rheology

/rɪˈleɪdʒ/ is the study of the flow of matter, primarily in the liquid state, but also as 'soft solids' or solids under conditions in which they respond with plastic flow rather than deforming elastically in response to an applied force. Rheology generally accounts for the behavior of non-Newtonian fluids, by characterizing the minimum number of functions that are needed to relate stresses with rate of change of strains or strain rates.

Newtonian and Non-Newtonian Fluids

**Newtonian**
- characterized by constant viscosity
  \[ \sigma = -pI + 2\mu \nabla^s v \]

**Non-Newtonian**
- any fluid that does not obey the previous equation
  \[ \sigma = -pI + \zeta \]
**newtonian and non-newtonian fluids**

**non-newtonian** \[ \sigma = -p I + \varsigma \]

- example: Oldroyd 8 constant model

\[
\lambda_1 D_t \varsigma + \frac{1}{2} \lambda_3 (A_1 \cdot \varsigma + \varsigma \cdot A_1) + \frac{1}{2} \lambda_5 (\text{tr}(\varsigma)) A_1 + \frac{1}{2} \lambda_6 (\text{tr}(\varsigma \cdot A_1)) I = \\
\mu \left( A_1 + \lambda_2 D_t A_1 + \lambda_4 A_1^2 + \frac{1}{2} \lambda_7 (\text{tr}(A_1^2)) I \right)
\]

where

\[
A_1 = 2\nabla_s^v
\]

\[
D_t \varsigma = \frac{\partial \varsigma}{\partial t} + \nabla_s \varsigma + (\varsigma \cdot \nabla_s v - \nabla_s^T v \cdot \varsigma)
\]

**complex fluid interfaces**

key examples:
- emulsions and foams in industrial processes

**complex fluid interfaces**

key examples:
- membranes of living cells
15 - interfaces between continua

**complex fluid interfaces**

**interfacial rheology**

Interfacial tension (quasi-static)

\[ \sigma n = p_{int} n = p_{ext} n + 2\gamma Hn \]

\[ 2H = -\nabla \cdot n = (I - n \otimes n)n \nabla n \]

Interfacial dilatational rheology

\[ \tilde{\gamma} e^{i(\omega t + \phi(\omega))} = E^*(\omega) \frac{\Delta A}{A_0} e^{i\omega t} \]

Interfacial shear rheology

\[ \sigma e^{i(\omega t + \phi(\omega))} = G^*(\omega) \epsilon e^{i\omega t} \]

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**Interfacial tension (quasi-static)**

**Laplace-Young equation**

\[ \sigma n = p_{int} n = p_{ext} n + 2\gamma Hn \]

\[ 2H = -\nabla \cdot n = (I - n \otimes n)n \nabla n \]

**Interfacial dilatational rheology**

\[ \tilde{\gamma} e^{i(\omega t + \phi(\omega))} = E^*(\omega) \frac{\Delta A}{A_0} e^{i\omega t} \]

**Interfacial shear rheology**

\[ \sigma e^{i(\omega t + \phi(\omega))} = G^*(\omega) \epsilon e^{i\omega t} \]

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**Experimental setup**

Xu et al. (2005)

Fuller and Vermant (2012)

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**Experimental setup**

Xu et al. (2005)

Fuller and Vermant (2012)
• fluid-solid interface
• fluid-fluid interface
• fluid-membrane-fluid interface

15 - interfaces between continua

fluid-solid interaction

recall: master balance law in spatial setting

\[
\frac{d}{dt} \int_{\Omega_s} \alpha \, d\Omega_s = \int_{\partial \Omega_s} \gamma^T \mathbf{n} \, d\partial \Omega_s + \int_{\Omega_s} \beta \, d\Omega_s \quad \text{scalar}
\]

\[
\frac{d}{dt} \int_{\Omega_s} \mathbf{a} \, d\Omega_s = \int_{\partial \Omega_s} \Gamma^T \mathbf{n} \, d\partial \Omega_s + \int_{\Omega_s} \beta \, d\Omega_s \quad \text{vector}
\]

15 - interfaces between continua

fluid-solid interaction

Arbitrary Lagrangian Eulerian (ALE) methods

• balance equations in advective form
• used for moving domains

\[
\int_{\Omega_x} \left[ \frac{\partial \alpha}{\partial t} + (\mathbf{v} - \mathbf{v}_x) \cdot \nabla_x \alpha + \alpha \nabla_x \cdot \mathbf{v} - \nabla_x \cdot \gamma - \beta \right] \, d\Omega_x = 0
\]

\[
\int_{\Omega_x} \left[ \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v} - \mathbf{v}_x) \cdot \nabla_x \mathbf{a} + \alpha \nabla_x \cdot \mathbf{v} - \nabla_x \cdot \Gamma - \beta \right] \, d\Omega_x = 0
\]

where \( \mathbf{v}_x \) is the velocity of the domain
fluid-solid interaction
Solid problem
- balance equations in Lagrangian description
- balance of linear momentum

