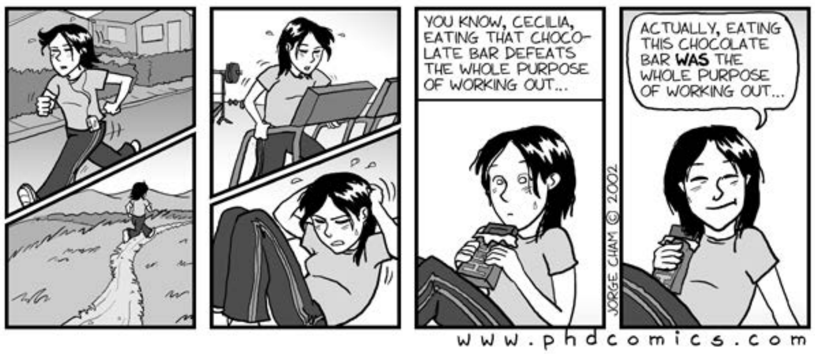
# 10 - hyperelastic materials

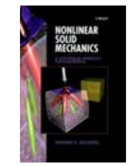



holzapfel 'nonlinear solid mechanics' [2000], chapter 6, pages 205-222



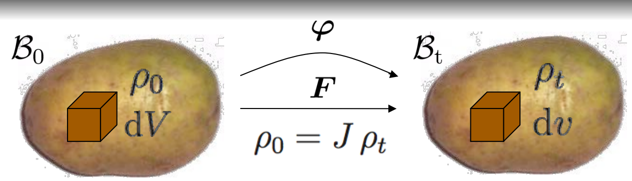
# me338 - syllabus

| day | date | | topic | chapters | pages |
|---|---|---|---|---|---|
| tue | sep | 25 | why continuum mechanics? | | |
| thu | sep | 27 | introduction to vectors and tensors | 1.1-1.5 | 1-32 |
| tue | oct | 02 | introduction to vectors and tensors | 1.6-1.9 | 32-55 |
| thu | oct | 04 | kinematics | 2.1-2.4 | 55-76 |
| tue | oct | 09 | kinematics | 2.5-2.8 | 76-109 |
| thu | oct | 11 | concept of stress | 3.1-3.4 | 109-131 |
| tue | oct | 16 | balance principles | 4.1-4.4 | 131-161 |
| thu | oct | 18 | balance principles | 4.5-4.7 | 161-179 |
| tue | oct | 23 | aspects of objectivity | 5.1-5.4 | 179-205 |
| thu | oct | 25 | hyperelastic materials | 6.1-6.2 | 205-222 |
| tue | oct | 30 | hyperelastic materials | 6.3-6.5 | 222-252 |
| thu | nov | 01 | hyperelastic materials | 6.6-6.8 | 252-278 |
| tue | nov | 06 | hyperelastic materials | 6.9-6.11 | 278-305 |
| thu | nov | 08 | thermodynamics of materials | 7.1-7.6 | 305-337 |
| tue | nov | 13 | midterm prep | | |
| thu | nov | 15 | midterm | | |
| tue | nov | 27 | thermodynamics of materials | 7.7-7.9 | 337-371 |
| thu | nov | 29 | variational principles | 8.1-8.3 | 371-392 |
| tue | dec | 04 | variational principles | 8.4-8.6 | 392-414 |
| thu | dec | 06 | final project discussion | | |






# entropy inequality




## local entropy inequality

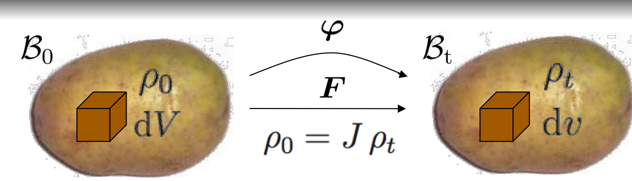
$$\mathcal{D}_0^{\mathrm{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi}_0 - \eta_0\,\dot{\theta} - \frac{1}{\theta}\boldsymbol{Q}\cdot\nabla_X\theta \geq 0$$

