continuum mechanics is the branch of mechanics concerned with the stress in solids, liquids and gases and the deformation or flow of these materials. the adjective continuous refers to the simplifying concept underlying the analysis: we disregard the molecular structure of matter and picture it as being without gaps or empty spaces. we suppose that all the mathematical functions entering the theory are continuous functions. this hypothetical continuous material we call a continuum.

malvern ‘introduction to the mechanics of a continuous medium.’ [1969]
kinematic equations - why not $\epsilon = \frac{\Delta l}{l}$?  
inhomogeneous deformation ∙ non-constant  
finito deformation ∙ non-linear  
inelastic deformation ∙ growth tensor  

F = $\nabla_X \phi$  
F = F$_b$ · F$_g$

balance equations - why not $\sigma = \frac{F}{A}$?  
equilibrium in deformed configuration ∙ multiple stresses  

constitutive equations - why not $\sigma = E \epsilon$?  
finite deformation ∙ non-linear  
inelastic deformation ∙ internal variables  
P = P(F)  
P = P(\rho, F, F_g)

continuum mechanics

Tensor calculus

- vector algebra  
  notation, euclidean vector space, scalar product, vector product, scalar triple product

- tensor algebra  
  notation, scalar products, dyadic product, invariants, trace, determinant, inverse, spectral decomposition, sym-skew decomposition, vol-dev decomposition, orthogonal tensor

- tensor analysis  
  derivatives, gradient, divergence, laplace operator, integral transformations

Tensor calculus

tensor [tensor] the word tensor was introduced in 1846 by william rowan hamilton. it was used in its current meaning by woldemar voigt in 1899. tensor calculus was developed around 1890 by gregorio ricci-curbastro under the title absolute differential calculus. in the 20th century, the subject came to be known as tensor analysis, and achieved broader acceptance with the introduction of einstein's theory of general relativity around 1915. tensors are used also in other fields such as continuum mechanics.

vector algebra - notation

- einstein's summation convention

\[ u_i = \sum_{j=1}^3 A_{ij} x_j + b_i = A_{ij} x_j + b_i \]

- summation over any indices that appear twice in a term

\[ u_1 = A_{11} x_1 + A_{12} x_2 + A_{13} x_3 + b_1 \]
\[ u_2 = A_{21} x_1 + A_{22} x_2 + A_{23} x_3 + b_2 \]
\[ u_3 = A_{31} x_1 + A_{32} x_2 + A_{33} x_3 + b_3 \]
tensor calculus

• kronecker symbol

\[ \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \]

\[ u_i = \delta_{ij} u_j \]

• permutation symbol

\[ \varepsilon_{ijk} = \begin{cases} 1 & \text{for } \{i, j, k\} \text{ ... even permutation} \\ -1 & \text{for } \{i, j, k\} \text{ ... odd permutation} \\ 0 & \text{... else} \end{cases} \]

tensor calculus

• euclidean vector space \( V^3 \)

\( \alpha, \beta \in \mathbb{R} \quad \mathbb{R} \quad \text{... real numbers} \)

\( u, v \in V^3 \quad V^3 \quad \text{... linear vector space} \)

• \( V^3 \) is defined through the following axioms

\[ \alpha (u + v) = \alpha u + \alpha v \]
\[ (\alpha + \beta) u = \alpha u + \beta u \]
\[ (\alpha \beta) u = \alpha (\beta u) \]

• zero element and identity

\[ 0 u = 0, \quad 1 u = u \]

• linear independence of \( e_1, e_2, e_3 \in V^3 \) if \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \)

is the only (trivial) solution to \( \alpha_i e_i = 0 \)

vector algebra - euclidean vector space

• euclidean vector space \( V^3 \) equipped with norm

\[ n : V^3 \to \mathbb{R} \quad \text{... norm} \]

• norm defined through the following axioms

\[ n(u) \geq 0 \quad n(u) = 0 \iff u = 0 \]

\[ n(\alpha u) = |\alpha| n(u) \]

\[ n(u + v) \leq n(u) + n(v) \]

\[ n^2(u + v) + n^2(u - v) = 2[n^2(u) + n^2(v)] \]

tensor calculus

• euclidean vector space \( E^3 \) equipped with euclidean norm

\[ n : E^3 \to \mathbb{R} \quad \text{... euclidean norm} \]

\[ n(u) = ||u|| = \sqrt{u \cdot u} = [u_1^2 + u_2^2 + u_3^2]^{1/2} \]