Fluid problem
- balance equations in ALE description
- continuity equation
- balance of linear momentum

\[
\rho \frac{\partial V}{\partial t} - \nabla \cdot P = B_0 \\
\rho \frac{\partial v}{\partial t} + \rho(v - \hat{v}) \cdot \nabla v - \nabla \cdot \sigma = b
\]

fluid-solid interaction: interface
coupled problem compatibility conditions

\[ v \big|_{\Gamma_t^f} = \frac{\partial u}{\partial t} \circ \hat{\phi}^{-1} \bigg|_{\Gamma_t^f} \quad \text{compatibility of kinematics} \]

\[ (\sigma^f n_t^f + \sigma^n n_t^f) \big|_{\Gamma_t^f} = 0 \quad \text{compatibility of stresses} \]

\[ \hat{u} \big|_{\Gamma_t^f} = u \circ \hat{\phi}^{-1} \bigg|_{\Gamma_t^f} \quad \text{compatibility of ALE map} \]
fluid-fluid interface: surface tension

schematic representation:

\[ \phi \]

\[ \Omega_f^2 \]

\[ \Gamma_{ff} \]

\[ \Omega_f^1 \]

\[ \Omega_0^1 \]

\[ \Omega_0^2 \]

\[ n = p_{int} n = p_{ext} n + 2\gamma H n \]

Young-Laplace equation at the interface

fluid problem in eulerian description inside the domain

\[ \nabla_x \cdot \sigma + \rho b = 0 \]

• determine the motion of only one fluid
• example of application: pendant drop experiments

numerical examples

incremental kinematics to use Lagrangian description and to avoid ALE mapping

\[ x_{t+1} = \phi(x_t) \]

\[ \phi(x_t) = x_t + u \]

\[ d = \frac{1}{\lambda_t} \sum \ln(\lambda_i) n_{(i)} \otimes n_{(i)} \]

incremental constitutive equation for nearly incompressible newtonian fluid

\[ \sigma = -K(J - 1) I + 2\mu d \]

Saksono and Peric (2006)
\[ A_\alpha \cdot A^\beta = \delta_\alpha^\beta \]  
\[ A_\alpha \cdot A_\beta = A_{\alpha\beta} \]  
\[ J_A = \sqrt{\det A_{\alpha\beta}} \]  
\[ V_{\alpha;\beta} = A_\beta \cdot \nabla V_\alpha = DV_\alpha[A_\beta] \]  
\[ \tilde{F} = a_\alpha \otimes A^\alpha \]  
\[ \tilde{I} = I - N \otimes N \]  
\[ \tilde{F}^{-1} \cdot \tilde{F} = \tilde{I} \]  
\[ \tilde{F}^t \cdot \tilde{F}^{-1} = \tilde{I} \]  
\[ 0 = \text{Div} \tilde{P} + \tilde{B} \]  
\[ \text{Div} \tilde{P} = \tilde{P}_{\alpha;\beta} \]  
\[ \tilde{\psi} = \tilde{\psi}(\tilde{F}) \]  
\[ \tilde{P} = \frac{\partial \tilde{\psi}}{\partial \tilde{F}} \]
fluid-membrane-fluid interface

**coupled problem**

- Local equilibrium for the volume

\[ 0 = \text{Div} \, P + B \]

- Local equilibrium for the surface

\[ 0 = \text{Div} \, \widehat{P} + \widehat{B} \]

\[ \widehat{B} = t_0^p - P \cdot N \]

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**continua with boundary energies**

fluid-membrane-fluid interface

**coupled problem**

- Global equilibrium for the volume

\[ \int_{S_0} \delta \varphi \cdot P \cdot N \, dA_0 = \int_{B_0} \nabla \delta \varphi : P \, dV_0 - \int_{B_0} \delta \varphi \cdot B \, dV_0 \]

- Global equilibrium for the closed surface

\[ \int_{S_0} \delta \varphi \cdot P \cdot N \, dA_0 = - \int_{S_0} \nabla \delta \varphi : \widehat{P} \, dA_0 + \int_{S_0} \delta \varphi \cdot t_0^p \, dA_0 \]

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**continua with boundary energies**

fluid-membrane-fluid interface

**coupled problem**

- Constitutive equation for the volume

\[ P = \frac{\partial \psi}{\partial F} \]

- Constitutive equation for the surface

\[ \widehat{P} = \frac{\partial \widehat{\psi}}{\partial F} \]