13 - constitutive equations -

constitutive equations

constitutive equations [kənˈstɪ.tu.tɪv ɪˈkweɪ.ʒəns] in structural analysis, constitutive relations connect applied stresses or forces to strains or deformations. the constitutive relations for linear materials are linear. more generally, in physics, a constitutive equation is a relation between two physical quantities (often tensors) that is specific to a material, and does not follow directly from physical law. some constitutive equations are simply phenomenological; others are derived from first principles.

Wikipedia
The Free Encyclopedia

constitutive equations

constitutive equations [kənˈstɪ.tu.tɪv ɪˈkweɪ.ʒəns] or equations of state bring in the characterization of particular materials within continuum mechanics. mathematically, the purpose of these relations is to supply connections between kinematic, mechanical and thermal fields. physically, constitutive equations represent the various forms of idealized material response which serve as models of the behavior of actual substances.

Chadwick „Continuum mechanics" [1976]

tensor analysis - basic derivatives

$$\{\bullet \otimes \circ\}_{ijkl} = \{\bullet\}_{ij} \, \{\circ\}_{kl}$$

$$\{\bullet \,\overline{\otimes}\, \circ\}_{ijkl} = \{\bullet\}_{ik} \, \{\circ\}_{jl}$$

$$\{\bullet \,\underline{\otimes}\, \circ\}_{ijkl} = \{\bullet\}_{il} \, \{\circ\}_{jk}$$

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}} = \boldsymbol{I} \,\overline{\otimes}\, \boldsymbol{I} \qquad \frac{\partial F_{ij}}{\partial F_{kl}} = \delta_{ik} \, \delta_{jl}$$

$$\frac{\partial \boldsymbol{F}^{\text{-1}}}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{\text{-1}} \,\overline{\otimes}\, \boldsymbol{F}^{\text{-t}} \qquad \frac{\partial F^{\text{-1}}_{ij}}{\partial F_{kl}} = -F^{\text{-1}}_{ik} \, F^{\text{-1}}_{lj}$$

$$\frac{\partial \boldsymbol{F}^{\text{t}}}{\partial \boldsymbol{F}} = \boldsymbol{I} \,\underline{\otimes}\, \boldsymbol{I} \qquad \frac{\partial F_{ji}}{\partial F_{kl}} = \delta_{il} \, \delta_{jk}$$

$$\frac{\partial \boldsymbol{F}^{\text{-t}}}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{\text{-t}} \,\underline{\otimes}\, \boldsymbol{F}^{\text{-1}} \qquad \frac{\partial F^{\text{-1}}_{ji}}{\partial F_{kl}} = -F^{\text{-1}}_{li} \, F^{\text{-1}}_{jk}$$

tensor analysis - basic derivatives

$$\frac{\partial \det(\boldsymbol{F})}{\partial \boldsymbol{F}} = \det(\boldsymbol{F})\, \boldsymbol{F}^{\text{-t}} \qquad \frac{\partial \ln(\det(\boldsymbol{F}))}{\partial \boldsymbol{F}} = \boldsymbol{F}^{\text{-t}}$$

$$\frac{\partial \det(\boldsymbol{F}^{\text{-1}})}{\partial \boldsymbol{F}} = -\frac{1}{\det(\boldsymbol{F})}\, \boldsymbol{F}^{\text{-t}} \qquad \frac{\partial \ln(\det(\boldsymbol{F})^{\text{-1}})}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{\text{-t}}$$

$$\frac{\partial \ln(\det(\boldsymbol{F}))}{\partial(\det(\boldsymbol{F}))} = \frac{1}{\det(\boldsymbol{F}))} \qquad \frac{\partial \ln^2(\det(\boldsymbol{F}))}{\partial \boldsymbol{F}} = 2\, \ln(\det(\boldsymbol{F}))\boldsymbol{F}^{\text{-t}}$$

