02 - introduction to vectors and tensors

holzapfel 'nonlinear solid mechanics' [2000], chapter 1.1-1.5, pages 1-32

me338 - syllabus

day	date		topic	chapters	pages
tue	sep	25	why continuum mechanics?		
thu	sep	27	introduction to vectors and tensors	1.1-1.5	1-32
tue	oct	02	introduction to vectors and tensors	1.6-1.9	32-55
thu	oct	04	kinematics	2.1-2.4	55-76
tue	oct	09	kinematics	2.5-2.8	76-109
thu	oct	11	concept of stress	3.1-3.4	109-131
tue	oct	16	balance principles	4.1-4.4	131-161
thu	oct	18	balance principles	4.5-4.7	161-179
tue	oct	23	aspects of objectivity	5.1-5.4	179-205
thu	oct	25	hyperelastic materials	6.1-6.2	205-222
tue	oct	30	hyperelastic materials	6.3-6.5	222-252
thu	nov	01	hyperelastic materials	6.6-6.8	252-278
tue	nov	06	hyperelastic materials	6.9-6.11	278-305
thu	nov	08	thermodynamics of materials	7.1-7.6	305-337
tue	nov	13	midterm prep		
thu	nov	15	midterm		
tue	nov	27	thermodynamics of materials	7.7-7.9	337-371
thu	nov	29	variational principles	8.1-8.3	371-392
tue	dec	04	variational principles	8.4-8.6	392-414
thu	dec	06	final project discussion		

continuum mechanics

continuum mechanics [kənˈtɪn.ju.əm məˈkæn.ɪks] is the branch of mechanics concerned with the stress in solids, liquids and gases and the deformation or flow of these materials. the adjective continuous refers to the simplifying concept underlying the analysis: we disregard the molecular structure of matter and picture it as being without gaps or empty spaces. we suppose that all the mathematical functions entering the theory are continuous functions. this hypothetical continuous material we call a continuum.

malvern 'introduction to the mechanics of a continuous medium' [1969]

the potato  equations

- **kinematic equations - what's strain?** $\epsilon = \frac{\Delta l}{l}$

  general equations that characterize the deformation of a physical body without studying its physical cause

- **balance equations - what's stress?** $\sigma = \frac{F}{A}$

  general equations that characterize the cause of motion of any body

- **constitutive equations - how are they related?** $\sigma = E\,\epsilon$

  material specific equations that complement the set of governing equations

the potato equations

- **kinematic equations - why not** $\epsilon = \frac{\Delta l}{l}$ **?**
  inhomogeneous deformation » non-constant
  finite deformation » non-linear $\boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$
  inelastic deformation » growth tensor $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{g}}$
- **balance equations - why not** $\sigma = \frac{F}{A}$**?** $\mathrm{Div}(\boldsymbol{P}) + \rho\, \boldsymbol{b}_0 = \boldsymbol{0}$
  equilibrium in deformed configuration » multiple stresses
- **constitutive equations - why not** $\sigma = E\,\epsilon$ **?**
  finite deformation » non-linear $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F})$
  inelastic deformation » internal variables $\boldsymbol{P} = \boldsymbol{P}(\rho, \boldsymbol{F}, \boldsymbol{F}_{\mathrm{g}})$

tensor calculus

tensor [ˈten.sor] the word tensor was introduced in 1846 by william rowan hamilton. it was used in its current meaning by woldemar voigt in 1899. tensor calculus was developed around 1890 by gregorio ricci-curbastro under the title absolute differential calculus. in the 20th century, the subject came to be known as tensor analysis, and achieved broader acceptance with the introduction of einsteins's theory of general relativity around 1915. tensors are used also in other fields such as continuum mechanics.