• representation of 3d vector \( u \in E^3 \)

\[ u = u_1 e_1 + u_2 e_2 + u_3 e_3 \]

with \( u_1, u_2, u_3 \) coordinates (components) of \( u \) relative to the basis \( e_1, e_2, e_3 \)

\[ u = [u_1, u_2, u_3]^t \]
vector algebra - scalar product

- Euclidean norm enables definition of scalar (inner) product
  \[ u \cdot v = \alpha \quad \alpha \in \mathbb{R} \]
  \[ u \cdot v = ||u|| \cdot ||v|| \cos \theta \]
  \[ ||u \cdot v|| \leq ||u|| \cdot ||v|| \]

- Properties of scalar product
  \[ u \cdot v = v \cdot u \]
  \[ (\alpha u + \beta v) \cdot w = \alpha (u \cdot w) + \beta (v \cdot w) \]
  \[ w \cdot (\alpha u + \beta v) = \alpha (w \cdot u) + \beta (w \cdot v) \]

- Positive definiteness
  \[ u \cdot u \geq 0, \quad u \cdot u = 0 \Leftrightarrow u = 0 \]

- Orthogonality
  \[ u \cdot v = 0 \Leftrightarrow u \perp v \]

vector algebra - vector product

- Vector product
  \[ u \times v = w \quad w \in \mathbb{E}^3 \]
  \[ u \times v = ||u|| \cdot ||v|| \sin \theta \cdot n \]
  \[ u \times v = 0 \Leftrightarrow u \parallel v \]
  \[ \begin{vmatrix}
    w_1 \\
    w_2 \\
    w_3 \\
  \end{vmatrix} =
  \begin{vmatrix}
    u_2 v_3 - u_3 v_2 \\
    u_3 v_1 - u_1 v_3 \\
    u_1 v_2 - u_2 v_1 \\
  \end{vmatrix} \]

- Properties of vector product
  \[ u \times v = -v \times u \]
  \[ (\alpha u + \beta v) \times w = \alpha (u \times w) + \beta (v \times w) \]
  \[ u \cdot (u \times v) = 0 \]
  \[ (u \times v) \cdot (u \times v) = (u \cdot u)(v \cdot v) - (u \cdot v)^2 \]

tensor calculus

vector algebra - scalar triple product

- Scalar triple product
  \[ [u, v, w] = u \cdot (v \times w) = \alpha \quad \alpha \in \mathbb{R} \]
  \[ u \times v = ||u|| \cdot ||v|| \sin \theta \cdot n \quad \text{area} \]
  \[ [u, v, w] = u \cdot (v \times w) \quad \text{volume} \]
  \[ \alpha = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) \]

- Properties of scalar triple product
  \[ [u, v, w] = [v, w, u] = [w, u, v] \]
  \[ = -[u, w, v] = -[v, u, w] = -[w, v, u] \]
  \[ [\alpha u + \beta v, w, d] = \alpha [u, w, d] + \beta [v, w, d] \]

- Linear independence
  \[ [u, v, w] \neq 0 \]

tensor algebra - second order tensors

- Second order tensor
  \[ A = u \otimes v \]
  \[ u = u_i e_i \quad \text{and} \quad v = v_j e_j \]
  \[ A = A_{ij} e_i \otimes e_j \]
  \[ [A_{ij}] = \begin{bmatrix}
    A_{11} & A_{12} & A_{13} \\
    A_{21} & A_{22} & A_{23} \\
    A_{31} & A_{32} & A_{33} \\
  \end{bmatrix} \]

  with \( A_{ij} = u_i v_j \) coordinates (components) of \( A \) relative to the basis \( e_1, e_2, e_3 \)

- Transpose of second order tensor
  \[ A^t = (u \otimes v)^t = v \otimes u \]
  \[ A^t = A_{ji} e_j \otimes e_i \]
  \[ [A_{ji}] = \begin{bmatrix}
    A_{11} & A_{21} & A_{31} \\
    A_{12} & A_{22} & A_{32} \\
    A_{13} & A_{23} & A_{33} \\
  \end{bmatrix} \]
**Tensor Calculus**

**Tensor Algebra - Second Order Tensors**

- Second order unit tensor in terms of Kronecker symbol
  \[ I = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \]
  with \(\delta_{ij}\) coordinates (components) of \(I\) relative to the basis \(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\)

- Matrix representation of coordinates
  \[
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 
  \end{bmatrix}
  \]

- Identity
  \[ I \cdot \mathbf{u} = \mathbf{u} \quad \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \cdot u_j \mathbf{e}_j = u_i \mathbf{e}_i \]

**Tensor Algebra - Third Order Tensors**

- Third order tensor
  \[ \mathbf{a} = A \otimes \mathbf{v} \quad A = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad \mathbf{v} = v_k \mathbf{e}_k \]
  \[ \mathbf{a} = a_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \]
  with \(a_{ijk} = A_{ij} v_k\) coordinates (components) of \(A\) relative to the basis \(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\)

