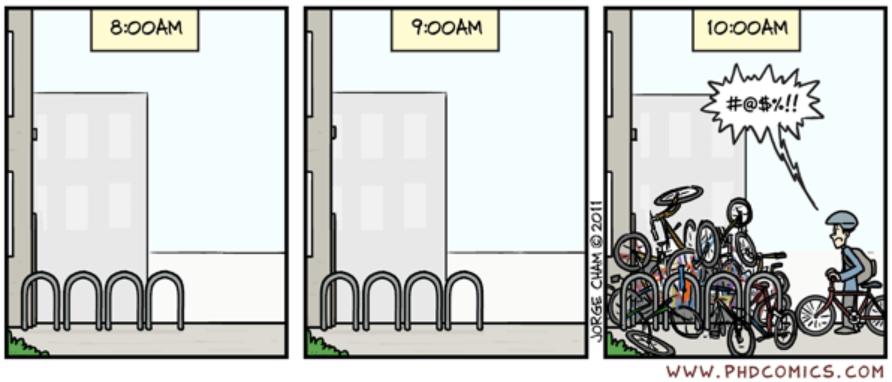
07 - balance principles

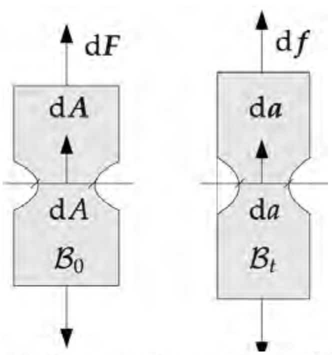


holzapfel 'nonlinear solid mechanics' [2000], chapter 4, pages 131-161

me338 - syllabus

day	date		topic	chapters	pages
tue	sep	25	why continuum mechanics?		
thu	sep	27	introduction to vectors and tensors	1.1-1.5	1-32
tue	oct	02	introduction to vectors and tensors	1.6-1.9	32-55
thu	oct	04	kinematics	2.1-2.4	55-76
tue	oct	09	kinematics	2.5-2.8	76-109
thu	oct	11	concept of stress	3.1-3.4	109-131
tue	oct	16	balance principles	4.1-4.4	131-161
thu	oct	18	balance principles	4.5-4.7	161-179
tue	oct	23	aspects of objectivity	5.1-5.4	179-205
thu	oct	25	hyperelastic materials	6.1-6.2	205-222
tue	oct	30	hyperelastic materials	6.3-6.5	222-252
thu	nov	01	hyperelastic materials	6.6-6.8	252-278
tue	nov	06	hyperelastic materials	6.9-6.11	278-305
thu	nov	08	thermodynamics of materials	7.1-7.6	305-337
tue	nov	13	midterm prep		
thu	nov	15	midterm		
tue	nov	27	thermodynamics of materials	7.7-7.9	337-371
thu	nov	29	variational principles	8.1-8.3	371-392
tue	dec	04	variational principles	8.4-8.6	392-414
thu	dec	06	final project discussion		

stress tensors



cauchy / true stress
relates spatial force to spatial area

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{t}\,\mathrm{d}a = \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}a = \boldsymbol{\sigma}\cdot\mathrm{d}\boldsymbol{a}$$

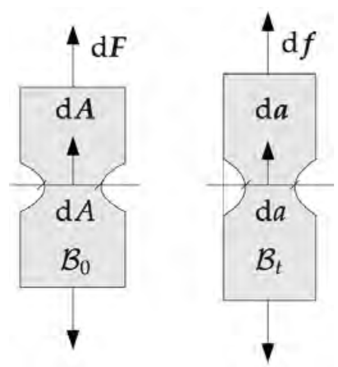
first piola kirchhoff / nominal stress
relates spatial force to material area