neo hooke'ian elasticity

- free energy $$\psi_0^{\text{neo}} = \tfrac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F})) + \tfrac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$$
- definition of stress
$$\boldsymbol{P}^{\text{neo}} = \mathrm{D}_{\boldsymbol{F}}\psi_0^{\text{neo}} = \tfrac{1}{2}\,\lambda_0\,2\,\ln(\det\boldsymbol{F})\,\boldsymbol{F}^{\text{-t}} + \tfrac{1}{2}\mu_0\,2\,\boldsymbol{F} - \mu_0\boldsymbol{F}^{\text{-t}} = \mu_0\,\boldsymbol{F} + [\,\lambda_0\ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{\text{-t}}$$
- definition of tangent operator
$$\mathbf{A}^{\text{neo}} = \mathrm{D}_{\boldsymbol{F}\boldsymbol{F}}\psi_0^{\text{neo}} = \mathrm{D}_{\boldsymbol{F}}\boldsymbol{P}^{\text{neo}} = \lambda_0\boldsymbol{F}^{\text{-t}}\otimes\boldsymbol{F}^{\text{-t}} + \mu_0\boldsymbol{I}\overline{\otimes}\boldsymbol{I} + [\mu_0 - \lambda_0\ln(\det(\boldsymbol{F}))]\,\boldsymbol{F}^{\text{-t}}\underline{\otimes}\boldsymbol{F}^{\text{-1}}$$

neo hooke'ian elasticity

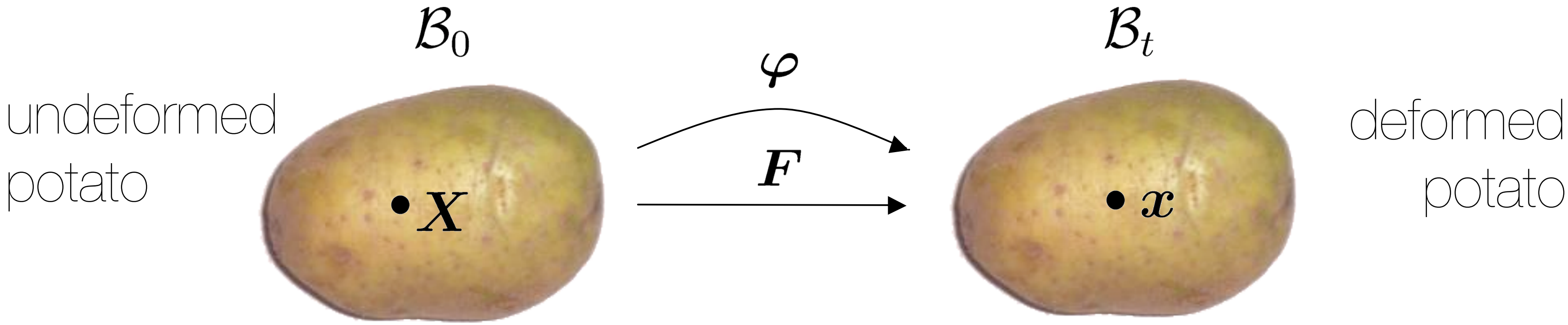
- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(F_{ij}))$
  $+ \frac{1}{2}\,\mu_0 [\, F_{ij}F_{ij} - n^{\text{dim}} - 2\,\ln(\det(F_{ij}))\,]$
- definition of stress

$$
\begin{aligned}
P_{ij}^{\text{neo}} &= \mathrm{D}_{F_{ij}}\psi_0^{\text{neo}} \\
&= \tfrac{1}{2}\,\lambda_0\, 2\,\ln(\det F_{ij}) F_{ji}^{-1} + \tfrac{1}{2}\mu_0\, 2\, F_{ij} - \mu_0 F_{ji}^{-1} \\
&= \mu_0\, F_{ij} + [\, \lambda_0 \ln(\det(F_{ij})) - \mu_0\,] F_{ji}^{-1}
\end{aligned}
$$

