isotropic hyperelastic materials

\[ \psi = \psi(b) = \psi(I_1^b, I_2^b, I_3^b) \]

Cauchy stress

\[ \sigma = \frac{2}{J} \frac{\partial \psi(b)}{\partial b} \cdot b = \frac{2}{J} b \cdot \frac{\partial \psi(b)}{\partial b} \]

Invariants in terms of principal stretches

\[
\begin{align*}
I_1^b &= \text{tr}(b) = \lambda_1 + \lambda_{II} + \lambda_{III} \\
I_2^b &= \frac{1}{2} \left[ \text{tr}^2(b) - \text{tr}(b^2) \right] = \lambda_1 \cdot \lambda_{II} + \lambda_{II} \cdot \lambda_{III} + \lambda_{III} \cdot \lambda_1 \\
I_3^b &= \det(b) = \lambda_1 \cdot \lambda_{II} \cdot \lambda_{III} \\
\end{align*}
\]

Principal cauchy stresses

\[ \sigma_I = -p + \lambda_I \frac{\partial \psi}{\partial \lambda_I} \quad I = 1, 2, 3 \]
% pressure vs stretch plot
% loop over all stretches lambda from 1.0 to 10.0
for i=1:901
    lam(i) = 1.0+(i-1)/100;
    p_og(i) = 2*H/R *(m1_og * (lam(i)^((a1_og-3)*lam(i))^(-2*lam(i)-3)) ... 
               + m2_og * (lam(i)^((a2_og-3)*lam(i))^(-2*lam(i)-3)) ... 
               + m3_og * (lam(i)^((a3_og-3)*lam(i))^(-2*lam(i)-3)));
    p_mr(i) = 2*H/R *(m1_mr * (lam(i)^((a1_mr-3)*lam(i))^(-2*lam(i)-3)) ... 
                      + m2_mr * (lam(i)^((a2_mr-3)*lam(i))^(-2*lam(i)-3)));
    p_nh(i) = 2*H/R *(m1_nh * (lam(i)^((a1_nh-3)*lam(i))^(-2*lam(i)-3)));
    p_vg(i) = 2*H/R *(m1_vg * (lam(i)^((a1_vg-3)*lam(i))^(-2*lam(i)-3)));
end
plot(lam,p_og/10^2,'-k','LineWidth',2.0)
plot(lam,p_mr/10^2,'-k','LineWidth',2.0)
plot(lam,p_nh/10^2,'-k','LineWidth',2.0)
plot(lam,p_vg/10^2,'-k','LineWidth',2.0)
14 - hyperelastic materials

inflation of a spherical rubber balloon

special case: gateau derivative

consider smooth, differentiable scalar field $\Phi$ with

- scalar argument $\Phi: \mathcal{R} \to \mathcal{R}; \quad \Phi(x) = \alpha$
- vectorial argument $\Phi: \mathcal{R}^3 \to \mathcal{R}; \quad \Phi(x) = \alpha$
- tensorial argument $\Phi: \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}; \quad \Phi(X) = \alpha$

- scalar $D \Phi(x) | u = \frac{d}{d\epsilon} \Phi(x + \epsilon u) |_{\epsilon=0} \forall u \in \mathcal{R}$
- vectorial $D \Phi(x) \cdot u = \frac{d}{d\epsilon} \Phi(x + \epsilon u) |_{\epsilon=0} \forall u \in \mathcal{R}^3$
- tensorial $D \Phi(X):U = \frac{d}{d\epsilon} \Phi(X + \epsilon U) |_{\epsilon=0} \forall U \in \mathcal{R}^3 \otimes \mathcal{R}^3$

14 - tensor analysis

frechet derivative

consider smooth, differentiable scalar field $\Phi$ with

- scalar argument $\Phi: \mathcal{R} \to \mathcal{R}; \quad \Phi(x) = \alpha$
- vectorial argument $\Phi: \mathcal{R}^3 \to \mathcal{R}; \quad \Phi(x) = \alpha$
- tensorial argument $\Phi: \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}; \quad \Phi(X) = \alpha$