- Third order permutation tensor in terms of permutation symbol \(e_{ijk}\)
  \[ e = e_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \]

**Tensor Algebra - Fourth Order Tensors**

- Fourth order tensor
  \[ A = A \otimes B \quad A = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad B = B_{kl} \mathbf{e}_k \otimes \mathbf{e}_l \]
  \[ A = A_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \]
  with \(A_{ijkl} = A_{ij} B_{kl}\) coordinates (components) of \(A\) relative to the basis \(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\)

- Fourth order unit tensor
  \[ \mathbf{I} = \delta_{ik} \delta_{jl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \]
  \[ \mathbf{I} : A = A \]

- Transpose of fourth order unit tensor
  \[ \mathbf{I}^t = \delta_{il} \delta_{jk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \]
  \[ \mathbf{I}^t : A = A^t \]

**Tensor Calculus**

**Tensor Algebra - Symmetric Tensors**

- Symmetric fourth order unit tensor
  \[ \mathbf{I}_{sym} = \frac{1}{2} [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \]
  \[ \mathbf{I}_{sym} : A = A_{sym} \]

- Screw-symmetric fourth order unit tensor
  \[ \mathbf{I}_{skw} = \frac{1}{2} [\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \]
  \[ \mathbf{I}_{skw} : A = A_{skw} \]

- Volumetric fourth order unit tensor
  \[ \mathbf{I}_{vol} = \frac{1}{3} \delta_{ij} \delta_{kl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \]
  \[ \mathbf{I}_{vol} : A = A_{vol} \]

- Deviatoric fourth order unit tensor
  \[ \mathbf{I}_{dev} = \left( -\frac{1}{3} \delta_{ij} \delta_{kl} + \frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} \right) \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \]
  \[ \mathbf{I}_{dev} : A = A_{dev} \]
tensor algebra - scalar product

- scalar (inner) product
  \[ A \cdot u = (A_{ij} e_i \otimes e_j) \cdot (u_k e_k) \]
  \[ = A_{ij} u_k \delta_{jk} e_i = A_{ij} u_j e_i = v_i e_i = v \]
  of second order tensor \( A \) and vector \( u \)

- zero and identity
  \[ 0 \cdot u = 0 \quad I \cdot u = u \]

- positive definiteness
  \[ a \cdot A \cdot a > 0 \]

- properties of scalar product
  \[ A \cdot (\alpha a + \beta b) = \alpha (A \cdot a) + \beta (A \cdot b) \]
  \[ (A + B) \cdot a = A \cdot a + B \cdot a \]
  \[ (\alpha A) \cdot a = \alpha (A \cdot a) \]

tensor calculus

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tensor algebra - dyadic product

- dyadic (outer) product
  \[ A = u \otimes v = u_i v_j e_i \otimes e_j = v_i v_j e_i \otimes e_j = A_{ij} e_i \otimes e_j \]
  of two vectors \( u, v \) introduces second order tensor \( A \)

- properties of dyadic product (tensor notation)
  \[ (u \otimes v) \cdot w = (v \cdot w) u \]
  \[ (\alpha u + \beta v) \otimes w = \alpha (u \otimes w) + \beta (v \otimes w) \]
  \[ u \otimes (\alpha v + \beta w) = \alpha (u \otimes v) + \beta (u \otimes w) \]
  \[ (u \otimes v) \cdot (w \otimes x) = (v \otimes w) (u \otimes x) \]
  \[ A \cdot (u \otimes v) = (A \cdot u) \otimes v \]
  \[ (u \otimes v) \cdot A = u \otimes (A^t \cdot v) \]
**Tensor algebra - dyadic product**

- Dyadic (outer) product

\[ A = u \otimes v = u_i e_i \otimes v_j e_j = u_i v_j e_i \otimes e_j = A_{ij} e_i \otimes e_j \]

of two vectors \( u, v \) introduces second order tensor \( A \)

- Properties of dyadic product (index notation)

\[ (u_i v_j) w_j = (v_j w_j) u_i \]
\[ (\alpha u_i + \beta v_i) w_j = \alpha (u_i w_j) + \beta (v_i w_j) \]
\[ u_i (\alpha v_j + \beta w_j) = \alpha (u_i v_j) + \beta (u_i w_j) \]
\[ (u_i v_j)(w_j x_k) = (v_j w_j)(u_i x_k) \]
\[ A_{ij}(u_j v_k) = (A_{ij} u_i)v_k \]
\[ (u_i v_j)A_{jk} = u_i(A_{kj} v_j) \]