$$\mathcal{D}_t^{\mathrm{int}} = \boldsymbol{\sigma} : \boldsymbol{d} - \dot{\psi}_t - \eta_t\,\dot{\theta} - \frac{1}{\theta}\boldsymbol{q}\cdot\nabla_x\theta \geq 0$$

$$\mathcal{D}_0^{\mathrm{int}} = J\,\mathcal{D}_t^{\mathrm{int}} \qquad \psi_0 = J\,\psi_t \qquad \eta_0 = J\,\eta_t \qquad \boldsymbol{Q} = J\,\boldsymbol{q}\cdot\boldsymbol{F}^{-\mathrm{t}}$$



# entropy inequality




## consequence I: heat flux

$$-\frac{1}{\theta}\boldsymbol{Q}\cdot\nabla_X\theta \geq 0 \qquad \boldsymbol{Q} = -\boldsymbol{\kappa}\cdot\nabla_X\theta \qquad \nabla_X\theta\cdot\boldsymbol{\kappa}\cdot\nabla_X\theta \geq 0$$

$$-\frac{1}{\theta}\boldsymbol{q}\cdot\nabla_x\theta \geq 0 \qquad \boldsymbol{q} = -\boldsymbol{\kappa}\cdot\nabla_x\theta \qquad \nabla_x\theta\cdot\boldsymbol{\kappa}\cdot\nabla_x\theta \geq 0$$

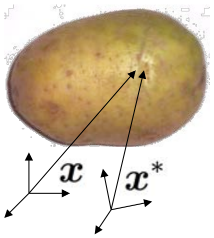
## consequence II: stress

$$\mathcal{D}_0^{\mathrm{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi}_0 - \eta_0\,\dot{\theta} \geq 0$$

$$\mathcal{D}_t^{\mathrm{int}} = \boldsymbol{\sigma} : \boldsymbol{d} - \dot{\psi}_t - \eta_t\,\dot{\theta} \geq 0$$

09 - balance principles

## objectivity




**change of observer**

$$x^* = Q(t) \cdot x + c(t)$$

**material objectivity / frame indifference**

$$A^*(x^*, t^*) = Q(t) \cdot A(x, t) \cdot Q^t(t)$$

$$u^*(x^*, t^*) = Q(t) \cdot u(x, t)$$

$$\alpha^*(x^*, t^*) = \alpha(x, t)$$

... are objective 2nd, 1st, 0th order tensors

$C, E, b, e$ ... are objective strain measures

$S, P, \sigma$ ... are objective stress measures



## objectivity

**principle of material objectivity**
**principle of frame-indifference**

the constitutive laws governing the internal conditions of a physical system and the interactions between its parts should not depend on whatever external frame of reference is used to describe them.

"i was responsible for introducing the obsolete term in 1958 and now regret that I misled a lot of people." walter noll



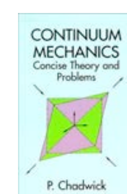
## constitutive equations

**constitutive equations** [kənˈstɪ.tu.tɪv ɪˈkweɪ.ʒəns] in structural analysis, constitutive relations **connect applied stresses** or forces **to strains** or deformations. the constitutive relations for linear materials are linear. more generally, in physics, a constitutive equation is a relation between two physical quantities (often tensors) that is specific to a material, and does not follow directly from physical law. some constitutive equations are **simply phenomenological;** others are **derived from first principles**.



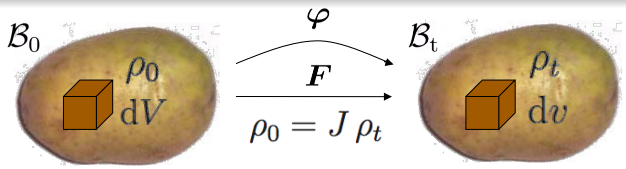
## constitutive equations

**constitutive equations** [kənˈstɪ.tu.tɪv ɪˈkweɪ.ʒəns] or equations of state bring in the **characterization of particular materials** within continuum mechanics. mathematically, the purpose of these relations is to supply connections between kinematic, mechanical and thermal fields. physically, constitutive equations represent the various forms of **idealized material response** which serve as **models** of the behavior of actual substances.




