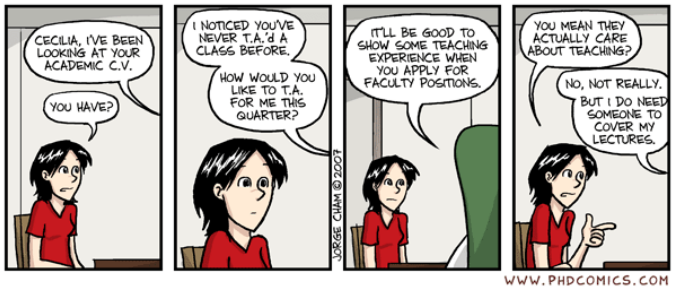
15 - interfaces between continua



rheology

rheology

/riːˈɒlədʒi/ is the study of the flow of matter, primarily in the liquid state, but also as 'soft solids' or solids under conditions in which they respond with plastic flow rather than deforming elastically in response to an applied force rheology generally accounts for the behavior of non-newtonian fluids, by characterizing the minimum number of functions that are needed to relate stresses with rate of change of strains or strain rates.

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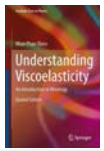
rheology

rheology

is the science of deformation and flow. this definition of was due to bingham, who, together with scott blair and reiner, helped form the society of rheology in 1929. the idea that everything has a time scale, and that if we are prepared to wait long enough then everything will flow was known to the greek philosopher heraclitus, and prior to him, to the prophetess deborah – *the mountains flowed before the lord*. not surprisingly, the motto of the society of rheology is παντα ρει (everything flows), a saying attributed to heraclitus .



newtonian and non-newtonian fluids

newtonian

- characterized by constant viscosity

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\nabla_x^s\boldsymbol{v}$$

non-newtonian

- any fluid that does not obey the previous equation

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \varsigma$$

newtonian and non-newtonian fluids

non-newtonian $\boxed{\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\varsigma}}$

- example: Oldroyd 8 constant model

$$\lambda_1 D_t \boldsymbol{\varsigma} + \frac{1}{2}\lambda_3(\boldsymbol{A}_1 \cdot \boldsymbol{\varsigma} + \boldsymbol{\varsigma} \cdot \boldsymbol{A}_1) + \frac{1}{2}\lambda_5(\mathrm{tr}(\boldsymbol{\varsigma}))\boldsymbol{A}_1 + \frac{1}{2}\lambda_6(\mathrm{tr}(\boldsymbol{\varsigma} \cdot \boldsymbol{A}_1))\boldsymbol{I} = \mu\left(\boldsymbol{A}_1 + \lambda_2 D_t \boldsymbol{A}_1 + \lambda_4 \boldsymbol{A}_1^2 + \frac{1}{2}\lambda_7(\mathrm{tr}(\boldsymbol{A}_1^2))\boldsymbol{I}\right)$$

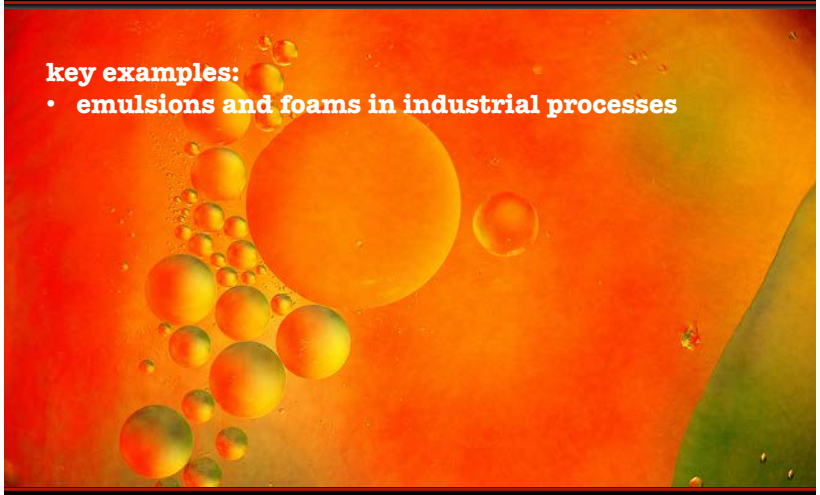
where

$$\boldsymbol{A}_1 = 2\nabla_x^s \boldsymbol{v}$$

$$D_t \boldsymbol{\varsigma} = \frac{\partial \boldsymbol{\varsigma}}{\partial t} + \boldsymbol{v} \cdot \nabla_x \boldsymbol{\varsigma} + (\boldsymbol{\varsigma} \cdot \nabla_x \boldsymbol{v} - \nabla_x^T \boldsymbol{v} \cdot \boldsymbol{\varsigma})$$

