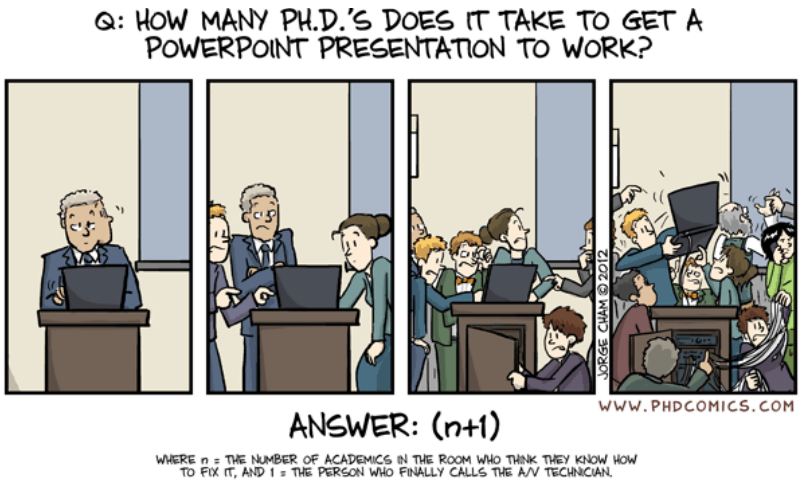
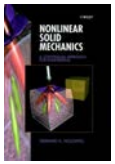
14 - hyperelastic materials



holzapfel 'nonlinear solid mechanics' [2000], chapter 6, pages 205-305

me338 - syllabus

day	date		topic	chapters	pages
tue	sep	25	why continuum mechanics?		
thu	sep	27	introduction to vectors and tensors	1.1-1.5	1-32
tue	oct	02	introduction to vectors and tensors	1.6-1.9	32-55
thu	oct	04	kinematics	2.1-2.4	55-76
tue	oct	09	kinematics	2.5-2.8	76-109
thu	oct	11	concept of stress	3.1-3.4	109-131
tue	oct	16	balance principles	4.1-4.4	131-161
thu	oct	18	balance principles	4.5-4.7	161-179
tue	oct	23	aspects of objectivity	5.1-5.4	179-205
thu	oct	25	hyperelastic materials	6.1-6.2	205-222
tue	oct	30	hyperelastic materials	6.3-6.5	222-252
thu	nov	01	hyperelastic materials	6.6-6.8	252-278
tue	nov	06	hyperelastic materials	6.9-6.11	278-305
thu	nov	08	thermodynamics of materials	7.1-7.6	305-337
tue	nov	13	midterm prep		
thu	nov	15	midterm		
tue	nov	27	thermodynamics of materials	7.7-7.9	337-371
thu	nov	29	variational principles	8.1-8.3	371-392
tue	dec	04	variational principles	8.4-8.6	392-414
thu	dec	06	final project discussion		



isotropic hyperelastic materials

$$\psi = \psi(\boldsymbol{b}) = \psi(I_1^b, I_2^b, I_3^b)$$

cauchy stress

$$\boldsymbol{\sigma} = \frac{2}{J}\frac{\partial \psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} = \frac{2}{J}\boldsymbol{b} \cdot \frac{\partial \psi(\boldsymbol{b})}{\partial \boldsymbol{b}}$$

invariants in terms of principal stretches

$$\begin{aligned} I_1^b &= \mathrm{tr}(\boldsymbol{b}) &&= \lambda_I + \lambda_{II} + \lambda_{III} \\ I_2^b &= \tfrac{1}{2}[\mathrm{tr}^2(\boldsymbol{b}) - \mathrm{tr}(\boldsymbol{b}^2)] &&= \lambda_I \cdot \lambda_{II} + \lambda_{II} \cdot \lambda_{III} + \lambda_{III} \cdot \lambda_I \\ I_3^b &= \det(\boldsymbol{b}) &&= \lambda_I \cdot \lambda_{II} \cdot \lambda_{III} \end{aligned}$$

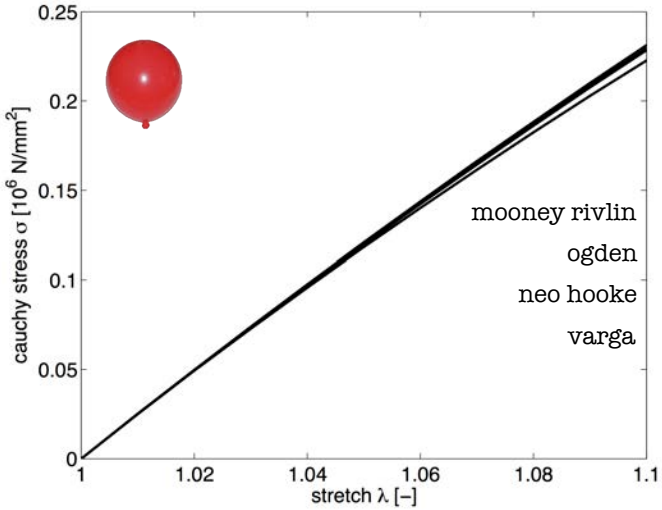
principal cauchy stresses

$$\sigma_I = -p + \lambda_I \frac{\partial \psi}{\partial \lambda_I} \qquad I = 1, 2, 3$$