WIKIPEDIA
The Free Encyclopedia

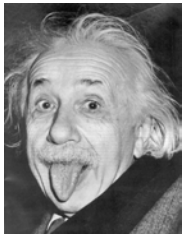
tensor calculus

- vector algebra
  notation, euklidian vector space, scalar product, vector product, scalar triple product
- tensor algebra
  notation, scalar products, dyadic product, invariants, trace, determinant, inverse, spectral decomposition, sym-skew decomposition, vol-dev decomposition, orthogonal tensor
- tensor analysis
  derivatives, gradient, divergence, laplace operator, integral transformations

vector algebra - notation

- einstein's summation convention

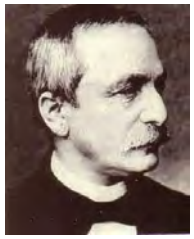
$$u_i = \sum_{j=1}^{3} A_{ij}\, x_j + b_i = A_{ij}\, x_j + b_i$$



- summation over any indices that appear twice in a term

$$\begin{aligned} u_1 &= A_{11}\, x_1 + A_{12}\, x_2 + A_{13}\, x_3 + b_1 \\ u_2 &= A_{21}\, x_1 + A_{22}\, x_2 + A_{23}\, x_3 + b_2 \\ u_3 &= A_{31}\, x_1 + A_{32}\, x_2 + A_{33}\, x_3 + b_3 \end{aligned}$$

vector algebra - notation



- kronecker symbol

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

$$u_i = \delta_{ij}\, u_j$$

- permutation symbol

$$\overset{3}{e}_{ijk} = \begin{cases} 1 & \text{for} \quad \{i,j,k\} & \text{... even permutation} \\ -1 & \text{for} \quad \{i,j,k\} & \text{... odd permutation} \\ 0 & & \text{... else} \end{cases}$$

vector algebra - euklidian vector space

- euklidian vector space $\mathcal{V}^3$

$$\begin{array}{lll} \alpha, \beta & \in \mathcal{R} & \mathcal{R} \quad \text{... real numbers} \\ \boldsymbol{u}, \boldsymbol{v} & \in \mathcal{V}^3 & \mathcal{V}^3 \quad \text{... linear vector space} \end{array}$$

- $\mathcal{V}^3$ is defined through the following axioms

$$\begin{aligned} \alpha\,(\boldsymbol{u} + \boldsymbol{v}) &= \alpha\,\boldsymbol{u} + \alpha\,\boldsymbol{v} \\ (\alpha + \beta)\,\boldsymbol{u} &= \alpha\,\boldsymbol{u} + \beta\,\boldsymbol{u} \\ (\alpha\,\beta)\,\boldsymbol{u} &= \alpha\,(\beta\,\boldsymbol{u}) \end{aligned}$$

- zero element and identity

$$0\,\boldsymbol{u} = \boldsymbol{0} \qquad 1\,\boldsymbol{u} = \boldsymbol{u}$$

- linear independence of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathcal{V}^3$ if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is the only (trivial) solution to $\alpha_i\,\boldsymbol{e}_i = 0$

vector algebra - euklidian vector space

- euklidian vector space $\mathcal{V}^3$equipped with norm

$$n : \mathcal{V}^3 \to \mathcal{R} \qquad \text{... norm}$$

- norm defined through the following axioms

$$n(\boldsymbol{u}) \geq 0 \qquad n(\boldsymbol{u}) = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$$

$$n(\alpha\,\boldsymbol{u}) = |\alpha|\, n(\boldsymbol{u})$$

$$n(\boldsymbol{u} + \boldsymbol{v}) \leq n(\boldsymbol{u}) + n(\boldsymbol{v})$$

$$n^2(\boldsymbol{u} + \boldsymbol{v}) + n^2(\boldsymbol{u} - \boldsymbol{v}) = 2\,[n^2(\boldsymbol{u}) + n^2(\boldsymbol{v})]$$

vector algebra - euklidian vector space

- euklidian vector space $\mathcal{E}^3$equipped with euklidian norm

$$n : \mathcal{E}^3 \to \mathcal{R} \qquad \text{... euklidian norm}$$

$$n(\boldsymbol{u}) = ||\boldsymbol{u}|| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = [u_1^2 + u_2^2 + u_3^2]^{1/2}$$