chadwick 'continuum mechanics' [1976]

## entropy inequality




### consequence I: heat flux

$$-\frac{1}{\theta}\boldsymbol{Q}\cdot\nabla_X\theta \geq 0 \qquad \boldsymbol{Q} = -\boldsymbol{\kappa}\cdot\nabla_X\theta \qquad \nabla_X\theta\cdot\boldsymbol{\kappa}\cdot\nabla_X\theta \geq 0$$

$$-\frac{1}{\theta}\boldsymbol{q}\cdot\nabla_x\theta \geq 0 \qquad \boldsymbol{q} = -\boldsymbol{\kappa}\cdot\nabla_x\theta \qquad \nabla_x\theta\cdot\boldsymbol{\kappa}\cdot\nabla_x\theta \geq 0$$

### consequence II: stress

$$\mathcal{D}_0^{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi}_0 - \eta_0\,\dot{\theta} \geq 0$$
$$\mathcal{D}_t^{\text{int}} = \boldsymbol{\sigma} : \boldsymbol{d} - \dot{\psi}_t - \eta_t\,\dot{\theta} \geq 0$$



## coleman noll evaluation

entropy $\quad \mathcal{D}^{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi} - \cancel{\eta\dot{\theta}} \geq 0$

assume $\psi = \psi(\boldsymbol{F}) \quad \dot{\psi} = \dfrac{\partial\psi(\boldsymbol{F})}{\partial t} = \dfrac{\partial\psi(\boldsymbol{F})}{\partial\boldsymbol{F}} : \dfrac{\partial\boldsymbol{F}}{\partial t} = \dfrac{\partial\psi(\boldsymbol{F})}{\partial\boldsymbol{F}} : \dot{\boldsymbol{F}}$

$$\mathcal{D}^{\text{int}} = \left[\boldsymbol{P} - \frac{\partial\psi(\boldsymbol{F})}{\partial\boldsymbol{F}}\right] : \dot{\boldsymbol{F}} \geq 0$$

with $\quad \mathcal{D}^{\text{int}} = 0 \quad$ and $\quad \forall\,\dot{\boldsymbol{F}}$

**first PK** $\quad \boldsymbol{P} = \dfrac{\partial\psi}{\partial\boldsymbol{F}} \qquad P_{aA} = \dfrac{\partial\psi}{\partial F_{aA}}$

**cauchy** $\quad \boldsymbol{\sigma} = \dfrac{1}{J}\dfrac{\partial\psi}{\partial\boldsymbol{F}}\cdot\boldsymbol{F}^{\text{t}} \qquad \sigma_{ab} = \dfrac{1}{J}\dfrac{\partial\psi}{\partial F_{aA}}F_{bA}$



## transport mechanisms

### covariant / strains

$\boldsymbol{E} = \boldsymbol{F}^{\text{t}}\cdot\boldsymbol{e}\cdot\boldsymbol{F} \qquad \longleftarrow$ pull back $\qquad \boldsymbol{e} = \boldsymbol{F}^{-\text{t}}\cdot\boldsymbol{E}\cdot\boldsymbol{F}^{-1}$

$E_{IJ} = F^{\text{t}}_{Ik}\,e_{kl}\,F_{lJ} \qquad$ push forward $\longrightarrow \qquad e_{ij} = F^{-\text{t}}_{iK}\,E_{KL}\,F^{-1}_{Lj}$

### contravariant / stresses

$\boldsymbol{S} = J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-\text{t}} \qquad \longleftarrow$ pull back $\qquad \boldsymbol{\sigma} = \dfrac{1}{J}\boldsymbol{F}\cdot\boldsymbol{S}\cdot\boldsymbol{F}^{\text{t}}$

$S_{IJ} = J\,F^{-1}_{Ik}\,\sigma_{kl}\,F^{-\text{t}}_{lJ} \qquad$ push forward $\longrightarrow \qquad \sigma_{ij} = \dfrac{1}{J}F_{iK}\,S_{KL}\,F^{\text{t}}_{Lj}$