newtonian and non-newtonian fluids

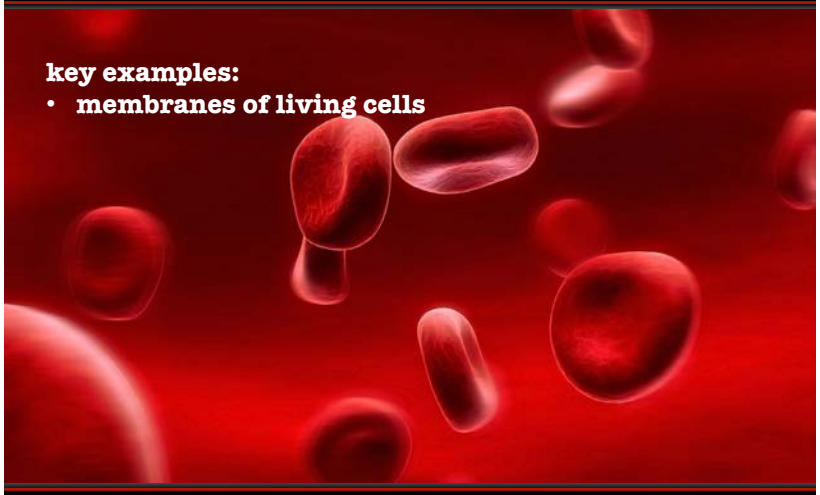
complex fluid interfaces



key examples:

- **emulsions and foams in industrial processes**

complex fluid interfaces



key examples:

- **membranes of living cells**

complex fluid interfaces

interfacial rheology

Interfacial tension (quasi-static)

- Laplace-Young equation

$$\boldsymbol{\sigma n} = p_{int}\boldsymbol{n} = p_{ext}\boldsymbol{n} + 2\gamma \boldsymbol{H n}$$

$$2\boldsymbol{H} = -\nabla_s \cdot \boldsymbol{n} = (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\nabla_x \boldsymbol{n}$$

Interfacial dilatational rheology

$$\bar{\gamma} e^{i(\omega t + \phi(\omega))} = E^*(\omega)\frac{\Delta A}{A_0} e^{i\omega t}$$

Interfacial shear rheology

$$\sigma e^{i(\omega t + \phi(\omega))} = G^*(\omega)\epsilon e^{i\omega t}$$

interfacial rheology

Interfacial tension (quasi-static)

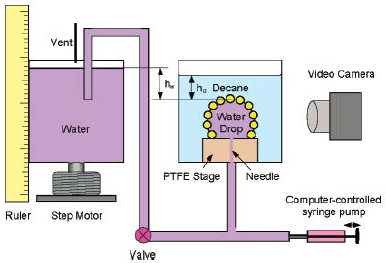
$$\boldsymbol{\sigma n} = p_{int}\boldsymbol{n} = p_{ext}\boldsymbol{n} + 2\gamma \boldsymbol{H n}$$

$$2\boldsymbol{H} = -\nabla_s \cdot \boldsymbol{n} = (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\nabla_x \boldsymbol{n}$$

Interfacial dilatational rheology

- experimental setup

$$\bar{\gamma} e^{i(\omega t + \phi(\omega))} = E^*(\omega)\frac{\Delta A}{A_0} e^{i\omega t}$$