- representation of 3d vector $\boldsymbol{u} \in \mathcal{E}^3$

$$\boldsymbol{u} = u_i\,\boldsymbol{e}_i = u_1\,\boldsymbol{e}_1 + u_2\,\boldsymbol{e}_2 + u_3\,\boldsymbol{e}_3$$

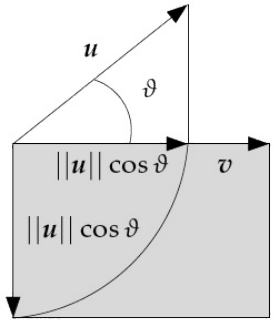
with u_1, u_2, u_3 coordinates (components) of $\boldsymbol{u}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

$$\boldsymbol{u} = [\,u_1, u_2, u_3\,]^{\mathrm{t}}$$

vector algebra - scalar product

- euklidian norm enables definition of scalar (inner) product

$$\boldsymbol{u}\cdot\boldsymbol{v}=\alpha \qquad \alpha\in\mathcal{R}$$
$$\boldsymbol{u}\cdot\boldsymbol{v}=||\boldsymbol{u}||\,||\boldsymbol{v}||\,\cos\vartheta$$
$$||\boldsymbol{u}\cdot\boldsymbol{v}||\le||\boldsymbol{u}||\,||\boldsymbol{v}||$$



- properties of scalar product

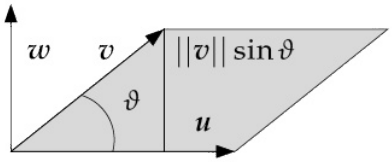
$$\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{v}\cdot\boldsymbol{u}$$
$$(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v})\cdot\boldsymbol{w}=\alpha\,(\boldsymbol{u}\cdot\boldsymbol{w})+\beta\,(\boldsymbol{v}\cdot\boldsymbol{w})$$
$$\boldsymbol{w}\cdot(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v})=\alpha\,(\boldsymbol{w}\cdot\boldsymbol{u})+\beta\,(\boldsymbol{w}\cdot\boldsymbol{v})$$

- positive definiteness $\quad \boldsymbol{u}\cdot\boldsymbol{u}\ge 0,\quad \boldsymbol{u}\cdot\boldsymbol{u}=0\Leftrightarrow\boldsymbol{u}=\boldsymbol{0}$
- orthogonality $\quad \boldsymbol{u}\cdot\boldsymbol{v}=0 \quad\Leftrightarrow\quad \boldsymbol{u}\perp\boldsymbol{v}$

vector algebra - vector product

- vector product

$$\boldsymbol{u}\times\boldsymbol{v}=\boldsymbol{w} \qquad \boldsymbol{w}\in\mathcal{E}^3$$
$$\boldsymbol{u}\times\boldsymbol{v}=||\boldsymbol{u}||\,||\boldsymbol{v}||\,\sin\vartheta\,\boldsymbol{n}$$
$$\boldsymbol{u}\times\boldsymbol{v}=\boldsymbol{0} \qquad\Leftrightarrow\qquad \boldsymbol{u}\parallel\boldsymbol{v}$$



$$\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix}=\begin{bmatrix} u_2\,v_3 & - & u_3\,v_2\\ u_3\,v_1 & - & u_1\,v_3\\ u_1\,v_2 & - & u_2\,v_1\end{bmatrix}$$

