continuum mechanics is the branch of mechanics concerned with the stress in solids, liquids and gases and the deformation or flow of these materials. the adjective continuous refers to the simplifying concept underlying the analysis: we disregard the molecular structure of matter and picture it as being without gaps or empty spaces. we suppose that all the mathematical functions entering the theory are continuous functions. this hypothetical continuous material we call a continuum.

malvern 'introduction to the mechanics of a continuous medium' [1969]
**kinematic equations - why not** $\epsilon = \frac{\Delta l}{l}$?

inhomogeneous deformation » non-constant
finite deformation » non-linear
inelastic deformation » growth tensor

$F = \nabla_\lambda \varphi$
$F = F_c \cdot F_g$

**balance equations - why not** $\sigma = \frac{F}{A}$?

$\text{Div}(P) + \rho b_0 = 0$
equilibrium in deformed configuration » multiple stresses

**constitutive equations - why not** $\sigma = E \epsilon$?

finite deformation » non-linear
inelastic deformation » internal variables

$P = P(F)$
$P = P(\rho, F, F_g)$

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**continuum mechanics**

**tensor calculus**

tensor [tensor] the word tensor was introduced in 1846 by william rowan hamilton. it was used in its current meaning by woldemar voigt in 1899. tensor calculus was developed around 1890 by gregorio ricci-curbastro under the title absolute differential calculus. in the 20th century, the subject came to be known as tensor analysis, and achieved broader acceptance with the introduction of einstein's theory of general relativity around 1915. tensors are used also in other fields such as continuum mechanics.

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**tensor calculus**

**vector algebra - notation**

einstein’s summation convention

$$u_i = \sum_{j=1}^{3} A_{ij} x_j + b_i = A_{ij} x_j + b_i$$

summation over any indices that appear twice in a term

$$u_1 = A_{11} x_1 + A_{12} x_2 + A_{13} x_3 + b_1$$
$$u_2 = A_{21} x_1 + A_{22} x_2 + A_{23} x_3 + b_2$$
$$u_3 = A_{31} x_1 + A_{32} x_2 + A_{33} x_3 + b_3$$
vector algebra - notation

- kronecker symbol

\[ \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \]

\[ u_i = \delta_{ij} u_j \]

- permutation symbol

\[ \varepsilon_{ijk} = \begin{cases} 1 & \text{for } \{i, j, k\} \text{ ... even permutation} \\ -1 & \text{for } \{i, j, k\} \text{ ... odd permutation} \\ 0 & \text{... else} \end{cases} \]

vector algebra - euklidian vector space

- euklidian vector space \( \mathcal{V}^3 \)

\[ \alpha, \beta \in \mathbb{R} \quad \mathbb{R} \quad \text{... real numbers} \]

\[ u, v \in \mathcal{V}^3 \quad \mathcal{V}^3 \quad \text{... linear vector space} \]

- \( \mathcal{V}^3 \) is defined through the following axioms

\[ \alpha \ (u + v) = \alpha \ u + \alpha \ v \]

\[ (\alpha + \beta) \ u = \alpha \ u + \beta \ u \]

\[ (\alpha \beta) \ u = \alpha \ (\beta \ u) \]

- zero element and identity

\[ 0 \ u = 0 \quad 1 \ u = u \]

- linear independence of \( e_1, e_2, e_3 \in \mathcal{V}^3 \) if \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) is the only (trivial) solution to \( \alpha_1 \ e_1 = 0 \)

tensor calculus

- euklidian vector space \( \mathcal{V}^3 \) equipped with norm

\[ n : \mathcal{V}^3 \to \mathbb{R} \quad \text{... norm} \]

- norm defined through the following axioms

\[ n(u) \geq 0 \quad n(u) = 0 \Leftrightarrow u = 0 \]

\[ n(\alpha \ u) = |\alpha| \ n(u) \]

\[ n(u + v) \leq n(u) + n(v) \]

\[ n^2(u + v) + n^2(u - v) = 2 \left(n^2(u) + n^2(v)\right) \]

vector algebra - euklidian vector space

- euklidian vector space \( \mathcal{V}^3 \) equipped with euklidian norm

\[ n : \mathcal{V}^3 \to \mathbb{R} \quad \text{... euklidian norm} \]

\[ n(u) = ||u|| = \sqrt{u \cdot u} = [u_1^2 + u_2^2 + u_3^2]^{1/2} \]