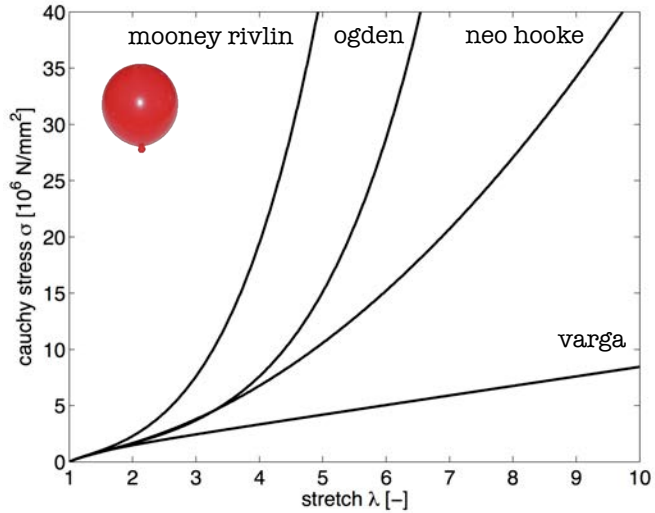
inflation of a spherical rubber balloon

```
% cauchy stress vs stretch plot

% loop over all stretches lambda from 1.0 to 10.0
for i=1:901
  lam(i) = 1.0+(i-1)/100;
  sig_og(i)  = m1_og * (lam(i)^a1_og-lam(i)^(-2*a1_og)) ...
             + m2_og * (lam(i)^a2_og-lam(i)^(-2*a2_og)) ...
             + m3_og * (lam(i)^a3_og-lam(i)^(-2*a3_og));
  sig_mr(i) = m1_mr * (lam(i)^a1_mr-lam(i)^(-2*a1_mr)) ...
             + m2_mr * (lam(i)^a2_mr-lam(i)^(-2*a2_mr));
  sig_nh(i) = m1_nh * (lam(i)^a1_nh-lam(i)^(-2*a1_nh));
  sig_vg(i) = m1_vg * (lam(i)^a1_vg-lam(i)^(-2*a1_vg));
endt

plot(lam,sig_og/10^6,'-k','LineWidth',2.0)
plot(lam,sig_mr/10^6,'-k','LineWidth',2.0)
plot(lam,sig_nh/10^6,'-k','LineWidth',2.0)
plot(lam,sig_vg/10^6,'-k','LineWidth',2.0)
```

inflation of a spherical rubber balloon



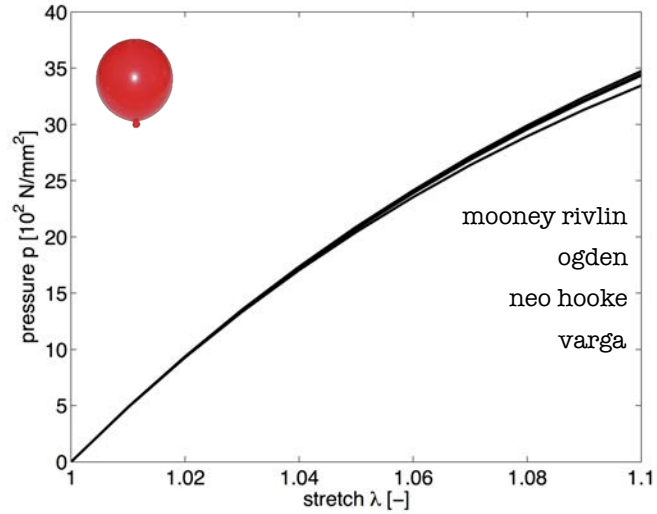
inflation of a spherical rubber balloon



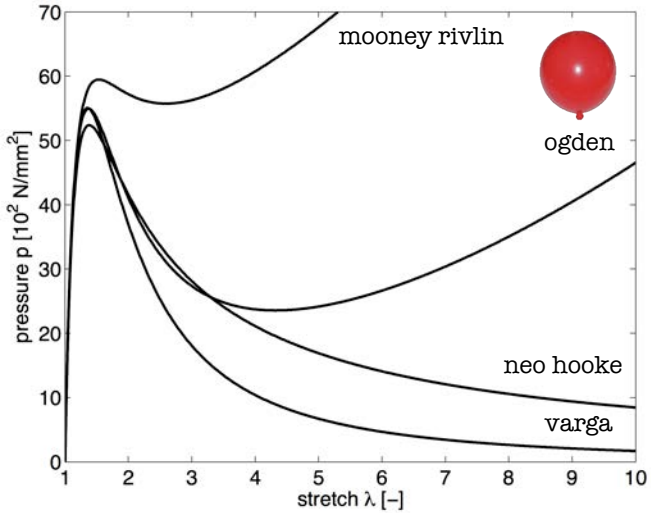
inflation of a spherical rubber balloon

```
% pressure vs stretch plot

% loop over all stretches lambda from 1.0 to 10.0
for i=1:901
  lam(i) = 1.0+(i-1)/100;
  p_og(i) = 2*H/R *(m1_og * (lam(i)^(a1_og-3)-lam(i)^(-2*a1_og-3)) ...
                    + m2_og * (lam(i)^(a2_og-3)-lam(i)^(-2*a2_og-3)) ...
                    + m3_og * (lam(i)^(a3_og-3)-lam(i)^(-2*a3_og-3)));
  p_mr(i) = 2*H/R *(m1_mr * (lam(i)^(a1_mr-3)-lam(i)^(-2*a1_mr-3)) ...
                     + m2_mr * (lam(i)^(a2_mr-3)-lam(i)^(-2*a2_mr-3)));
  p_nh(i) = 2*H/R *(m1_nh * (lam(i)^(a1_nh-3)-lam(i)^(-2*a1_nh-3)));
  p_vg(i) = 2*H/R *(m1_vg * (lam(i)^(a1_vg-3)-lam(i)^(-2*a1_vg-3)));
end

plot(lam,p_og/10^2,'-k','LineWidth',2.0)
plot(lam,p_mr/10^2,'-k','LineWidth',2.0)
plot(lam,p_nh/10^2,'-k','LineWidth',2.0)
plot(lam,p_vg/10^2,'-k','LineWidth',2.0)
```