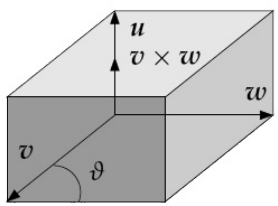
- properties of vector product

$$\boldsymbol{u}\times\boldsymbol{v}=-\boldsymbol{v}\times\boldsymbol{u}$$
$$(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v})\times\boldsymbol{w}=\alpha\,(\boldsymbol{u}\times\boldsymbol{w})+\beta\,(\boldsymbol{v}\times\boldsymbol{w})$$
$$\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{v})=0$$
$$(\boldsymbol{u}\times\boldsymbol{v})\cdot(\boldsymbol{u}\times\boldsymbol{v})=(\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{v}\cdot\boldsymbol{v})-(\boldsymbol{u}\cdot\boldsymbol{v})^2$$

vector algebra - scalar triple product

- scalar triple product

$$[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]=\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})=\alpha \qquad \alpha\in\mathcal{R}$$
$$\boldsymbol{u}\times\boldsymbol{v}=||\boldsymbol{u}||\,||\boldsymbol{v}||\,\sin\vartheta\,\boldsymbol{n} \quad \text{area}$$
$$[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]=\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) \qquad \text{volume}$$



$$\alpha=u_1(v_2w_3-v_3w_2)+u_2(v_3w_1-v_1w_3)+u_3(v_1w_2-v_3w_1)$$

- properties of scalar triple product

$$[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]=[\boldsymbol{v},\boldsymbol{w},\boldsymbol{u}]=[\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}]$$
$$=-[\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}]=-[\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}]=-[\boldsymbol{w},\boldsymbol{v},\boldsymbol{u}]$$
$$[\alpha\boldsymbol{u}+\beta\boldsymbol{v},\boldsymbol{w},\boldsymbol{d}]=\alpha\,[\boldsymbol{u},\boldsymbol{w},\boldsymbol{d}]+\beta\,[\boldsymbol{v},\boldsymbol{w},\boldsymbol{d}]$$

- linear independency $\quad [\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]\neq 0$

tensor algebra - second order tensors

- second order tensor

$$\boldsymbol{A}=\boldsymbol{u}\otimes\boldsymbol{v} \qquad \boldsymbol{u}=u_i\,\boldsymbol{e}_i \quad\text{and}\quad \boldsymbol{v}=v_j\,\boldsymbol{e}_j$$
$$\boldsymbol{A}=A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j \qquad [A_{ij}]=\begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$$

with $A_{ij}=u_i\,v_j$ coordinates (components) of $\boldsymbol{A}$ relative to the basis $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3$

- transpose of second order tensor

$$\boldsymbol{A}^{\mathrm{t}}=(\boldsymbol{u}\otimes\boldsymbol{v})^{\mathrm{t}}=\boldsymbol{v}\otimes\boldsymbol{u}$$
$$\boldsymbol{A}^{\mathrm{t}}=A_{ji}\,\boldsymbol{e}_j\otimes\boldsymbol{e}_i \qquad [A_{ji}]=\begin{bmatrix} A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}$$

tensor algebra - second order tensors

- second order unit tensor in terms of kronecker symbol

$$\boldsymbol{I} = \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

with δ_{ij} coordinates (components) of $\boldsymbol{I}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- matrix representation of coordinates

$$[\delta_{ji}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- identity

$$\boldsymbol{I} \cdot \boldsymbol{u} = \boldsymbol{u} \qquad \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot u_j\, \boldsymbol{e}_j = u_i\, \boldsymbol{e}_i$$

tensor algebra - third order tensors

- third order tensor

$$\overset{3}{\boldsymbol{a}} = \boldsymbol{A} \otimes \boldsymbol{v} \qquad \boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \text{ and } \boldsymbol{v} = v_k\, \boldsymbol{e}_k$$

$$\overset{3}{\boldsymbol{a}} = a_{ijk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$$

with $a_{ijk} = A_{ij}\, v_k$ coordinates (components) of $\boldsymbol{A}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- third order permutation tensor in terms of permutation symbol $\overset{3}{e}_{ijk}$

$$\overset{3}{\boldsymbol{e}} = e_{ijk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$$

tensor algebra - fourth order tensors

- fourth order tensor

$$\mathsf{A} = \boldsymbol{A} \otimes \boldsymbol{B} \qquad \boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \text{ and } \boldsymbol{B} = B_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

$$\mathsf{A} = A_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

with $A_{ijkl} = A_{ij}\, B_{kl}$ coordinates (components) of A relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- fourth order unit tensor

$$\mathsf{I} = \delta_{ik}\, \delta_{jl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I} : \boldsymbol{A} = \boldsymbol{A}$$

