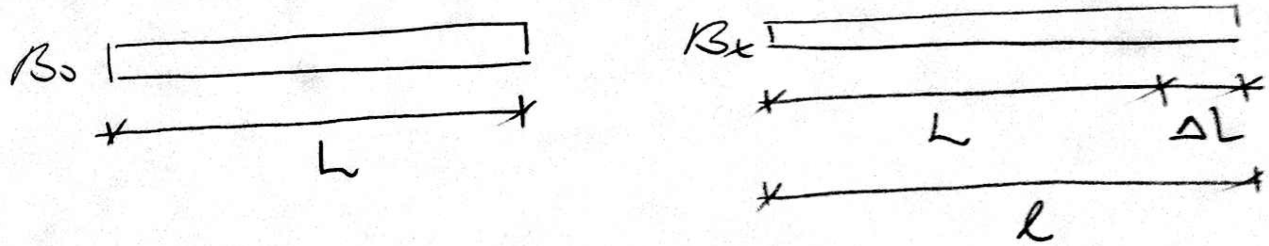
… Some more examples…

#01 Directional derivative, 1D



$\varepsilon_E = \frac{\Delta L}{L} = \frac{l-L}{L}$ … engineering strain

$\varepsilon_L = \ln\left(\frac{l}{L}\right)$ … logarithmic strain

$\varepsilon_G = \frac{l^2 - L^2}{2L^2} = E$ … Green-Lagrange strain

$\varepsilon_A = \frac{l^2 - L^2}{2\,l^2} = e$ … Euler-Almansi strain

determine directional derivatives $D\varepsilon(l)[u]$, where u is a small increment in the direction l
(i.e. replace "l" by "$l+\epsilon u$")

$$D\varepsilon_E(l)[u] = \frac{d}{d\epsilon}\left[\frac{(l+\epsilon u)-L}{L}\right]\Bigg|_{\epsilon=0} = \frac{u}{L}\Bigg|_{\epsilon=0} = \underline{\underline{\frac{u}{L}}}$$

$$D\varepsilon_L(l)[u] = \frac{d}{d\epsilon}\left[\ln(l+\epsilon u) - \ln(L)\right]\Big|_{\epsilon=0} = \frac{u}{l+\epsilon u}\Bigg|_{\epsilon=0} = \underline{\underline{\frac{u}{l}}}$$

$$D\varepsilon_G(l)[u] = \frac{d}{d\epsilon}\left[\frac{(l+\epsilon u)^2 - L^2}{2L^2}\right]\Bigg|_{\epsilon=0}$$

$$= \frac{1}{2L^2}\left[2(l+\epsilon u)u\right]\Big|_{\epsilon=0} = \underline{\underline{\frac{ul}{L^2}}}$$

$$= \left(\frac{l}{L}\right)\left(\frac{u}{l}\right)\left(\frac{l}{L}\right)$$

in 3D: $D\mathbb{E}[u_i] = \mathbb{F}^t \cdot \varepsilon \cdot \mathbb{F}$ w/ $\mathbb{F} = \frac{l}{L}$

$$D\mathcal{E}_A(l)[u] = \frac{d}{d\epsilon}\left[\frac{(l+\epsilon u)^2 - L^2}{2(l+\epsilon u)^2}\right]\Bigg|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\left[\frac{1}{2} - \frac{L^2}{2}(l+\epsilon u)^{-2}\right]\Bigg|_{\epsilon=0}$$

$$= \frac{L^2 u}{(l+\epsilon u)^3}\Bigg|_{\epsilon=0} = \underline{\underline{\frac{L^2 u}{l^3}}} = \left(\frac{L}{l}\right)\left(\frac{u}{l}\right)\left(\frac{L}{l}\right)$$