## stress tensors

gustav robert kirchhoff [1824-1887]

**first piola kirchhoff**

$$\boldsymbol{P} = \boldsymbol{F}\cdot\boldsymbol{S}$$
$$\boldsymbol{P} = J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-\text{t}}$$

augustin louis caucy [1789-1857]

**second piola kirchhoff**

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\cdot\boldsymbol{P}$$
$$\boldsymbol{S} = J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-\text{t}}$$

**cauchy**

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{P}\cdot\boldsymbol{F}^{\text{t}}$$
$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\cdot\boldsymbol{S}\cdot\boldsymbol{F}^{\text{t}}$$

## stress tensors

gustav robert kirchhoff
[1824-1887]

$$\frac{\partial \psi(\boldsymbol{F})}{\partial \boldsymbol{F}} = \frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}} : \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{F}}$$

$$= 2\,\frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^{\mathrm{t}} = 2\,\boldsymbol{F} \cdot \frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}}$$

augustin louis caucy
[1789-1857]

**first piola kirchhoff**

$$\boldsymbol{P} = \frac{\partial \psi(\boldsymbol{F})}{\partial \boldsymbol{F}} = 2\,\boldsymbol{F} \cdot \frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}}$$

**cauchy**

$$\boldsymbol{\sigma} = \frac{1}{J}\,\frac{\partial \psi(\boldsymbol{F})}{\partial \boldsymbol{F}} \cdot \boldsymbol{F}^{\mathrm{t}} = \frac{2}{J}\,\boldsymbol{F} \cdot \frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^{\mathrm{t}}$$

**second piola kirchhoff**

$$\boldsymbol{S} = \boldsymbol{F}^{-1} \cdot \frac{\partial \psi(\boldsymbol{F})}{\partial \boldsymbol{F}} = 2\,\frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = \frac{\partial \psi(\boldsymbol{E})}{\partial \boldsymbol{E}}$$



## tensor algebra - basic derivatives

- (principal) invariants of second order tensor

$$\begin{aligned} I_A &= \mathrm{tr}\,(\boldsymbol{A}) \\ II_A &= \tfrac{1}{2}\,[\mathrm{tr}^2\,(\boldsymbol{A}) - \mathrm{tr}\,(\boldsymbol{A}^2)] \\ III_A &= \det\,(\boldsymbol{A}) \end{aligned}$$

- derivatives of invariants wrt second order tensor

$$\begin{aligned} \partial_{\boldsymbol{A}}\ I_A &= \boldsymbol{I} \\ \partial_{\boldsymbol{A}}\ II_A &= I_A\,\boldsymbol{I} - \boldsymbol{A} \\ \partial_{\boldsymbol{A}}\ III_A &= III_A\,\boldsymbol{A}^{-\mathrm{t}} \end{aligned}$$



## tensor analysis - basic derivatives

$$\{\bullet \otimes \circ\}_{ijkl} = \{\bullet\}_{ij}\,\{\circ\}_{kl}$$
$$\{\bullet \,\overline{\otimes}\, \circ\}_{ijkl} = \{\bullet\}_{ik}\,\{\circ\}_{jl}$$
$$\{\bullet \,\underline{\otimes}\, \circ\}_{ijkl} = \{\bullet\}_{il}\,\{\circ\}_{jk}$$

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}} = \boldsymbol{I} \,\overline{\otimes}\, \boldsymbol{I} \qquad \frac{\partial F_{ij}}{\partial F_{kl}} = \delta_{ik}\,\delta_{jl}$$

$$\frac{\partial \boldsymbol{F}^{-1}}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{-1} \,\overline{\otimes}\, \boldsymbol{F}^{-\mathrm{t}} \qquad \frac{\partial F^{-1}_{ij}}{\partial F_{kl}} = -F^{-1}_{ik}\,F^{-1}_{lj}$$

$$\frac{\partial \boldsymbol{F}^{\mathrm{t}}}{\partial \boldsymbol{F}} = \boldsymbol{I} \,\underline{\otimes}\, \boldsymbol{I} \qquad \frac{\partial F_{ji}}{\partial F_{kl}} = \delta_{il}\,\delta_{jk}$$

$$\frac{\partial \boldsymbol{F}^{-\mathrm{t}}}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{-\mathrm{t}} \,\underline{\otimes}\, \boldsymbol{F}^{-1} \qquad \frac{\partial F^{-1}_{ji}}{\partial F_{kl}} = -F^{-1}_{li}\,F^{-1}_{jk}$$



