text{with} \quad \bar{\psi}_i = \partial\bar{\psi}/\partial\bar{I}_i$$

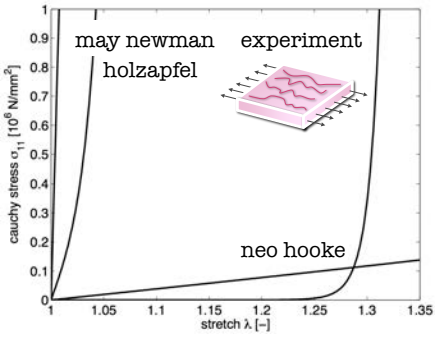
$$\bar{\psi}_1 = 2\,c_0 c_1[\bar{I}_1 - 3]\,\exp(c_1[\bar{I}_1 - 3]^2 + c_2[\bar{I}_4 - 1]^2)$$
$$\bar{\psi}_4 = 2\,c_0 c_2[\bar{I}_4 - 1]\,\exp(c_1[\bar{I}_1 - 3]^2 + c_2[\bar{I}_4 - 1]^2)$$

example 03 - holzapfel model

$$\bar{\psi} = c_0[\,\bar{I}_1 - 3\,] + \frac{c_1}{2c_2}[\exp(c_2\,[\bar{I}_4 - 1]^2) - 1\,]$$



$$\bar{\boldsymbol{S}} = 2\,\frac{\partial\bar{\psi}}{\partial\bar{\boldsymbol{C}}} = 2\,\bar{\psi}_1\,\boldsymbol{I} + 2\,\bar{\psi}_4\,\boldsymbol{N} \quad \text{with} \quad \bar{\psi}_i = \partial\bar{\psi}/\partial\bar{I}_i$$

$$\bar{\psi}_1 = c_0$$
$$\bar{\psi}_4 = c_1[\,\bar{I}_4 - 1\,]\exp(c_2\,[\bar{I}_4 - 1\,]^2)$$

uniaxial stretching of anisotropic sheet