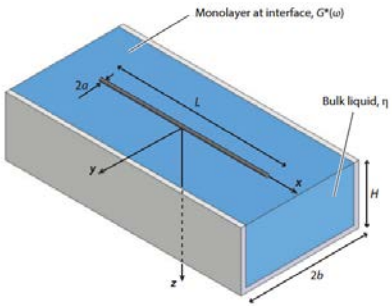
Xu et al. (2005)

interfacial rheology

Interfacial shear rheology:

$$\sigma e^{i(\omega t + \phi(\omega))} = G^*(\omega)\epsilon e^{i\omega t}$$

- experimental setup

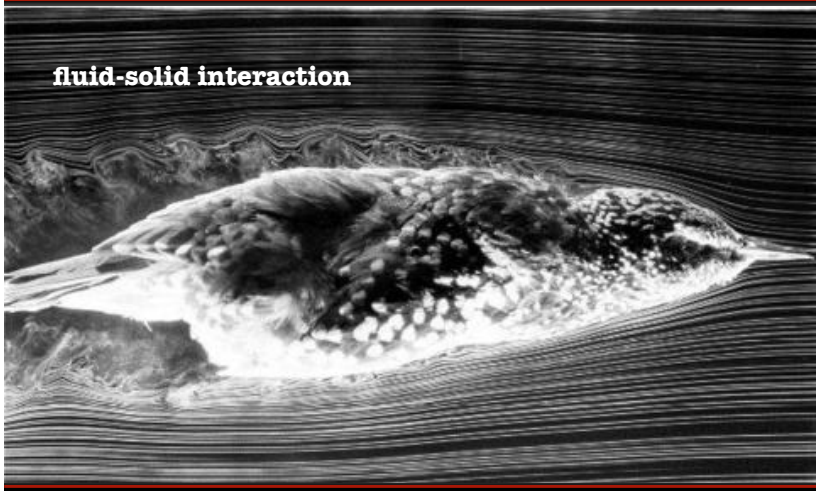


Fuller and Vermant (2012)

interfaces between continua

- fluid-solid interface
- fluid-fluid interface
- fluid-membrane-fluid interface

interfaces between continua



interfaces between continua

fluid-solid interaction

recall: master balance law in spatial setting

$$\frac{d}{dt}\int_{\Omega_x} \alpha d\Omega_x = \int_{\partial\Omega_x} \boldsymbol{\gamma}_x^T \boldsymbol{n}_x d\partial\Omega_x + \int_{\partial\Omega_x} \beta d\partial\Omega_x \qquad \text{scalar}$$

$$\frac{d}{dt}\int_{\Omega_x} \boldsymbol{\alpha} d\Omega_x = \int_{\partial\Omega_x} \boldsymbol{\Gamma}_x^T \boldsymbol{n}_x d\partial\Omega_x + \int_{\partial\Omega_x} \boldsymbol{\beta} d\partial\Omega_x \qquad \text{vector}$$

interfaces between continua

fluid-solid interaction

Arbitrary Lagrangian Eulerian (ALE) methods
- balance equations in advective form
- used for moving domains

$$\int_{\Omega_x} \left[\frac{\partial \alpha}{\partial t} + (\boldsymbol{v} - \hat{\boldsymbol{v}}) \cdot \nabla_x \alpha + \alpha \nabla_x \cdot \boldsymbol{v} - \nabla_x \cdot \boldsymbol{\gamma} - \beta \right] d\Omega_x = 0$$

$$\int_{\Omega_x} \left[\frac{\partial \boldsymbol{\alpha}}{\partial t} + (\boldsymbol{v} - \hat{\boldsymbol{v}}) \cdot \nabla_x \boldsymbol{\alpha} + \boldsymbol{\alpha} \nabla_x \cdot \boldsymbol{v} - \nabla_x \cdot \boldsymbol{\Gamma} - \boldsymbol{\beta} \right] d\Omega_x = 0$$

where $\hat{\boldsymbol{v}}$ is the velocity of the domain

interfaces between continua

fluid-solid interaction

Solid problem

- balance equations in Lagrangian description
- balance of linear momentum

$$\rho_0 \frac{\partial \boldsymbol{V}}{\partial t} - \nabla_X \cdot \boldsymbol{P} = \boldsymbol{B}_0$$