- scalar $D \Phi(x) = \frac{\partial \Phi(x)}{\partial x} = \partial_x \Phi(x)$
- vectorial $D \Phi(x) = \frac{\partial \Phi(x)}{\partial x} = \partial_x \Phi(x)$
- tensorial $D \Phi(X) = \frac{\partial \Phi(X)}{\partial X} = \partial_X \Phi(X)$

14 - tensor analysis

example: derivative of invariants

use the Gateaux derivative

$$D\Phi(A) : \Delta A = \frac{\partial \Phi(A + \epsilon \Delta A)}{\partial \epsilon} |_{\epsilon=0}$$

to determine the derivatives of the three invariants $I_A, II_A, III_A$ of the second order tensor $A$ with respect to $A$ itself!

in words: the Gateaux derivative $D$ of a scalar field $\Phi$ (in this case the invariant $I_A, II_A, III_A$) along a given direction $U$ (in this case $\Delta A$) is the derivative of the field $\Phi$ at position $X$ (in this case $A$) "perturbed" by the direction $\epsilon U$ (in this case $\Delta A$) with respect to $\epsilon$ evaluated at $\epsilon = 0$

setting $\epsilon = 0$ filters out the linear terms, i.e., all higher order terms are set to zero, the Gateaux derivative is therefore also referred to as linearization, it corresponds to the first term of a Taylor series

14 - tensor analysis
example: derivative of 1st invariant

\[ I_A = \text{tr} (A) = A : I \]

\[ DI_A(A) : \Delta A = \frac{d}{d \epsilon} \text{tr}(A + \epsilon \Delta A) |_{\epsilon=0} = \frac{d}{d \epsilon} [A + \epsilon \Delta A] : I |_{\epsilon=0} = \Delta A : I |_{\epsilon=0} = I : \Delta A \]

\[ DI_A(A) = \frac{\partial I_A(A)}{\partial A} = I \]

remark: \( I_A \) is linear in \( A \), its Gateaux derivative is constant, there are no terms in \( \epsilon \) once we take the derivative \( d/d \epsilon \)

14 - tensor analysis

example: derivative of 2nd invariant

\[ \text{derivative of second invariant } \ II_A = \frac{1}{2} ([A : I]^2 + A : A') \]

\[ DII_A(A) : \Delta A = \frac{d}{d \epsilon} \frac{1}{2} \text{tr}^2(A + \epsilon \Delta A) - \frac{1}{2} \text{tr}(A + \epsilon \Delta A)^2 |_{\epsilon=0} = \frac{d}{d \epsilon} \frac{1}{2} [A + \epsilon \Delta A] : [A + \epsilon \Delta A]^T |_{\epsilon=0} = [A + \epsilon \Delta A] : I \Delta A : I = \frac{1}{2} \Delta A : A' - \frac{1}{2} \Delta A : \Delta A' - \epsilon \Delta A : \Delta A' |_{\epsilon=0} = [\text{tr}(A) I - A'] : \Delta A \]

\[ DII_A(A) = \frac{\partial II_A(A)}{\partial A} = \frac{\text{tr}(A) I - A'}{2} \]

here, we have used the following identity

\[ \text{tr}(A^2) = (A : A) : I = A : A' \]

remark: \( II_A \) is quadratic in \( A \), its Gateaux derivative is linear, once we took the derivative \( d/d \epsilon \), the higher order term in \( \epsilon \) is filtered out by setting \( \epsilon = 0 \)