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{t}\,\mathrm{d}a = \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}a = \boldsymbol{\sigma}\cdot\mathrm{d}\boldsymbol{a} = J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-t}\cdot\mathrm{d}\boldsymbol{A} = \boldsymbol{P}\cdot\mathrm{d}\boldsymbol{A}$$

second piola kirchhoff stress
relates material force to material area

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{F}^{-1}\cdot\mathrm{d}\boldsymbol{f} = \boldsymbol{F}^{-1}\cdot\boldsymbol{P}\cdot\mathrm{d}\boldsymbol{A} = J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-t}\cdot\mathrm{d}\boldsymbol{A} = \boldsymbol{S}\cdot\mathrm{d}\boldsymbol{A}$$

stress tensors



cauchy / true stress
relates spatial force to spatial area

$$\boldsymbol{t} = \boldsymbol{\sigma}\cdot\boldsymbol{n} \qquad t_i = \sigma_{ij}\, n_j$$

first piola kirchhoff / nominal stress
relates spatial force to material area

$$\boldsymbol{P} = J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-t} \qquad P_{iJ} = J\,\sigma_{ik}\,F^{-t}_{kJ}$$

second piola kirchhoff stress
relates material force to material area

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\cdot\boldsymbol{P} = J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-t}$$

$$S_{IJ} = F^{-1}_{Ik}\,P_{kJ} = J\,F^{-1}_{Ik}\,\sigma_{kl}\,F^{-t}_{lJ}$$

stress tensors

gustav robert kirchhoff
[1824-1887]

first piola kirchhoff

$$P = F \cdot S$$
$$P = J\,\sigma \cdot F^{-t}$$

augustin louis caucy
[1789-1857]

second piola kirchhoff

$$S = F^{-1} \cdot P$$
$$S = J\,F^{-1} \cdot \sigma \cdot F^{-t}$$

cauchy

$$\sigma = \frac{1}{J}\,P \cdot F^{t}$$
$$\sigma = \frac{1}{J}\,F \cdot S \cdot F^{t}$$

transport mechanisms

covariant / strains

$$E = F^{t} \cdot e \cdot F \qquad E_{IJ} = F^{t}_{Ik}\, e_{kl}\, F_{lJ}$$

← pull back
push forward →

$$e = F^{-t} \cdot E \cdot F^{-1} \qquad e_{ij} = F^{-t}_{iK}\, E_{KL}\, F^{-1}_{Lj}$$

contravariant / stresses

$$S = J\,F^{-1} \cdot \sigma \cdot F^{-t} \qquad S_{IJ} = J\,F^{-1}_{Ik}\, \sigma_{kl}\, F^{-t}_{lJ}$$

← pull back
push forward →

$$\sigma = \frac{1}{J}\,F \cdot S \cdot F^{t} \qquad \sigma_{ij} = \frac{1}{J}\,F_{iK}\, S_{KL}\, F^{t}_{Lj}$$

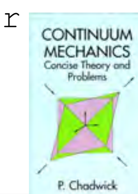
balance equations

balance equations [ˈbæl.əns ɪˈkweɪ.ʒəns] of mass, momentum, angular momentum and energy, supplemented with an entropy inequality constitute the set of conservation laws. the law of **conservation of mass**/matter states that the **mass of a closed system** of substances will remain **constant**, regardless of the processes acting inside the system. the principle of conservation of momentum states that the total momentum of a closed system of objects is constant.

Wikipedia
The Free Encyclopedia

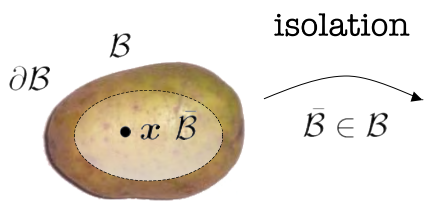
balance equations

balance equations [ˈbæl.əns ɪˈkweɪ.ʒəns] of mass, linear momentum, angular momentum and energy **apply to all material bodies.** each one gives rise to a field equation, holing on the configurations of a body in a sufficiently smooth motion and a jump condition on surfaces of discontinuity. like position, time and body, the concepts of mass, force, heating and internal energy which enter into the formulation of the balance equations are regarded as having primitive status in continuum mechanics.