inflation of a spherical rubber balloon



inflation of a spherical rubber balloon



frechet derivative

consider smooth, differentiable scalar field Φ with

- scalar argument $\Phi: \quad \mathcal{R} \rightarrow \mathcal{R}; \quad \Phi(x) = \alpha$
- vectorial argument $\Phi: \quad \mathcal{R}^3 \rightarrow \mathcal{R}; \quad \Phi(\boldsymbol{x}) = \alpha$
- tensorial argument $\Phi: \quad \mathcal{R}^3 \times \mathcal{R}^3 \rightarrow \mathcal{R}; \quad \Phi(\boldsymbol{X}) = \alpha$

- scalar $\quad \mathrm{D}\Phi(x) = \dfrac{\partial \Phi(x)}{\partial x} = \partial_x \Phi(x)$
- vectorial $\quad \mathrm{D}\Phi(\boldsymbol{x}) = \dfrac{\partial \Phi(\boldsymbol{x})}{\partial \boldsymbol{x}} = \partial_x \Phi(\boldsymbol{x})$
- tensorial $\quad \mathrm{D}\Phi(\boldsymbol{X}) = \dfrac{\partial \Phi(\boldsymbol{X})}{\partial \boldsymbol{X}} = \partial_X \Phi(\boldsymbol{X})$

special case: gateau derivative

consider smooth, differentiable scalar field Φ with

- scalar argument $\Phi: \quad \mathcal{R} \rightarrow \mathcal{R}; \quad \Phi(x) = \alpha$
- vectorial argument $\Phi: \quad \mathcal{R}^3 \rightarrow \mathcal{R}; \quad \Phi(\boldsymbol{x}) = \alpha$
- tensorial argument $\Phi: \quad \mathcal{R}^3 \times \mathcal{R}^3 \rightarrow \mathcal{R}; \quad \Phi(\boldsymbol{X}) = \alpha$

- scalar $\quad \mathrm{D}\Phi(x)\ u = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(x + \epsilon u)\,|_{\epsilon=0} \quad \forall\, u \in \mathcal{R}$
- vectorial $\quad \mathrm{D}\Phi(\boldsymbol{x}) \cdot \boldsymbol{u} = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(\boldsymbol{x} + \epsilon \boldsymbol{u})\,|_{\epsilon=0} \quad \forall\, \boldsymbol{u} \in \mathcal{R}^3$
- tensorial $\quad \mathrm{D}\Phi(\boldsymbol{X}) : \boldsymbol{U} = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(\boldsymbol{X} + \epsilon \boldsymbol{U})|_{\epsilon=0} \quad \forall\, \boldsymbol{U} \in \mathcal{R}^3 \otimes \mathcal{R}^3$

example: derivative of invariants

use the Gateaux derivative

$$\mathrm{D}\Phi(\boldsymbol{A}) : \Delta\boldsymbol{A} = \frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}} : \Delta\boldsymbol{A} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0}$$

to determine the derivatives of the three invariants I_A, II_A, III_A of the second order tensor $\boldsymbol{A}$ with respect to $\boldsymbol{A}$ itself!

in words: the Gateaux derivative D of a scalar field Φ (in this case the invariant I_A, II_A, III_A) along a given direction $\boldsymbol{U}$ (in this case $\Delta\boldsymbol{A}$) is the derivative of the field Φ at position $\boldsymbol{X}$ (in this case $\boldsymbol{A}$) "perturbed" by the direction $\epsilon\boldsymbol{U}$ (in this case $\epsilon\,\Delta\boldsymbol{A}$) with respect to ϵ evaluated at $\epsilon = 0$

setting $\epsilon = 0$ filters out the linear terms, i.e., all higher order terms are set to zero, the Gateaux derivative is therefore also referred to as linearization, it corresponds to the first term of a Taylor series

example: derivative of 1st invariant

derivative of first invariant $I_A = \mathrm{tr}\,(\boldsymbol{A}) = \boldsymbol{A} : \boldsymbol{I}$

$$\begin{aligned} \mathrm{D}I_A(\boldsymbol{A}) : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\mathrm{tr}(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}|_{\epsilon=0} \\ &= \quad \Delta\boldsymbol{A} : \boldsymbol{I}|_{\epsilon=0} = \boldsymbol{I} : \Delta\boldsymbol{A} \end{aligned}$$

$$\mathrm{D}I_A(\boldsymbol{A}) = \frac{\partial I_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \underline{\underline{\boldsymbol{I}}}$$