- transpose of fourth order unit tensor

$$\mathsf{I}^{\mathrm{t}} = \delta_{il}\, \delta_{jk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\mathrm{t}} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{t}}$$

tensor algebra - fourth order tensors

- symmetric fourth order unit tensor

$$\mathsf{I}^{\mathrm{sym}} = \tfrac{1}{2}\left[\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\mathrm{sym}} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{sym}}$$

- screw-symmetric fourth order unit tensor

$$\mathsf{I}^{\mathrm{skw}} = \tfrac{1}{2}\left[\delta_{ik}\, \delta_{jl} - \delta_{il}\, \delta_{jk}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\mathrm{skw}} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{skw}}$$

- volumetric fourth order unit tensor

$$\mathsf{I}^{\mathrm{vol}} = \tfrac{1}{3}\, \delta_{ij}\, \delta_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\mathrm{vol}} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{vol}}$$

- deviatoric fourth order unit tensor $\qquad \mathsf{I}^{\mathrm{dev}} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{dev}}$

$$\mathsf{I}^{\mathrm{dev}} = \left[-\tfrac{1}{3}\, \delta_{ij}\, \delta_{kl} + \tfrac{1}{2}\, \delta_{ik}\, \delta_{jl} + \tfrac{1}{2}\, \delta_{il}\, \delta_{jk}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

tensor algebra - scalar product

- scalar (inner) product

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{u} &= (A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot(u_k\,\boldsymbol{e}_k) \\ &= A_{ij}\,u_k\,\delta_{jk}\,\boldsymbol{e}_i = A_{ij}\,u_j\,\boldsymbol{e}_i = v_i\boldsymbol{e}_i = \boldsymbol{v} \end{aligned}$$

of second order tensor $\boldsymbol{A}$ and vector $\boldsymbol{u}$

- zero and identity $\quad \boldsymbol{0}\cdot\boldsymbol{u}=\boldsymbol{0} \qquad \boldsymbol{I}\cdot\boldsymbol{u}=\boldsymbol{u}$
- positive definiteness $\quad \boldsymbol{a}\cdot\boldsymbol{A}\cdot\boldsymbol{a}>0$
- properties of scalar product

$$\boldsymbol{A}\cdot(\alpha\boldsymbol{a}+\beta\boldsymbol{b}) = \alpha\,(\boldsymbol{A}\cdot\boldsymbol{a})+\beta\,(\boldsymbol{A}\cdot\boldsymbol{b})$$
$$(\boldsymbol{A}+\boldsymbol{B})\cdot\boldsymbol{a} = \boldsymbol{A}\cdot\boldsymbol{a}+\boldsymbol{B}\cdot\boldsymbol{a}$$
$$(\alpha\,\boldsymbol{A})\cdot\boldsymbol{a} = \alpha\,(\boldsymbol{A}\cdot\boldsymbol{a})$$

tensor algebra - scalar product

- scalar (inner) product

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B} &= (A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot(B_{kl}\,\boldsymbol{e}_k\otimes\boldsymbol{e}_l) \\ &= A_{ij}\,B_{kl}\,\delta_{ik}\boldsymbol{e}_i\otimes\boldsymbol{e}_l \\ &= A_{ij}\,B_{jl}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_l = C_{il}\boldsymbol{e}_i\otimes\boldsymbol{e}_l = \boldsymbol{C} \end{aligned}$$