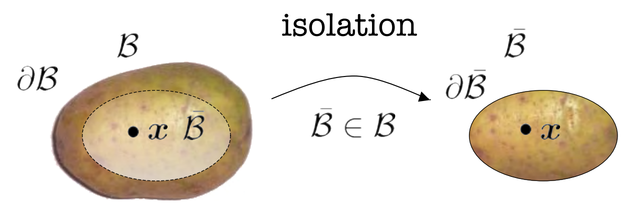
chadwick 'continuum mechanics' [1976]

potato balance equations



[1] isolate subset $\bar{\mathcal{B}}$ from $\mathcal{B}$

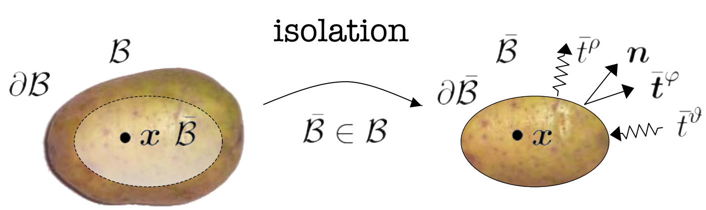
potato balance equations



[1] isolate subset $\bar{\mathcal{B}}$ from $\mathcal{B}$

[2] **characterize** influence of remaining body through phenomenological quantities - contact fluxes $\bar{t}^{\rho}$, $\bar{\boldsymbol{t}}^{\varphi}$& $\bar{t}^{\vartheta}$

potato balance equations

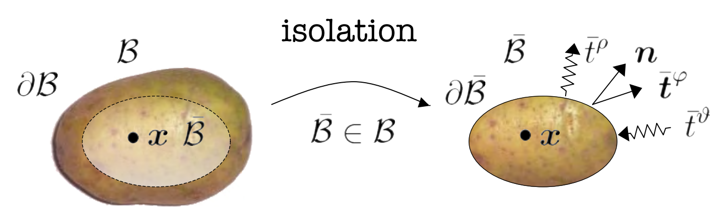


[1] isolate subset $\bar{\mathcal{B}}$ from $\mathcal{B}$

[2] characterize influence of remaining body through phenomenological quantities - contact fluxes $\bar{t}^{\rho}$, $\bar{\boldsymbol{t}}^{\varphi}$& $\bar{t}^{\vartheta}$

[3] **define** basic physical quantities - mass, linear and angular momentum, energy

potato balance equations



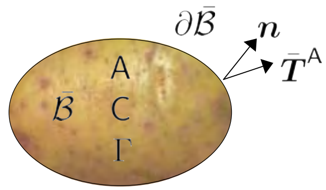
[1] isolate subset $\bar{\mathcal{B}}$ from $\mathcal{B}$

[2] characterize influence of remaining body through phenomenological quantities - contact fluxes $\bar{t}^{\rho}$, $\bar{\boldsymbol{t}}^{\varphi}$& $\bar{t}^{\vartheta}$

[3] define basic physical quantities - mass, linear and angular momentum, energy

[4] **postulate** balance of these quantities

generic balance equation



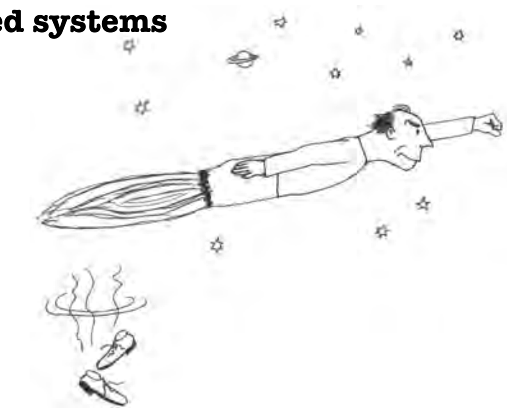
general format

A ... balance quantity
$\mathbf{B}$ ... flux $\mathbf{B}\cdot\boldsymbol{n} = \bar{\boldsymbol{T}}^{\mathsf{A}}$
C ... source
Γ ... production