## tensor analysis - basic derivatives

$$\frac{\partial \det(\boldsymbol{F})}{\partial \boldsymbol{F}} = \det(\boldsymbol{F})\,\boldsymbol{F}^{-\mathrm{t}} \qquad \frac{\partial \ln(\det(\boldsymbol{F}))}{\partial \boldsymbol{F}} = \boldsymbol{F}^{-\mathrm{t}}$$

$$\frac{\partial \det(\boldsymbol{F}^{-1})}{\partial \boldsymbol{F}} = -\frac{1}{\det(\boldsymbol{F})}\,\boldsymbol{F}^{-\mathrm{t}} \qquad \frac{\partial \ln(\det(\boldsymbol{F})^{-1})}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{-\mathrm{t}}$$

$$\frac{\partial \ln(\det(\boldsymbol{F}))}{\partial(\det(\boldsymbol{F}))} = \frac{1}{\det(\boldsymbol{F}))} \qquad \frac{\partial \ln^2(\det(\boldsymbol{F}))}{\partial \boldsymbol{F}} = 2\,\ln(\det(\boldsymbol{F}))\boldsymbol{F}^{-\mathrm{t}}$$

## example: neo hooke'ian elasticity

- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F}))$
$$+\,\frac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$$
- definition of stress
$$\begin{aligned}\boldsymbol{P}^{\text{neo}} &= \mathrm{D}_{\boldsymbol{F}}\psi_0^{\text{neo}}\\ &= \tfrac{1}{2}\,\lambda_0\,2\,\ln(\det\boldsymbol{F})\boldsymbol{F}^{\text{-t}} + \tfrac{1}{2}\mu_0\,2\,\boldsymbol{F} - \mu_0\boldsymbol{F}^{\text{-t}}\\ &= \mu_0\,\boldsymbol{F} + [\,\lambda_0\ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{\text{-t}}\end{aligned}$$
- definition of tangent operator
$$\begin{aligned}\mathsf{A}^{\text{neo}} &= \mathrm{D}_{\boldsymbol{FF}}\psi_0^{\text{neo}} = \mathrm{D}_{\boldsymbol{F}}\boldsymbol{P}^{\text{neo}}\\ &= \lambda_0\boldsymbol{F}^{\text{-t}}\otimes\boldsymbol{F}^{\text{-t}} + \mu_0\boldsymbol{I}\,\overline{\otimes}\,\boldsymbol{I}\\ &\quad+[\mu_0 - \lambda_0\ln(\det(\boldsymbol{F}))]\,\boldsymbol{F}^{\text{-t}}\,\underline{\otimes}\,\boldsymbol{F}^{\text{-1}}\end{aligned}$$