remark: I_A is linear in $\boldsymbol{A}$, its Gateaux derivative is constant, there are no terms in ϵ once we take the derivative $\mathrm{d}/\mathrm{d}\epsilon$

example: derivative of 2nd invariant

derivative of second invariant $II_A = \frac{1}{2}\left[[\boldsymbol{A} : \boldsymbol{I}]^2 + \boldsymbol{A} : \boldsymbol{A}^{\mathrm{t}}\right]$

$$\begin{aligned} \mathrm{D}II_A(\boldsymbol{A}) : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\tfrac{1}{2}\mathrm{tr}^2(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}) - \tfrac{1}{2}\mathrm{tr}(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})^2|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\tfrac{1}{2}[[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}]^2 \\ &- \quad \tfrac{1}{2}[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : [\boldsymbol{A}^{\mathrm{t}} + \epsilon\Delta\boldsymbol{A}^{\mathrm{t}}]|_{\epsilon=0} \\ &= \quad [\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}]\Delta\boldsymbol{A} : \boldsymbol{I} \\ &- \quad \tfrac{1}{2}\Delta\boldsymbol{A} : \boldsymbol{A}^{\mathrm{t}} - \tfrac{1}{2}\boldsymbol{A} : \Delta\boldsymbol{A}^{\mathrm{t}} - \epsilon\Delta\boldsymbol{A} : \Delta\boldsymbol{A}^{\mathrm{t}}|_{\epsilon=0} \\ &= \quad [\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} - \boldsymbol{A}^{\mathrm{t}}] : \Delta\boldsymbol{A} \end{aligned}$$

$$\mathrm{D}II_A(\boldsymbol{A}) = \frac{\partial II_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \underline{\underline{\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} - \boldsymbol{A}^{\mathrm{t}}}}$$

here, we have used the following identity

$$\mathrm{tr}(\boldsymbol{A}^2) = (\boldsymbol{A} \cdot \boldsymbol{A}) : \boldsymbol{I} = \boldsymbol{A} : \boldsymbol{A}^{\mathrm{t}}$$

remark: II_A is quadratic in $\boldsymbol{A}$, its Gateaux derivative is linear, once we took the derivative $\mathrm{d}/\mathrm{d}\epsilon$, the higher order term in ϵ is filtered out by setting $\epsilon = 0$

example: derivative of 3rd invariant

derivative of third invariant $III_A = \det(\boldsymbol{A})$

$$\begin{aligned} \mathrm{D}III_A(\boldsymbol{A}) : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A} \cdot [\boldsymbol{I} + \boldsymbol{A}^{-1} \cdot \epsilon\Delta\boldsymbol{A}])|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A}) \cdot \det(\epsilon\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A} + \boldsymbol{I})|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A}) \cdot [(\epsilon\lambda_{A^{-1}\cdot\Delta A1} + 1) \\ &\qquad (\epsilon\lambda_{A^{-1}\cdot\Delta A2} + 1)(\epsilon\lambda_{A^{-1}\cdot\Delta A3} + 1)]\,|_{\epsilon=0} \\ &= \det(\boldsymbol{A}) \cdot [\lambda_{A^{-1}\cdot\Delta A1} + \lambda_{A^{-1}\cdot\Delta A2} + \lambda_{A^{-1}\cdot\Delta A3}] \\ &= \det(\boldsymbol{A}) \cdot \mathrm{tr}(\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A}) \\ &= \det(\boldsymbol{A}) \cdot \boldsymbol{A}^{-\mathrm{t}} : \Delta\boldsymbol{A} \end{aligned}$$

$$\mathrm{D}III_A(\boldsymbol{A}) = \frac{\partial III_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \underline{\underline{\det(\boldsymbol{A}) \cdot \boldsymbol{A}^{-\mathrm{t}}}}$$

example: derivative of 3rd invariant

here, we have used the following expression for the determinant of $(\epsilon\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A})$ expressed through the characteristic polynom for the eigenvalue $\lambda = -1$

$$\begin{aligned} \det(\epsilon\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A} - \lambda\boldsymbol{I}) \;=\; & (\epsilon\lambda_{A^{-1}\cdot\Delta A1} - \lambda) \\ & (\epsilon\lambda_{A^{-1}\cdot\Delta A2} - \lambda)(\epsilon\lambda_{A^{-1}\cdot\Delta A3} - \lambda) \end{aligned}$$

reformulation with the help of index notation

$$\begin{aligned} \mathrm{tr}(\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A}) &= (\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A}) : \boldsymbol{I} \\ &= (A^{-1}_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (\Delta A_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l) : (\delta_{mn}\boldsymbol{e}_m \otimes \boldsymbol{e}_n) \\ &= (A^{-1}_{ij}\Delta A_{jl}\boldsymbol{e}_i \otimes \boldsymbol{e}_l) : (\delta_{mn}\boldsymbol{e}_m \otimes \boldsymbol{e}_n) \\ &= A^{-1}_{ij}\Delta A_{jl}\delta_{im}\delta_{ln}\delta_{mn} = \underline{\underline{\boldsymbol{A}^{-\mathrm{t}} : \Delta\boldsymbol{A}}} \end{aligned}$$

remark: III_A is cubic in $\boldsymbol{A}$

derivative of function wrt tensor

derivative of function with tensorial argument show that the derivative of the function $\Phi(\boldsymbol{A})$ with respect to its tensorial arguement $\boldsymbol{A}$ is given through the component-wise derivative with respect to the individual tensorial entries A_{ij}