14 - tensor analysis

example: derivative of 3rd invariant

\[ \text{derivative of third invariant } \ III_A = \text{det}(A) \]

\[ DIII_A(A) : \Delta A = \frac{d}{d \epsilon} \text{det}(A + \epsilon \Delta A) |_{\epsilon=0} = \frac{d}{d \epsilon} \text{det}(A \cdot [I + A^{-1} \cdot \epsilon \Delta A]) |_{\epsilon=0} = \frac{d}{d \epsilon} \text{det}(A) \cdot \text{det}(A^{-1} \cdot \Delta A + I) |_{\epsilon=0} = \frac{d}{d \epsilon} \text{det}(A) \cdot [(\epsilon \lambda_{A^{-1} \cdot \Delta A} + 1) (\epsilon \lambda_{A^{-1} \cdot \Delta A} + 1)] |_{\epsilon=0} = \text{det}(A) \cdot \left[ \lambda_{A^{-1} \cdot \Delta A} + \lambda_{A^{-1} \cdot \Delta A} + \lambda_{A^{-1} \cdot \Delta A} \right] = \text{det}(A) \cdot \text{tr}(A^{-1} \cdot \Delta A) \]

\[ DIII_A(A) = \frac{\partial III_A(A)}{\partial A} = \frac{\text{det}(A) \cdot A^{-1}}{2} \]

14 - tensor analysis

example: derivative of 3rd invariant

here, we have used the following expression for the determinant of \((\epsilon A^{-1} \cdot \Delta A)\) expressed through the characteristic polynom for the eigenvalue \( \lambda = -1 \)

\[ \text{det}(\epsilon A^{-1} \cdot \Delta A - \lambda I) = (\epsilon \lambda_{A^{-1} \cdot \Delta A1} - \lambda) (\epsilon \lambda_{A^{-1} \cdot \Delta A2} - \lambda) (\epsilon \lambda_{A^{-1} \cdot \Delta A3} - \lambda) \]

reformulation with the help of index notation

\[ \text{tr}(A^{-1} \cdot \Delta A) = (A^{-1} \cdot \Delta A) : I = (A^{-1} \cdot \Delta A) : (\delta_{mn} e_m \otimes e_n) \]

\[ = (A^{-1} \Delta A_{ij} e_i \otimes e_j) : (\delta_{mn} e_m \otimes e_n) = A^{-1} \Delta A_{ij} \delta_{im} \delta_{jn} = A^{-1} : \Delta A \]

remark: \( III_A \) is cubic in \( A \)
derivative of function wrt tensor

**derivative of function with tensorial argument** show that the derivative of the function $\Phi(A)$ with respect to its tensorial argument $A$ is given through the component-wise derivative with respect to the individual tensorial entries $A_{ij}$.

Index representation of second order tensor $A$

$$A = A_{ij} e_i \otimes e_j$$

Extraction of individual components $A_{ij}$ through “projection” onto the base vectors $e_i$ and $e_j$:

$$A_{ij} = e_i \cdot A \cdot e_j = e_i \cdot (A_{kl} e_k \otimes e_l) \cdot e_j = \delta_{ik} A_{kl} \delta_{lj} = A_{ij}$$

example: derivative of 1st invariant

**derivative of first invariant** $I_A$ wrt $A$ use the component-wise derivation to determine the derivative of the first invariant $I_A$ with respect to its tensor and validate the result derived previously:

$$I_A = \text{tr}A = A_{11} + A_{22} + A_{33}$$

$$\frac{\partial I_A(A)}{\partial A} = \begin{bmatrix} \frac{\partial I_A}{\partial A_{11}} & \frac{\partial I_A}{\partial A_{12}} & \frac{\partial I_A}{\partial A_{13}} \\ \frac{\partial I_A}{\partial A_{21}} & \frac{\partial I_A}{\partial A_{22}} & \frac{\partial I_A}{\partial A_{23}} \\ \frac{\partial I_A}{\partial A_{31}} & \frac{\partial I_A}{\partial A_{32}} & \frac{\partial I_A}{\partial A_{33}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial I_A(A)}{\partial A} = I$$

Remark: derivatives of functions with respect to tensors can be derived individually for each component.