of two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$

- zero and identity $\quad \boldsymbol{0}\cdot\boldsymbol{A}=\boldsymbol{A} \qquad \boldsymbol{I}\cdot\boldsymbol{A}=\boldsymbol{A}$
- properties of scalar product $\quad (\boldsymbol{A}\cdot\boldsymbol{B})^{\mathrm{t}} = \boldsymbol{B}^{\mathrm{t}}\cdot\boldsymbol{A}^{\mathrm{t}}$

$$\alpha\,(\boldsymbol{A}\cdot\boldsymbol{B}) = (\alpha\,\boldsymbol{A})\cdot\boldsymbol{B} = \boldsymbol{A}\cdot(\alpha\,\boldsymbol{B})$$
$$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}$$
$$(\boldsymbol{A}+\boldsymbol{B})\cdot\boldsymbol{C} = \boldsymbol{A}\cdot\boldsymbol{C}+\boldsymbol{B}\cdot\boldsymbol{C}$$

tensor algebra - scalar product

- scalar (inner) product

$$\begin{aligned} \boldsymbol{A}:\boldsymbol{B} &= (A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j):(B_{kl}\,\boldsymbol{e}_k\otimes\boldsymbol{e}_l) \\ &= A_{ij}\,B_{kl}\,\delta_{ik}\,\delta_{jl} = A_{ij}\,B_{ij} = \alpha \end{aligned}$$

of two second order tensors $\boldsymbol{A},\boldsymbol{B}$

- scalar (inner) product

$$\begin{aligned} \mathsf{A}:\boldsymbol{B} &= (A_{ijkl}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l):(B_{mn}\,\boldsymbol{e}_m\otimes\boldsymbol{e}_n) \\ &= A_{ijkl}\,B_{mn}\,\delta_{km}\,\delta_{ln}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j \\ &= A_{ijkl}\,B_{kl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \boldsymbol{A} \end{aligned}$$

of fourth order tensors A and second order tensor $\boldsymbol{B}$

- zero and identity $\quad \mathsf{0}:\boldsymbol{A}=\boldsymbol{0} \qquad \mathsf{I}:\boldsymbol{A}=\boldsymbol{A}$

tensor algebra - dyadic product

- dyadic (outer) product

$$\boldsymbol{A}=\boldsymbol{u}\otimes\boldsymbol{v}=u_i\,\boldsymbol{e}_i\otimes v_j\boldsymbol{e}_j=u_i\,v_j\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j=A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j$$

of two vectors $\boldsymbol{u},\boldsymbol{v}$ introduces second order tensor $\boldsymbol{A}$

- properties of dyadic product (tensor notation)

$$(\boldsymbol{u}\otimes\boldsymbol{v})\cdot\boldsymbol{w} = (\boldsymbol{v}\cdot\boldsymbol{w})\,\boldsymbol{u}$$
$$(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v})\otimes\boldsymbol{w} = \alpha\,(\boldsymbol{u}\otimes\boldsymbol{w})+\beta\,(\boldsymbol{v}\otimes\boldsymbol{w})$$
$$\boldsymbol{u}\otimes(\alpha\,\boldsymbol{v}+\beta\,\boldsymbol{w}) = \alpha\,(\boldsymbol{u}\otimes\boldsymbol{v})+\beta\,(\boldsymbol{u}\otimes\boldsymbol{w})$$
$$(\boldsymbol{u}\otimes\boldsymbol{v})\cdot(\boldsymbol{w}\otimes\boldsymbol{x}) = (\boldsymbol{v}\cdot\boldsymbol{w})\,(\boldsymbol{u}\otimes\boldsymbol{x})$$
$$\boldsymbol{A}\cdot(\boldsymbol{u}\otimes\boldsymbol{v}) = (\boldsymbol{A}\cdot\boldsymbol{u})\otimes\boldsymbol{v}$$
$$(\boldsymbol{u}\otimes\boldsymbol{v})\cdot\boldsymbol{A} = \boldsymbol{u}\otimes(\boldsymbol{A}^{\mathrm{t}}\cdot\boldsymbol{v})$$