#02 DIRECTIONAL DERIVATIVE, 3D

GIVEN ANY SECOND ORDER TENSOR $\mathbb{A}$, LINEARIZE $\mathbb{A}^2 = \mathbb{A}\cdot\mathbb{A}$ IN AN INCREMENT $\mathbb{U}$

$$D\mathbb{A}^2[\mathbb{U}] = \frac{d}{d\epsilon}\left[[\mathbb{A}+\epsilon\mathbb{U}]\cdot[\mathbb{A}+\epsilon\mathbb{U}]\right]\Bigg|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\left[\underbrace{\mathbb{A}\cdot\mathbb{A}}_{\text{CONSTANT TERMS DROP}} + \epsilon\mathbb{U}\cdot\mathbb{A} + \mathbb{A}\cdot\epsilon\mathbb{U} + \epsilon^2\mathbb{U}\cdot\mathbb{U}\right]\Bigg|_{\epsilon=0}$$

$$= \mathbb{U}\cdot\mathbb{A} + \mathbb{A}\cdot\mathbb{U} + \underbrace{2\epsilon\mathbb{U}\cdot\mathbb{U}}_{\text{HIGHER ORDER TERMS DROP}}\Big|_{\epsilon=0}$$

$$= \underline{\underline{\mathbb{U}\cdot\mathbb{A} + \mathbb{A}\cdot\mathbb{U}}}$$

RECIPE:

$$\boxed{D\phi(\mathbb{X})[\mathbb{U}] = \frac{d}{d\epsilon}\left[\phi(\mathbb{X}+\epsilon\mathbb{U})\right]\Big|_{\epsilon=0}}$$

(1) REPLACE $\mathbb{X} \leftarrow \mathbb{X} + \epsilon\mathbb{U}$

(2) EVALUATE $\phi(\mathbb{X}+\epsilon\mathbb{U})$

(3) TAKE DERIVATIVE WRT ϵ (TO REMOVE C's)

(4) SET $\epsilon = 0$ (TO REMOVE HOT's)

#3 DIRECTIONAL DERIVATIVE w/ NUMBERS

$$\Phi(\mathbb{X})[u] = X_1^2 + 3X_2X_3$$

$$u = \frac{1}{\sqrt{3}}[1,1,1]^t \; ; \quad \mathbb{X} = [2,-1,0]^t$$

$$D\Phi(\mathbb{X})[u] = \frac{d}{d\epsilon}\left[(X_1+\epsilon u_1)^2 + 3(X_2+\epsilon u_2)(X_3+\epsilon u_3)\right]_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\left[X_1^2 + 2\epsilon X_1 u_1 + \epsilon^2 u_1^2 + 3X_2X_3 + 3X_2\epsilon u_3 + 3\epsilon u_2 X_3 + 3\epsilon^2 u_2 u_3\right]$$

$$= \left[2X_1u_1 + 2\epsilon u_1^2 + 3X_2u_3 + 3u_2X_3 + 6\epsilon u_2u_3\right]_{\epsilon=0}$$

$$= 2X_1u_1 + 3X_2u_3 + 3u_2X_3$$

$$D\phi(\mathbb{X})[u] = 2[2]\left[\frac{1}{\sqrt{3}}1\right] + 3[-1]\left[\frac{1}{\sqrt{3}}1\right] + 3\left[\frac{1}{\sqrt{3}}\right]\overset{=0}{[0]} = \frac{1}{\sqrt{3}}$$

#4 Kinematics - 1D

UNIAXIAL STRETCH $\boxed{\lambda = \frac{dl}{dL}}$

REPHRASE STRAINS $\varepsilon_E, \varepsilon_G, \varepsilon_A$

$\varepsilon_E = \frac{l-L}{L} = \lambda - 1$ ...engineering strain

$\varepsilon_G = \frac{l^2-L^2}{2L^2} = \frac{1}{2}[\lambda^2-1]$ ... Green-Lagrange strain

$\varepsilon_A = \frac{l^2-L^2}{2l^2} = \frac{1}{2}[1-\lambda^{-2}]$ ... Euler-Almansi strain

SHOW THAT FOR SMALL STRAINS ε_E,

$\varepsilon_G \approx \varepsilon_A \approx \varepsilon_E$!