derivative of function wrt tensor

and thus

$$\frac{\partial \Phi(A)}{\partial A} = \frac{\partial \Phi}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial A} = \frac{\partial \Phi}{\partial A_{ij}} \frac{\partial e_i \cdot A \cdot e_j}{\partial A} = \frac{\partial \Phi}{\partial A_{ij}} e_i \otimes e_j$$

$$\frac{\partial \Phi(A)}{\partial A} = \frac{\partial \Phi}{\partial A_{11}} e_1 \otimes e_1 + \frac{\partial \Phi}{\partial A_{12}} e_1 \otimes e_2 + \frac{\partial \Phi}{\partial A_{13}} e_1 \otimes e_3$$

$$+ \frac{\partial \Phi}{\partial A_{21}} e_2 \otimes e_1 + \frac{\partial \Phi}{\partial A_{22}} e_2 \otimes e_2 + \frac{\partial \Phi}{\partial A_{23}} e_2 \otimes e_3$$

$$+ \frac{\partial \Phi}{\partial A_{31}} e_3 \otimes e_1 + \frac{\partial \Phi}{\partial A_{32}} e_3 \otimes e_2 + \frac{\partial \Phi}{\partial A_{33}} e_3 \otimes e_3$$

Remark: derivatives of functions with respect to tensors can be derived individually for each component.

example: derivative of $A$ wrt itself

**derivative of $A$ wrt itself**

$$A = A_{ij} e_i \otimes e_j$$

$$\frac{\partial A}{\partial A} = \begin{bmatrix} \frac{\partial A_{ij}}{\partial A_{kl}} & \frac{\partial A_{ij}}{\partial A_{k2}} & \frac{\partial A_{ij}}{\partial A_{kl}} \\ \frac{\partial A_{kl}}{\partial A_{1l}} & \frac{\partial A_{kl}}{\partial A_{2l}} & \frac{\partial A_{kl}}{\partial A_{3l}} \\ \frac{\partial A_{kl}}{\partial A_{12}} & \frac{\partial A_{kl}}{\partial A_{13}} & \frac{\partial A_{kl}}{\partial A_{23}} \end{bmatrix} = \delta_{ik} \delta_{jl}$$

$$\frac{\partial A}{\partial A} = I \otimes I = I$$

Remark: always keep in mind that the derivative of $A$ with respect to itself is not(!) 1 but the fourth order unity tensor $I$. 

14 - tensor analysis

Example: derivative of $A$ wrt inverse

The derivative of the inverse of a tensor $A$ with respect to $A$ is given by:

$$ \frac{\partial A^{-1}}{\partial A} = \left[ \frac{\partial A^{-1}}{\partial A_{ij}} \right] e_i \otimes e_j \otimes e_k \otimes e_l $$

The derivative of the identity with respect to $A$ is:

$$ \frac{\partial \delta_{ji}}{\partial A_{ij}} = \frac{\partial A^{-1} A_{jm}}{\partial A_{ij}} = A_{ji}^{-1} A_{jm} + A_{ij}^{-1} \frac{\partial A_{jm}}{\partial A_{ij}} = 0 $$

And thus,

$$ \left[ \frac{\partial A^{-1}}{\partial A_{ij}} \right] = -A_{ij}^{-1} \frac{\partial A_{jm}}{\partial A_{ij}} = -A_{ij}^{-1} \delta_{jm} A_{jm} = -A_{ij}^{-1} A_{jm} $$

Remark: the derivative of the inverse of a tensor $A$ with respect to $A$ is negative. It can be determined with a trick by using the definition of the inverse.

14 - hyperelastic materials

Transversely isotropic incompressible

Incompressible transversely isotropic material

$$ \psi = \psi^\text{vol}(J) + \tilde{\psi}(\tilde{I}_1, \tilde{I}_4) $$

$$ J = \det(F) \quad I_1 = \tilde{C} : I \quad I_4 = \tilde{C} : N $$

$$ \frac{\partial \psi}{\partial C} = \frac{1}{2} J C^{-1} \quad \frac{\partial \psi}{\partial C} \tilde{I}_1 = I \quad \frac{\partial \psi}{\partial C} \tilde{I}_4 = N $$