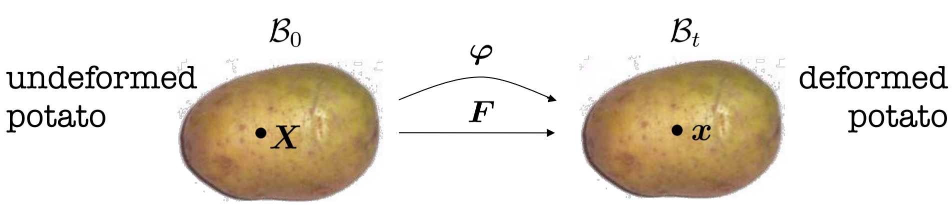


## example: neo hooke'ian elasticity

- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(F_{ij}))$
$$+\,\frac{1}{2}\,\mu_0[\,F_{ij}F_{ij} - n^{\text{dim}} - 2\,\ln(\det(F_{ij}))\,]$$
- definition of stress
$$\begin{aligned}P_{ij}^{\text{neo}} &= \mathrm{D}_{F_{ij}}\psi_0^{\text{neo}}\\ &= \tfrac{1}{2}\,\lambda_0\,2\,\ln(\det F_{ij})F^{\text{-1}}_{ji} + \tfrac{1}{2}\mu_0\,2\,F_{ij} - \mu_0 F^{\text{-1}}_{ji}\\ &= \mu_0\,F_{ij} + [\,\lambda_0\ln(\det(F_{ij})) - \mu_0\,]F^{\text{-1}}_{ji}\end{aligned}$$
- definition of tangent operator
$$\begin{aligned}\mathsf{A}^{\text{neo}}_{ijkl} &= \mathrm{D}_{F_{ij}F_{kl}}\psi_0^{\text{neo}} = \mathrm{D}_{F_{kl}}P_{ij}^{\text{neo}}\\ &= \lambda_0 F^{\text{-1}}_{ji}\;F^{\text{-1}}_{lk} + \mu_0 I_{ik}\;I_{jl}\\ &\quad+[\mu_0 - \lambda_0\ln(\det(F_{ij}))]\,F^{\text{-1}}_{li}\;F^{\text{-1}}_{jk}\end{aligned}$$



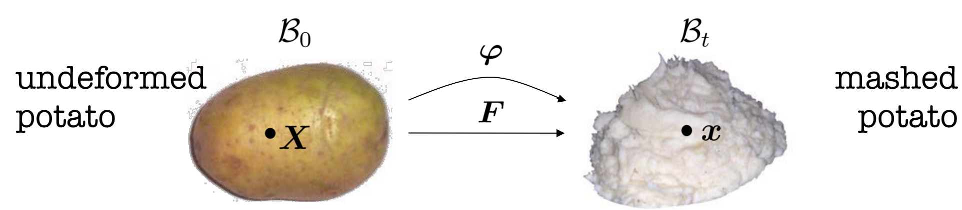
## example: neo hooke'ian elasticity




- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F}))$
$$+\,\frac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$$
- definition of stress - neo hookean elasticity
$$\begin{aligned}\boldsymbol{P}^{\text{neo}} &= \mathrm{D}_{\boldsymbol{F}}\psi_0^{\text{neo}}\\ &= \mu_0\,\boldsymbol{F} + [\,\lambda_0\ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{\text{-t}}\end{aligned}$$



## example: neo hooke'ian elasticity




- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F}))$
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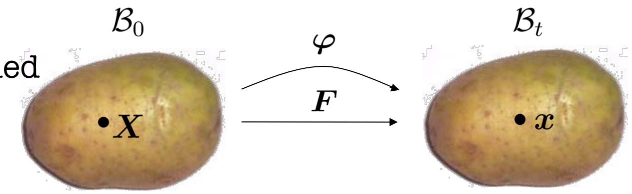
- definition of stress - neo hookean elasticity
$$\begin{aligned}\boldsymbol{P}^{\text{neo}} &= \mathrm{D}_{\boldsymbol{F}}\psi_0^{\text{neo}}\\ &= \mu_0\,\boldsymbol{F} + [\,\lambda_0\ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{\text{-t}}\end{aligned}$$
- remember! mashing potatoes is not an elastic process!

# example: neo hooke'ian elasticity

undeformed potato




deformed potato

- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F}))$
$$+ \tfrac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$$
- large strain - lamé parameters and bulk modulus
$$\lambda = \frac{E\,\nu}{[1+\nu][1-2\,\nu]} \qquad \mu = \frac{E}{2\,[1+\nu]} \qquad \kappa = \frac{E}{3\,[1-2\,\nu]}$$
- small strain - young ‘s modulus and poisson ‘s ratio
$$E = 3\,\kappa\,[1-2\,\nu] \qquad \nu = \frac{3\,\kappa - 2\,\mu}{2\,[3\,\kappa+\mu]}$$