$$\lambda = \varepsilon_E + 1$$

$$\varepsilon_G = \frac{1}{2}\left[(\varepsilon_E+1)^2 - 1\right] = \underline{\underline{\varepsilon_E + \frac{1}{2}\varepsilon_E^2}}$$

$$\varepsilon_A = \frac{1}{2}\left[1 - \frac{1}{(\varepsilon_E+1)^2}\right] = \underline{\underline{\frac{1}{2}\left[\frac{\varepsilon_E^2 + 2\varepsilon_E}{\varepsilon_E^2 + 2\varepsilon_E + 1}\right]}}$$

<u>FOR SMALL STRAINS ε_E</u>

$$\varepsilon_G = \varepsilon_E + \frac{1}{2}\underbrace{\varepsilon_E^2}_{\approx 0} \approx \underline{\underline{\varepsilon_E}} \quad ✓$$

$$\varepsilon_A = \frac{1}{2}\left[\frac{\cancel{\varepsilon_E^2}^{\approx 0} + 2\varepsilon_E}{\cancel{\varepsilon_E^2}^{\approx 0} + \cancel{2\varepsilon_E + 1}^{\approx 1}}\right] \approx \frac{\cancel{2}\varepsilon_E}{\cancel{2}\cdot 1} = \frac{\varepsilon_E}{1} = \underline{\underline{\varepsilon_E}} \quad ✓$$

#5 BALANCE EQUATIONS

$$\varphi = \begin{bmatrix} 5 - 3X_1 - X_2 \\ 2 + 5/4\, X_1 - 2X_2 \\ X_3 \end{bmatrix} \qquad \mathbb{P} = \begin{bmatrix} X_1 & X_1 & 0 \\ \alpha & X_2 & 0 \\ 0 & 0 & X_3 \end{bmatrix}$$

<u>5.1) ISOCHORIC?</u> $\rightarrow J = \det(\mathbb{F}) = 1$

$$\mathbb{F} = \frac{d\varphi}{d\mathbf{X}} = \begin{bmatrix} -3 & -1 & 0 \\ 5/4 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det \mathbb{F} = 6 - \left(-\frac{5}{4}\right) = \underline{\underline{7\frac{1}{4} \neq 1}}$$

$\rightsquigarrow$ <u>non-ISOCHORIC</u>

5.2) <u>α, SUCH THAT $\mathbb{P}$ SATISFIES BALANCE OF ANGULAR MOMENTUM</u>

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^t} \qquad \boxed{\boldsymbol{\sigma} = \frac{1}{J}\,\mathbb{P}\cdot\mathbb{F}^t}$$

$$J\boldsymbol{\sigma} = \begin{bmatrix} X_1 & X_1 & 0 \\ \alpha & X_2 & 0 \\ 0 & 0 & X_3 \end{bmatrix} \begin{bmatrix} -3 & -1 & 0 \\ +5/4 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4X_1 & -\frac{3}{4}X_1 & 0 \\ -(3\alpha + X_2) & (+\frac{5}{4}\alpha - 2X_2) & 0 \\ 0 & 0 & X_3 \end{bmatrix}$$

THUS $\quad -\frac{3}{4}X_1 \overset{!}{=} -3\alpha - X_2$

$$\rightsquigarrow \quad \underline{\underline{\alpha = \frac{1}{4}X_1 - \frac{1}{3}X_2}}$$

5.3) <u>BODY FORCE FIELD $\mathbb{f}$ TO SATISFY BALANCE OF LINEAR MOMENTUM</u>

$$\boxed{\mathrm{Div}\,\mathbb{P} + \mathbb{f} \overset{!}{=} \mathbb{0}} \rightsquigarrow \mathbb{f} = -\mathrm{Div}\,\mathbb{P} = \sum_{i,j=1}^{3} \frac{\partial P_{ij}}{\partial X_j}$$

$$\mathbb{P} = \begin{bmatrix} X_1 & X_1 & 0 \\ \left[\frac{X_1}{4} - \frac{X_2}{3}\right] & X_2 & 0 \\ 0 & 0 & X_3 \end{bmatrix}$$

$$\underline{\underline{\mathbb{f} = -\begin{bmatrix} 1 \\ \frac{1}{4} + 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 5/4 \\ 1 \end{bmatrix}}}$$