Second Piola-Kirchhoff stress

$$ S = 2 \frac{\partial \psi}{\partial C} = S^\text{vol} + S^\text{iso} $$

Volumetric part

$$ S^\text{vol} = 2 \frac{\partial (\psi^\text{vol})}{\partial C} = J p \ C^{-1} \quad S^\text{iso} = 2 \frac{\partial \tilde{\psi}}{\partial C} = J^{-2/3} \ P \ : \ \tilde{S} $$

$$ \tilde{S} = 2 \frac{\partial \tilde{\psi}}{\partial C} = 2 \tilde{\psi}_1 I + 2 \tilde{\psi}_4 N $$

Structural tensor

$$ \mathbf{N} = n_0 \otimes n_0 $$

14 - hyperelastic materials

Transversely isotropic incompressible

Incompressible transversely isotropic material

$$ F = F^\text{vol} \cdot \bar{F} $$

Volumetric part

$$ F^\text{vol} = J^{1/3} \ I $$

Isochoric part

$$ \bar{F} = J^{-1/3} F $$

Transversely isotropic material

Fiber orientation

$$ ||n_0|| = 1 $$

Structural tensor

$$ \mathbf{N} = n_0 \otimes n_0 $$
example 01: holzapfel model
\[ \tilde{\psi} = c_0 \left[ I_1 - 3 \right] + \frac{c_1}{2c_2} \exp(c_2 \left[ I_4 - 1 \right]^2) - 1 \]
\[ \tilde{S} = 2 \frac{\partial \tilde{\psi}}{\partial \tilde{C}} = 2 \tilde{\psi}_1 I + 2 \tilde{\psi}_4 N \quad \text{with} \quad \tilde{\psi}_i = \frac{\partial \tilde{\psi}}{\partial \tilde{I}_i} \]
\[ \tilde{\psi}_1 = c_0 \]
\[ \tilde{\psi}_4 = c_1 [ I_4 - 1 ] \exp(c_2 [ I_4 - 1 ]^2) \]

example 02: may newman model
\[ \tilde{\psi} = c_0 \left[ \exp(c_1 \left[ I_1 - 3 \right]^2 + c_2 \left[ I_4 - 1 \right]^2) - 1 \right] \]
\[ \tilde{S} = 2 \frac{\partial \tilde{\psi}}{\partial \tilde{C}} = 2 \tilde{\psi}_1 I + 2 \tilde{\psi}_4 N \quad \text{with} \quad \tilde{\psi}_i = \frac{\partial \tilde{\psi}}{\partial \tilde{I}_i} \]
\[ \tilde{\psi}_1 = 2 c_0 c_1 [ I_1 - 3 ] \exp(c_1 [ I_1 - 3 ]^2 + c_2 [ I_4 - 1 ]^2) \]
\[ \tilde{\psi}_4 = 2 c_0 c_2 [ I_4 - 1 ] \exp(c_1 [ I_1 - 3 ]^2 + c_2 [ I_4 - 1 ]^2) \]

example 03: neo hooke model
\[ \tilde{\psi} = c_0 \left[ I_1 - 3 \right] \]
\[ \tilde{S} = 2 \frac{\partial \tilde{\psi}}{\partial \tilde{C}} = 2 \tilde{\psi}_1 I + 2 \tilde{\psi}_4 N \quad \text{with} \quad \tilde{\psi}_i = \frac{\partial \tilde{\psi}}{\partial \tilde{I}_i} \]
\[ \tilde{\psi}_1 = c_0 \quad \text{and} \quad \tilde{\psi}_4 = 0 \]

---

**14 - hyperelastic materials**

**isotropic incompressible**

**uniaxial stretching of anisotropic sheet**

---

```
function [] = UniAxialTest()
lambda1 = [1:0.001:2.0]; lambda2 = lambda1;

%% MATERIAL PARAMETERS
% Holzapfel
c0_Hlz = 18364377.50;
c1_Hlz = 2499419166.42;
c2_Hlz = 97.44;
% May-Newman
c0_May = 8958355943.52;
c1_May = 0.89577484;
c2_May = 1.79619884;
% Neo Hooke
c0_Neo = 63700000;
% Experimental
c0_exp = 52.0;
c1_exp = 4.63;
c2_exp = 22.6;
```
%% DERIVATIVES OF STRAIN ENERGY WRT INVARIANTS