Fluid problem

- balance equations in ALE description
- continuity equation
- balance of linear momentum

$$\nabla_x \cdot \boldsymbol{v} = 0$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} - \hat{\boldsymbol{v}}) \cdot \nabla_x \boldsymbol{v} - \nabla_x \cdot \boldsymbol{\sigma} = \boldsymbol{b}$$

interfaces between continua

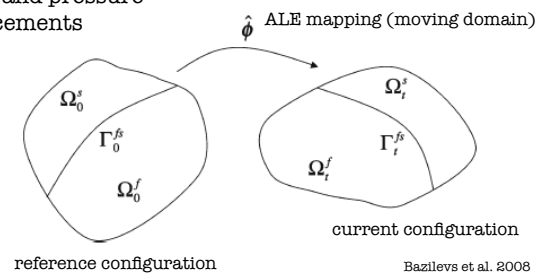
fluid-solid interaction

coupled problem

$\boldsymbol{u}$ solid displacements

$\boldsymbol{v}, p$ fluid velocities and pressure

$\hat{\boldsymbol{u}}$ domain displacements



Bazilevs et al. 2008

interfaces between continua

fluid-solid interaction: interface

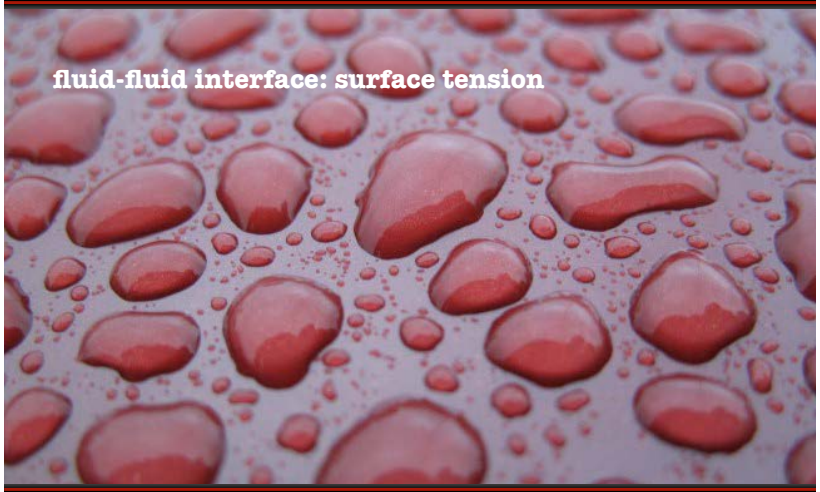
coupled problem compatibility conditions

$$\boldsymbol{v}|_{\Gamma_t^{fs}} = \frac{\partial \boldsymbol{u}}{\partial t} \circ \hat{\boldsymbol{\phi}}^{-1}\Big|_{\Gamma_t^{fs}} \qquad \text{compatibility of kinematics}$$

$$(\boldsymbol{\sigma}^f \boldsymbol{n}_t^{fs} + \boldsymbol{\sigma}^s \boldsymbol{n}_t^{fs})\Big|_{\Gamma_t^{fs}} = \boldsymbol{0} \qquad \text{compatibility of stresses}$$

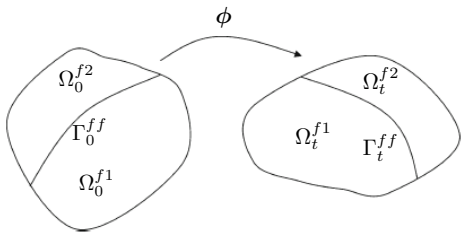
$$\hat{\boldsymbol{u}}|_{\Gamma_t^{fs}} = \boldsymbol{u} \circ \hat{\boldsymbol{\phi}}^{-1}\Big|_{\Gamma_t^{fs}} \qquad \text{compatibility of ALE map}$$

interfaces between continua



interfaces between continua

fluid-fluid interface: surface tension



interfaces between continua

fluid-fluid interface: surface tension

- determine the motion of only one fluid
- example of application: pendant drop experiments