index representation of second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

extraction of individual components A_{ij} through "projection" onto the base vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$

$$A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A} \cdot \boldsymbol{e}_j = \boldsymbol{e}_i \cdot (A_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l) \cdot \boldsymbol{e}_j = \delta_{ik}\, A_{kl}\, \delta_{lj} = A_{ij}$$

derivative of function wrt tensor

and thus

$$\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}} = \frac{\partial \Phi}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial \boldsymbol{A}} = \frac{\partial \Phi}{\partial A_{ij}} \frac{\partial \boldsymbol{e}_i \cdot \boldsymbol{A} \cdot \boldsymbol{e}_j}{\partial \boldsymbol{A}} = \frac{\partial \Phi}{\partial A_{ij}}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$\begin{aligned}\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}} = & \frac{\partial \Phi}{\partial A_{11}} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \frac{\partial \Phi}{\partial A_{12}} \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \frac{\partial \Phi}{\partial A_{13}} \boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\ & + \frac{\partial \Phi}{\partial A_{21}} \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \frac{\partial \Phi}{\partial A_{22}} \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \frac{\partial \Phi}{\partial A_{23}} \boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ & + \frac{\partial \Phi}{\partial A_{31}} \boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + \frac{\partial \Phi}{\partial A_{32}} \boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + \frac{\partial \Phi}{\partial A_{33}} \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\end{aligned}$$

remark: derivatives of functions with respect to tensors can be derived individually for each component

example: derivative of 1st invariant

derivative of first invariant I_A wrt A use the component-wise derivation to determine the derivative of the first invariant I_A with respect to its tensor and validate the result derived previously

$$I_A = \mathrm{tr}\boldsymbol{A} = A_{11} + A_{22} + A_{33}$$

$$\frac{\partial I_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \left[\frac{\partial I_A}{\partial A_{ij}}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$\left[\frac{\partial I_A}{\partial A_{ij}}\right] = \begin{bmatrix} \frac{\partial I_A}{\partial A_{11}} & \frac{\partial I_A}{\partial A_{12}} & \frac{\partial I_A}{\partial A_{13}} \\ \frac{\partial I_A}{\partial A_{21}} & \frac{\partial I_A}{\partial A_{22}} & \frac{\partial I_A}{\partial A_{23}} \\ \frac{\partial I_A}{\partial A_{31}} & \frac{\partial I_A}{\partial A_{32}} & \frac{\partial I_A}{\partial A_{33}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{\frac{\partial I_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \boldsymbol{I}}}$$

remark: derivatives of functions with respect to tensors can be derived individually for each component

example: derivative of A wrt itself

derivative of A wrt itself

$$\boldsymbol{A} = A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} = \left[\frac{\partial A_{ij}}{\partial A_{kl}}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \left[\frac{\partial A_{ij}}{\partial A_{kl}}\right] = \delta_{ik}\delta_{jl}$$

$$\underline{\underline{\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} = \boldsymbol{I} \overline{\otimes} \boldsymbol{I} = \mathbb{I}}}$$

remark: always keep in mind that the derivative of $\boldsymbol{A}$ with respect to itself is not(!) 1 but the fourth order unity tensor $\mathbb{I}$

example: derivative of A wrt inverse

derivative of inverse A^{-1} wrt A determine the derivative of the inverse of A wrt A by using the definition of the inverse $A^{-1} \cdot A = I$

$$\frac{\partial A^{-1}}{\partial A} = \left[\frac{\partial A^{-1}_{ij}}{\partial A_{kl}}\right] e_i \otimes e_j \otimes e_k \otimes e_l$$

derivative of identity with respect to A

$$\frac{\partial \delta_{im}}{\partial A_{kl}} = \frac{\partial A^{-1}_{ij} A_{jm}}{\partial A_{kl}} = \frac{\partial A^{-1}_{ij}}{\partial A_{kl}} A_{jm} + A^{-1}_{ij} \frac{\partial A_{jm}}{\partial A_{kl}} = \mathbf{0}$$

and thus

$$\left[\frac{\partial A^{-1}_{ij}}{\partial A_{kl}}\right] = -A^{-1}_{ij} \frac{\partial A_{jm}}{\partial A_{kl}} A^{-1}_{mj} = -A^{-1}_{ij} \delta_{jk} \delta_{ml} A^{-t}_{jm} = -A^{-1}_{ik} A^{-t}_{jl}$$

$$\frac{\partial A^{-1}}{\partial A} = -A^{-1} \overline{\otimes} A^{-t}$$

remark: the derivative of the inverse of a tensor wrt the tensor itself is negative. it can be determined with a trick by using the definition of the inverse.