$$\mathrm{D}_t\mathsf{A} = \mathrm{Div}(\mathbf{B}) + \mathsf{C} + \Gamma$$

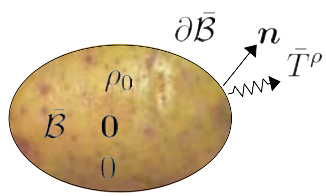
balance of mass

- here: **closed systems**



- unlike open systems **closed systems** have a **constant mass**
- examples of open systems:
  rocket propulsion and **biological growth** (me337)

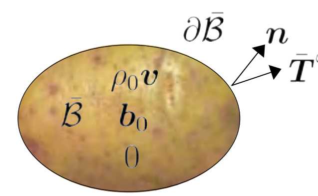
balance of mass



ρ_0 ... density
$\mathbf{0}$ ... no mass flux $\bar{T}^{\rho} = 0$
0 ... no mass source
0 ... no mass production

continuity equation $\mathrm{D}_t\rho_0 = 0$

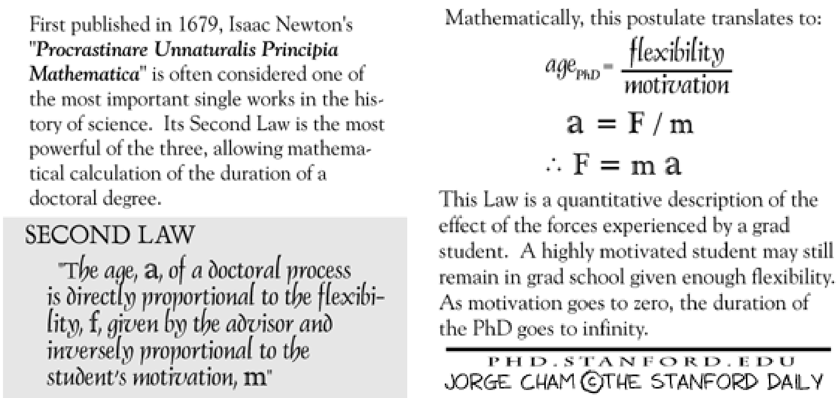
balance of (linear) momentum



$\rho_0\,\boldsymbol{v}$ ... linear momentum density
$\boldsymbol{P}$ ... momentum flux - stress $\boldsymbol{P}\cdot\boldsymbol{n} = \bar{\boldsymbol{T}}^{\varphi}$
$\boldsymbol{b}_0$ ... momentum source - force
0 ... no momentum production

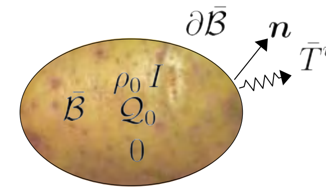
equilibrium equation $\mathrm{D}_t(\rho_0\boldsymbol{v}) = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0$

compare



$$\mathrm{D}_t(\rho_0 \boldsymbol{v}) = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0$$ **mass point** $$m\,\mathrm{D}_t \boldsymbol{v} = m\,\boldsymbol{a} = \boldsymbol{F}$$

balance of (internal) energy



$\rho_0\, I$ ... internal energy density

$\boldsymbol{Q}$ ... heat flux $\quad -\boldsymbol{Q}\cdot\boldsymbol{n} = \bar{T}^\vartheta$

$\mathcal{Q}_0$ ... heat source

0 ... no heat production

energy equation $$\mathrm{D}_t(\rho_0\, I) = \underbrace{\boldsymbol{P} : \mathrm{D}_t \boldsymbol{F} - \boldsymbol{v}\cdot\boldsymbol{b}_0}_{\text{internal mechanical power}} + \underbrace{\mathrm{Div}(-\boldsymbol{Q}) + \mathcal{Q}_0}_{\text{external thermal power}}$$