Young-Laplace equation at the interface

$$\boldsymbol{\sigma}\boldsymbol{n} = p_{int}\boldsymbol{n} = p_{ext}\boldsymbol{n} + 2\gamma\boldsymbol{H}\boldsymbol{n}$$

fluid problem in eulerian description inside the domain

$$\nabla_x \cdot \boldsymbol{\sigma} + \rho\boldsymbol{b} = 0$$

interfaces between continua

fluid-fluid interface: surface tension

incremental kinematics to use Lagrangian description and to avoid ALE mapping

$$\boldsymbol{x}_{t+1} = \boldsymbol{\phi}(\boldsymbol{x}_t)$$

$$\boldsymbol{\phi}(\boldsymbol{x}_t) = \boldsymbol{x}_t + \boldsymbol{u}$$

$$\bar{\boldsymbol{d}} = \frac{1}{\Delta t}\sum_i \ln(\lambda_{(i)})\boldsymbol{n}_{(i)} \otimes \boldsymbol{n}_{(i)}$$

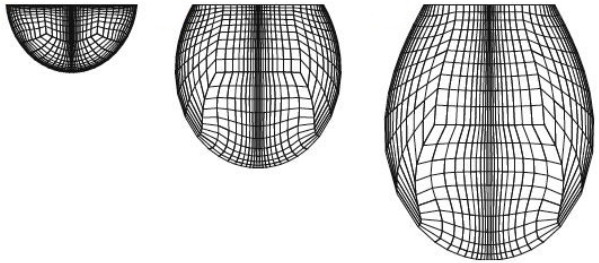
incremental constitutive equation for nearly incompressible newtonian fluid

$$\bar{\boldsymbol{\sigma}} = -K(J-1)\boldsymbol{I} + 2\mu\bar{\boldsymbol{d}}$$

interfaces between continua

fluid-fluid interface: surface tension

numerical examples

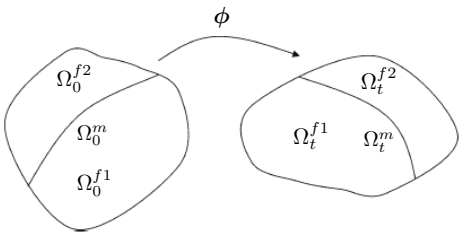


Saksono and Peric (2006)

interfaces between continua

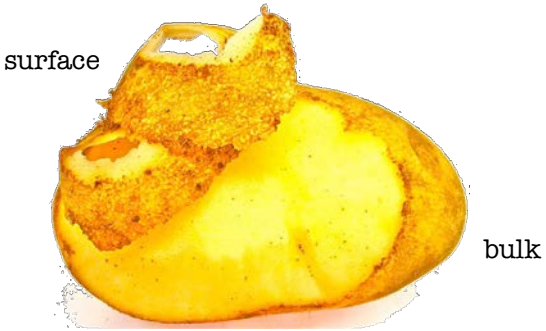
interfaces between continua

fluid-membrane-fluid interface



interfaces between continua

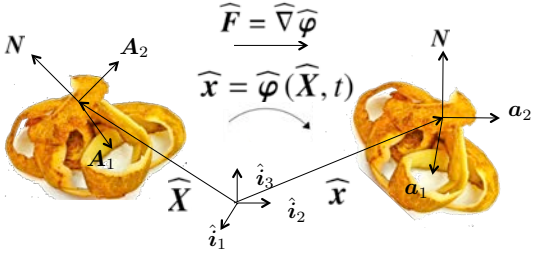
fluid-membrane-fluid interface



interfaces between continua

fluid-membrane-fluid interface

membrane kinematics



interfaces between continua

fluid-membrane-fluid interface

membrane kinematics: differential geometry of surfaces

$\boldsymbol{A}_\alpha \cdot \boldsymbol{A}^\beta = \delta_\alpha^\beta$ dual basis

$\boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta = A_{\alpha\beta}$ metric tensor

$J_A = \sqrt{\det A_{\alpha\beta}}$ area defined by basis vectors

$\boldsymbol{V}_{\alpha;\beta} = \boldsymbol{A}_\beta \cdot \nabla \boldsymbol{V}_\alpha = D\boldsymbol{V}_\alpha[\boldsymbol{A}_\beta]$ covariant derivative

interfaces between continua

fluid-membrane-fluid interface

membrane kinematics

$\widehat{\boldsymbol{F}} = \boldsymbol{a}_\alpha \otimes \boldsymbol{A}^\alpha$ surface deformation gradient