- representation of 3d vector \( u \in \mathcal{V}^3 \)

\[ u = u_1 \ e_1 = u_1 \ e_1 + u_2 \ e_2 + u_3 \ e_3 \]

with \( u_1, u_2, u_3 \) coordinates (components) of \( u \) relative to the basis \( e_1, e_2, e_3 \)

\[ u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^t \]

tensor calculus
• euklidian norm enables definition of scalar (inner) product
  \[ u \cdot v = \alpha \quad \alpha \in \mathbb{R} \]
  \[ u \cdot v = ||u|| ||v|| \cos \vartheta \]
  \[ ||u \cdot v|| \leq ||u|| ||v|| \]
• properties of scalar product
  \[ u \cdot v = v \cdot u \]
  \[ (\alpha u + \beta v) \cdot w = \alpha (u \cdot w) + \beta (v \cdot w) \]
  \[ w \cdot (\alpha u + \beta v) = \alpha (w \cdot u) + \beta (w \cdot v) \]
• positive definiteness \[ u \cdot u \geq 0, \quad u \cdot u = 0 \Leftrightarrow u = 0 \]
• orthogonality \[ u \cdot v = 0 \Leftrightarrow u \perp v \]

### vector algebra - scalar product

\[ \begin{align*}
  u \cdot v &= \alpha, \quad \alpha \in \mathbb{R} \\
  u \cdot v &= ||u|| ||v|| \cos \vartheta \\
  ||u \cdot v|| &\leq ||u|| ||v|| \\
  u \cdot v &= v \cdot u \\
  (\alpha u + \beta v) \cdot w &= \alpha (u \cdot w) + \beta (v \cdot w) \\
  w \cdot (\alpha u + \beta v) &= \alpha (w \cdot u) + \beta (w \cdot v) \\
  u \cdot u &\geq 0, \quad u \cdot u = 0 \Leftrightarrow u = 0 \\
  u \cdot v &= 0 \Leftrightarrow u \perp v
\end{align*} \]

### tensor calculus

• vector product
  \[ u \times v = w \]
  \[ w \in \mathcal{E}^3 \]
  \[ u \times v = ||u|| ||v|| \sin \vartheta \quad n \]
  \[ u \times v = 0 \Leftrightarrow u \parallel v \]
  \[ \begin{bmatrix}
    w_1 \\
    w_2 \\
    w_3
  \end{bmatrix} = \begin{bmatrix}
    u_2 v_3 - u_3 v_2 \\
    u_3 v_1 - u_1 v_3 \\
    u_1 v_2 - u_2 v_1
  \end{bmatrix} \]
• properties of vector product
  \[ u \times v = -v \times u \]
  \[ (\alpha u + \beta v) \times w = \alpha (u \times w) + \beta (v \times w) \]
  \[ u \cdot (u \times v) = 0 \]
  \[ (u \times v) \cdot (u \times v) = (u \cdot u) (v \cdot v) - (u \cdot v)^2 \]

### tensor algebra - second order tensors

• second order tensor
  \[ A = u \otimes v \]
  \[ A = A_{ij} e_i \otimes e_j \]
  \[ [A_{ij}] = \begin{bmatrix}
    A_{11} & A_{12} & A_{13} \\
    A_{21} & A_{22} & A_{23} \\
    A_{31} & A_{32} & A_{33}
  \end{bmatrix} \]
  with \[ A_{ij} = u_i v_j \]
  coordinates (components) of \( A \) relative to the basis \( e_1, e_2, e_3 \)
• transpose of second order tensor
  \[ A^t = (u \otimes v)^t = v \otimes u \]
  \[ A^t = A_{ji} e_j \otimes e_i \]
  \[ [A_{ji}] = \begin{bmatrix}
    A_{11} & A_{21} & A_{31} \\
    A_{12} & A_{22} & A_{32} \\
    A_{13} & A_{23} & A_{33}
  \end{bmatrix} \]
tensor calculus

- scalar (inner) product
  \[ A \cdot u = (A_{ij} e_i \otimes e_j) \cdot (u_k e_k) = \sum_{ij} A_{ij} u_k \delta_{jk} e_i = v_i e_i = v \]
  of second order tensor \( A \) and vector \( u \)

- zero and identity
  \[ 0 \cdot u = 0 \quad I \cdot u = u \]

- properties of scalar product
  \[ A \cdot (\alpha a + \beta b) = \alpha (A \cdot a) + \beta (A \cdot b) \]
  \[ (A + B) \cdot a = A \cdot a + B \cdot a \]
  \[ (\alpha A) \cdot a = \alpha (A \cdot a) \]

- scalar (inner) product
  \[ A \cdot B = (A_{ij} e_i \otimes e_j) \cdot (B_{kl} e_k \otimes e_l) = \sum_{ijkl} A_{ij} B_{kl} \delta_{ik} \delta_{jl} = \sum_{ij} A_{ij} B_{ij} \]
  of two second order tensors \( A, B \)

- zero and identity
  \[ 0 \cdot A = A \quad I \cdot A = A \]

- properties of scalar product
  \[ (A \cdot B) = (\alpha A) \cdot B = A \cdot (\alpha B) \]
  \[ A \cdot (B + C) = A \cdot B + A \cdot C \]
  \[ (A + B) \cdot C = A \cdot C + B \cdot C \]