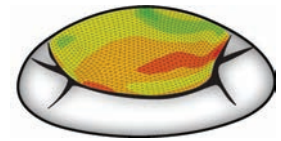
transversely isotropic incompressible

incompressible material

$$\boldsymbol{F} = \boldsymbol{F}^{\mathrm{vol}} \cdot \bar{\boldsymbol{F}}$$

volumetric part

$$\boldsymbol{F}^{\mathrm{vol}} = J^{1/3} \boldsymbol{I}$$



isochoric part

$$\bar{\boldsymbol{F}} = J^{-1/3} \boldsymbol{F}$$

transversely isotropic material

fiber orientation

$$||\boldsymbol{n}_0|| = 1$$

structural tensor

$$\boldsymbol{N} = \boldsymbol{n}_0 \otimes \boldsymbol{n}_0$$

transversely isotropic incompressible

incompressible transversely isotropic material

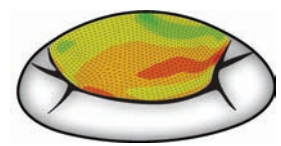
$$\psi = \psi^{\mathrm{vol}}(J) + \bar{\psi}(\bar{I}_1, \bar{I}_4)$$

$$\begin{aligned} J &= \det(\boldsymbol{F}) \\ \bar{I}_1 &= \bar{\boldsymbol{C}} : \boldsymbol{I} \\ \bar{I}_4 &= \bar{\boldsymbol{C}} : \boldsymbol{N} \end{aligned} \quad \text{with} \quad \begin{aligned} \partial_{\boldsymbol{C}} J &= \tfrac{1}{2} J \boldsymbol{C}^{-1} \\ \partial_{\bar{\boldsymbol{C}}} \bar{I}_1 &= \boldsymbol{I} \\ \partial_{\bar{\boldsymbol{C}}} I_4 &= \boldsymbol{N} \,. \end{aligned}$$

transversely isotropic incompressible

second piola kirchhoff stress

$$\boldsymbol{S} = 2 \frac{\partial \psi}{\partial \boldsymbol{C}} = \boldsymbol{S}^{\mathrm{vol}} + \boldsymbol{S}^{\mathrm{iso}}$$



volumetric part

$$\boldsymbol{S}^{\mathrm{vol}} = 2 \frac{\partial \psi^{\mathrm{vol}}}{\partial \boldsymbol{C}} = J\, p\, \boldsymbol{C}^{-1}$$

isochoric part

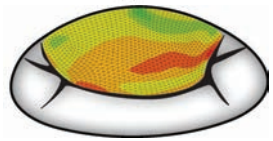
$$\boldsymbol{S}^{\mathrm{iso}} = 2 \frac{\partial \bar{\psi}}{\partial \boldsymbol{C}} = J^{-2/3}\, \mathbb{P} : \bar{\boldsymbol{S}}$$

$$\bar{\boldsymbol{S}} = 2 \frac{\partial \bar{\psi}}{\partial \bar{\boldsymbol{C}}} = 2\, \bar{\psi}_1\, \boldsymbol{I} + 2\, \bar{\psi}_4\, \boldsymbol{N}$$

transversely isotropic incompressible

example 01: holzapfel model

$$\bar{\psi} = c_0[\,\bar{I}_1 - 3\,] + \frac{c_1}{2c_2}[\,\exp(c_2\,[\bar{I}_4 - 1]^2) - 1\,]$$



$$\bar{\boldsymbol{S}} = 2\,\frac{\partial\bar{\psi}}{\partial\bar{\boldsymbol{C}}} = 2\,\bar{\psi}_1\,\boldsymbol{I} + 2\,\bar{\psi}_4\,\boldsymbol{N} \quad \text{with} \quad \bar{\psi}_i = \partial\bar{\psi}/\partial\bar{I}_i$$

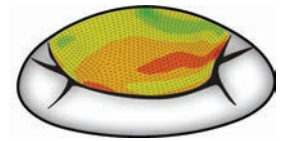
$$\bar{\psi}_1 = c_0$$

$$\bar{\psi}_4 = c_1[\,\bar{I}_4 - 1\,]\exp(c_2\,[\bar{I}_4 - 1\,]^2)$$

transversely isotropic incompressible

example 02: may newman model

$$\bar{\psi} = c_0\,[\,\exp(c_1[\,\bar{I}_1 - 3\,]^2 + c_2[\,\bar{I}_4 - 1\,]^2) - 1\,]$$



$$\bar{\boldsymbol{S}} = 2\,\frac{\partial\bar{\psi}}{\partial\bar{\boldsymbol{C}}} = 2\,\bar{\psi}_1\,\boldsymbol{I} + 2\,\bar{\psi}_4\,\boldsymbol{N} \quad \text{with} \quad \bar{\psi}_i = \partial\bar{\psi}/\partial\bar{I}_i$$

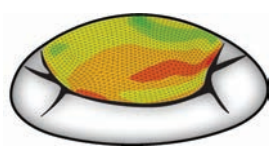
$$\bar{\psi}_1 = 2\,c_0 c_1[I_1 - 3]\,\exp(c_1[I_1 - 3]^2 + c_2[\bar{I}_4 - 1]^2)$$

$$\bar{\psi}_4 = 2\,c_0 c_2[\bar{I}_4 - 1]\,\exp(c_1[\bar{I}_1 - 3]^2 + c_2[\bar{I}_4 - 1]^2)$$

isotropic incompressible

example 03: neo hooke model

$$\bar{\psi} = c_0\,[\,\bar{I}_1 - 3\,]$$



$$\bar{\boldsymbol{S}} = 2\,\frac{\partial\bar{\psi}}{\partial\bar{\boldsymbol{C}}} = 2\,\bar{\psi}_1\,\boldsymbol{I} + 2\,\bar{\psi}_4\,\boldsymbol{N} \quad \text{with} \quad \bar{\psi}_i = \partial\bar{\psi}/\partial\bar{I}_i$$