function [Psi1] = Psi1_Hlz(c0, c1, c2, I1, I4)
    Psi1 = c0;
end

function [Psi4] = Psi4_Hlz(c0, c1, c2, I1, I4)
    Psi4 = c1.*(I4-1).*exp(c2.*(I4-1).^2);
end

function [Psi1] = Psi1_May(c0, c1, c2, I1, I4)
    Psi1 = c0.*exp(c1.*(I1-3).^2+c2.*(I4-1).^2)*2*c1.*(I1-3);
end

function [Psi4] = Psi4_May(c0, c1, c2, I1, I4)
    Psi4 = c0.*exp(c1.*(I1-3).^2+c2.*(I4-1).^2).*2.*c2.*(I4-1);
end

function [Psi1] = Psi1_Neo(c0, I1, I4)
    Psi1 = c0;
end

%% STRESS-STRETCH IN FIBER DIRECTION

I1 = lambda1.^2 + 2./lambda1;
I4 = lambda1.^2;

%Holzapfel
Hlz_Sigma_11 = 2 .* Psi1_Hlz(c0_Hlz, c1_Hlz, c2_Hlz, I1, I4) .* (lambda1.^2-1./lambda1) ... + 2 .* Psi4_Hlz(c0_Hlz, c1_Hlz, c2_Hlz, I1, I4) .* lambda1.^2;

%May-Newman
May_Sigma_11 = 2 .* Psi1_May(c0_May, c1_May, c2_May, I1, I4) .* (lambda1.^2-1./lambda1) ... + 2 .* Psi4_May(c0_May, c1_May, c2_May, I1, I4) .* lambda1.^2;

%Neo-Hookean
Neo_Sigma_11 = 2 .* Psi1_Neo(c0_Neo, I1, I4) .* (lambda1.^2-1./lambda1);

%Experimental
Exp_Sigma_11 = 2 .* Psi1_May(c0_exp, c1_exp, c2_exp, I1, I4) .* (lambda1.^2-1./lambda1) ... + 2 .* Psi4_May(c0_exp, c1_exp, c2_exp, I1, I4) .* lambda1.^2;

plot(lambda1, Hlz_Sigma_11/10^9,'g-','LineWidth',2.0)
plot(lambda1, May_Sigma_11/10^9,'b-','LineWidth',2.0)
plot(lambda1, Neo_Sigma_11/10^9,'r-','LineWidth',2.0)
plot(lambda1, Exp_Sigma_11/10^9,'k-','LineWidth',2.0)

%% STRESS-STRETCH IN CROSS-FIBER DIRECTION

I1 = lambda2.^2 + 2./lambda2;
I4 = 1./lambda2;

%Holzapfel
Hlz_Sigma_22 = 2 .* Psi1_Hlz(c0_Hlz, c1_Hlz, c2_Hlz, I1, I4) .* (lambda2.^2-1./lambda2);

%May-Newman
May_Sigma_22 = 2 .* Psi1_May(c0_May, c1_May, c2_May, I1, I4) .* (lambda2.^2-1./lambda2);

%Neo-Hookean
Neo_Sigma_22 = 2 .* Psi1_Neo(c0_Neo, I1, I4) .* (lambda2.^2-1./lambda2);

%Experimental
Exp_Sigma_22 = 2 .* Psi1_May(c0_exp, c1_exp, c2_exp, I1, I4) .* (lambda2.^2-1./lambda2);

plot(lambda1, Hlz_Sigma_22/10^9,'g-','LineWidth',2.0)
plot(lambda1, May_Sigma_22/10^9,'b-','LineWidth',2.0)
plot(lambda1, Neo_Sigma_22/10^9,'r-','LineWidth',2.0)
plot(lambda1, Exp_Sigma_22/10^9,'k-','LineWidth',2.0)
14 - hyperelastic materials

uniaxial stretching of anisotropic sheet

- Holzapfel
- May-Newman
- Neo-Hooke
- Experimental

in vivo strains in the mitral leaflet

- May Newman model
- Holzapfel model

maximum principal strain

0 4 8 [%]