$\widehat{\boldsymbol{I}} = \boldsymbol{I} - \boldsymbol{N} \otimes \boldsymbol{N}$ identity tensor on the surface

$\widehat{\boldsymbol{F}}^{-1} \cdot \widehat{\boldsymbol{F}} = \widehat{\boldsymbol{I}}$

$\widehat{\boldsymbol{F}}^{t} \cdot \widehat{\boldsymbol{F}}^{-t} = \widehat{\boldsymbol{I}}$

pseudo-inverse of the deformation gradient

interfaces between continua

fluid-membrane-fluid interface

membrane balance equations

$\boldsymbol{0} = \widehat{\mathrm{Div}}\,\widehat{\boldsymbol{P}} + \widehat{\boldsymbol{B}}$ linear momentum balance

$\widehat{\mathrm{Div}}\,\widehat{\boldsymbol{P}} = \widehat{P}_{\alpha\beta;\beta}$ divergence on the surface

$\widehat{\psi} = \widehat{\psi}(\widehat{\boldsymbol{F}})$ surface strain-energy function

$\widehat{\boldsymbol{P}} = \dfrac{\partial \widehat{\psi}}{\partial \widehat{\boldsymbol{F}}}$ first Piola-Kirchhoff stress tensor

continua with boundary energies

fluid-membrane-fluid interface

coupled problem
- standard problem inside the volume
- include membrane contribution on the boundary

volume

$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$

$\boldsymbol{F} = \nabla \boldsymbol{\varphi}$

surface

$\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{\varphi}}(\widehat{\boldsymbol{X}}, t)$

$\widehat{\boldsymbol{F}} = \widehat{\nabla}\widehat{\boldsymbol{\varphi}} = \boldsymbol{F} \cdot \widehat{\boldsymbol{I}}$

continua with boundary energies

fluid-membrane-fluid interface

coupled problem

- Local equilibrium for the volume

$$\mathbf{0} = \mathrm{Div}\,\boldsymbol{P} + \boldsymbol{B}$$

- Local equilibrium for the surface

$$\mathbf{0} = \widehat{\mathrm{Div}}\,\widehat{\boldsymbol{P}} + \widehat{\boldsymbol{B}}$$

$$\widehat{\boldsymbol{B}} = \boldsymbol{t}_0^{\mathrm{p}} - \boldsymbol{P}\cdot\boldsymbol{N}$$

continua with boundary energies

fluid-membrane-fluid interface

coupled problem

- global equilibrium for the volume

$$\int_{\mathcal{S}_0} \delta\boldsymbol{\varphi}\cdot\boldsymbol{P}\cdot\boldsymbol{N}\,\mathrm{d}A_0 = \int_{\mathcal{B}_0} \nabla\delta\boldsymbol{\varphi} : \boldsymbol{P}\,\mathrm{d}V_0 - \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi}\cdot\boldsymbol{B}\,\mathrm{d}V_0$$

- global equilibrium for the closed surface

$$\int_{\mathcal{S}_0} \delta\boldsymbol{\varphi}\cdot\boldsymbol{P}\cdot\boldsymbol{N}\,\mathrm{d}A_0 = -\int_{\mathcal{S}_0} \widehat{\nabla}\delta\boldsymbol{\varphi} : \widehat{\boldsymbol{P}}\,\mathrm{d}A_0 + \int_{\mathcal{S}_0} \delta\boldsymbol{\varphi}\cdot\boldsymbol{t}_0^{\mathrm{p}}\,\mathrm{d}A_0$$

continua with boundary energies

fluid-membrane-fluid interface

coupled problem

- constitutive equation for the volume

$$\boldsymbol{P} = \frac{\partial\psi}{\partial\boldsymbol{F}}$$

- constitutive equation for the surface

$$\widehat{\boldsymbol{P}} = \frac{\partial\widehat{\psi}}{\partial\widehat{\boldsymbol{F}}}$$