$$\bar{\psi}_1 = c_0 \quad \text{and} \quad \bar{\psi}_4 = 0$$

uniaxial stretching of anisotropic sheet

```
function [] = UniAxialTest()
lambda1 = [1:0.001:2.0];   lambda2 = lambda1;

%% MATERIAL PARAMETERS
% Holzapfel
c0_Hlz = 18364377.50;
c1_Hlz = 2499419166.42;
c2_Hlz = 97.44;
% May-Newman
c0_May = 8958355943.52;
c1_May = 0.89577484;
c2_May = 1.79619884;
% Neo Hooke
c0_Neo = 63700000;
% Experimental
c0_exp = 52.0;
c1_exp = 4.63;
c2_exp = 22.6;
```

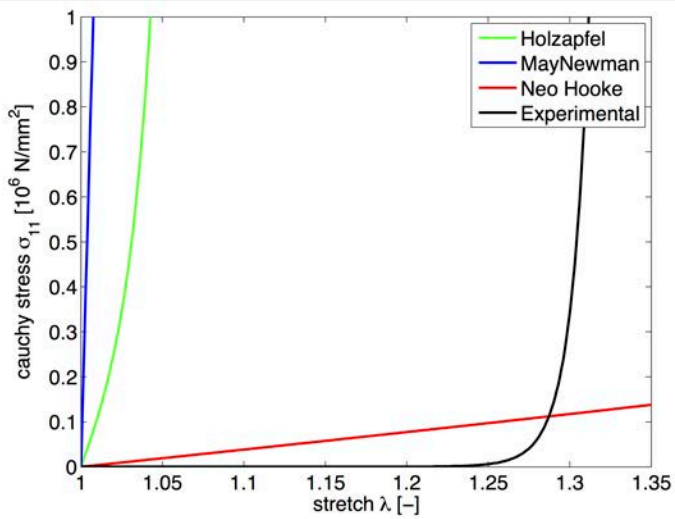
uniaxial stretching of anisotropic sheet

```
%% DERIVATIVES OF STRAIN ENERGY WRT INVARIANTS

function [Psi1] = Psi1_Hlz(c0,c1,c2,I1,I4)
  Psi1 = c0;
end

function [Psi4] = Psi4_Hlz(c0,c1,c2,I1,I4)
  Psi4 = c1.*(I4-1).* exp(c2.*(I4-1).^2);
endS

function [Psi1] = Psi1_May(c0,c1,c2,I1,I4)
  Psi1 = c0.*exp(c1.*(I1-3).^2+c2.*(I4-1).^2)*2*c1.*(I1-3);
end

function [Psi4] = Psi4_May(c0,c1,c2,I1,I4)
  Psi4 = c0.*exp(c1.*(I1-3).^2+c2.*(I4-1).^2).*2.*c2.*(I4-1);
end

function [Psi1] = Psi1_Neo(c0,I1,I4)
  Psi1 = c0;
end
```


uniaxial stretching of anisotropic sheet

```
%% STRESS-STRETCH IN FIBER DIRECTION
I1 = lambda1.^2 + 2./lambda1;
I4 = lambda1.^2;

%Holzapfel
Hlz_Sigma_11 = 2 .* Psi1_Hlz(c0_Hlz,c1_Hlz,c2_Hlz,I1,I4) .* (lambda1.^2-1./lambda1) ...
               + 2 .* Psi4_Hlz(c0_Hlz,c1_Hlz,c2_Hlz,I1,I4) .*  lambda1.^2;
%May-Newman
May_Sigma_11 = 2 .*Psi1_May(c0_May,c1_May,c2_May,I1,I4) .* (lambda1.^2-1./lambda1)
               + 2 .* Psi4_May(c0_May,c1_May,c2_May,I1,I4) .*  lambda1.^2;
%Neo-Hookean
Neo_Sigma_11 = 2 .* Psi1_Neo(c0_Neo,I1,I4) * (lambda1.^2-1./lambda1);
%Experimental
Exp_Sigma_11 = 2 .* Psi1_May(c0_exp,c1_exp,c2_exp,I1,I4) .* (lambda1.^2-1./lambda1)
               + 2 .* Psi4_May(c0_exp,c1_exp,c2_exp,I1,I4) .*  lambda1.^2;

plot(lambda1, Hlz_Sigma_11/10^9,'g-','LineWidth',2.0)
plot(lambda1, May_Sigma_11/10^9,'b-','LineWidth',2.0)
plot(lambda1, Neo_Sigma_11/10^9,'r-','LineWidth',2.0)
plot(lambda1, Exp_Sigma_11/10^9,'k-','LineWidth',2.0)
```

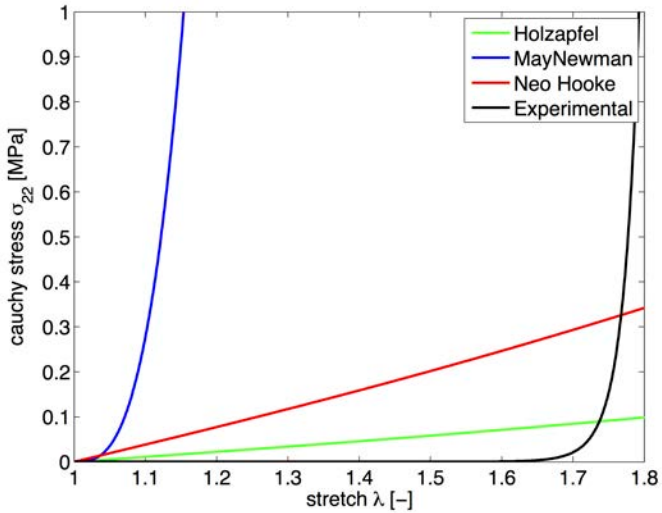

uniaxial stretching of anisotropic sheet



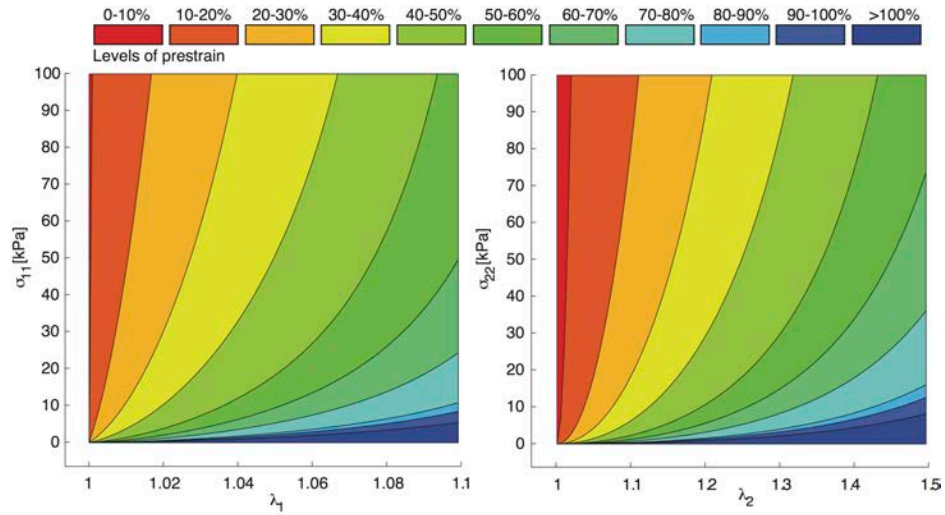
uniaxial stretching of anisotropic sheet

```
%% STRESS-STRETCH IN CROSS-FIBER DIRECTION
I1 = lambda2.^2 + 2./lambda2;
I4 = 1./lambda2;

%Holzapfel
Hlz_Sigma_22 = 2 .* Psi1_Hlz(c0_Hlz,c1_Hlz,c2_Hlz,I1,I4) .* (lambda2.^2-1./lambda2);
%May-Newman
May_Sigma_22=2 .*Psi1_May(c0_May,c1_May,c2_May,I1,I4) .* (lambda2.^2-1./lambda2);
%Neo-Hookean
Neo_Sigma_22 = 2 .* Psi1_Neo(c0_Neo,I1,I4) .* (lambda2.^2-1./lambda2);
%Experimental
Exp_Sigma_22 = 2 .* Psi1_May(c0_exp,c1_exp,c2_exp,I1,I4) .* (lambda2.^2-1./lambda2);

plot(lambda1, Hlz_Sigma_22/10^9,'g-','LineWidth',2.0)
plot(lambda1, May_Sigma_22/10^9,'b-','LineWidth',2.0)
plot(lambda1, Neo_Sigma_22/10^9,'r-','LineWidth',2.0)
plot(lambda1, Exp_Sigma_22/10^9,'k-','LineWidth',2.0)
```

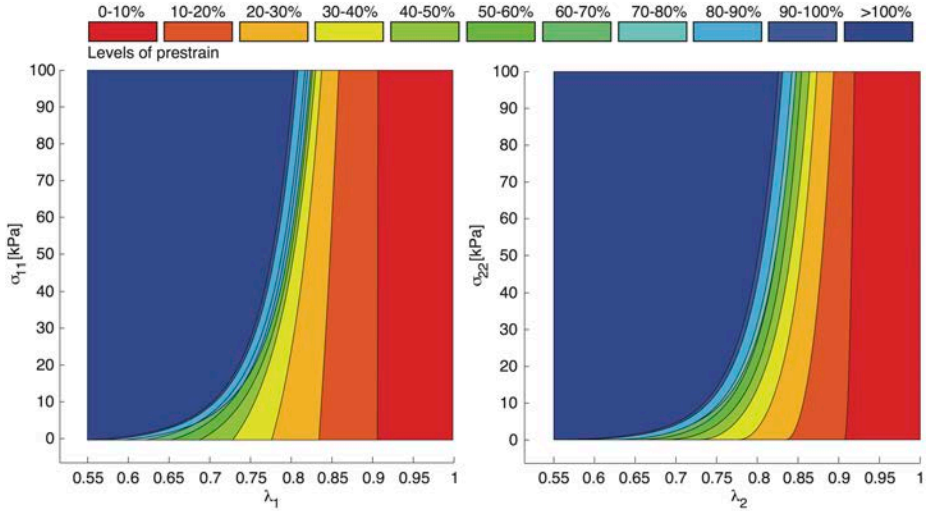
uniaxial stretching of anisotropic sheet



uniaxial stretching of anisotropic sheet



uniaxial stretching of anisotropic sheet



in vivo strains in the mitral leaflet