- definition of tangent operator

$$
\begin{aligned}
\mathsf{A}_{ijkl}^{\text{neo}} &= \mathrm{D}_{F_{ij}F_{kl}}\psi_0^{\text{neo}} = \mathrm{D}_{F_{kl}} P_{ij}^{\text{neo}} \\
&= \lambda_0 F_{ji}^{-1}\, F_{lk}^{-1} + \mu_0 I_{ik}\, I_{jl} \\
&\quad + [\mu_0 - \lambda_0 \ln(\det(F_{ij}))]\, F_{li}^{-1}\, F_{jk}^{-1}
\end{aligned}
$$

neo hooke'ian elasticity



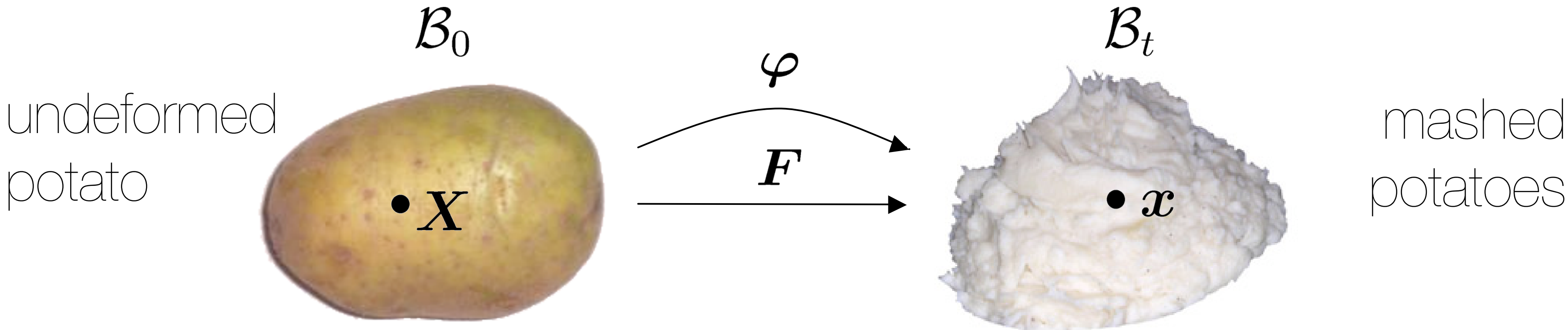
- free energy

$$\psi_0^{\text{neo}} = \tfrac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F})) + \tfrac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$$

- definition of stress

$$\boldsymbol{P}^{\text{neo}} = \text{D}_{\boldsymbol{F}}\psi_0^{\text{neo}} = \mu_0\,\boldsymbol{F} + [\,\lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{\text{-t}}$$

neo hooke'ian elasticity



- free energy

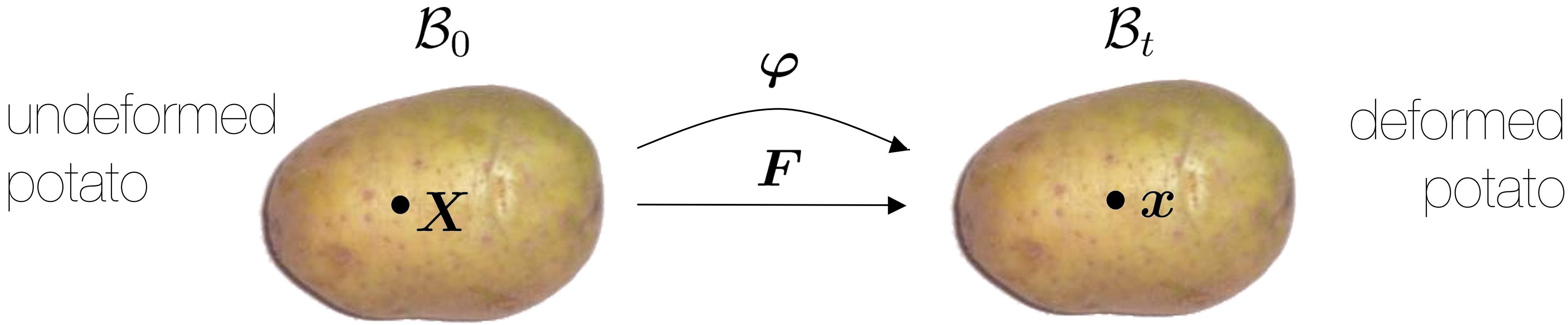
$$\psi^{\text{neo}} = \frac{1}{2}\,\lambda\,\ln^2(\det(\boldsymbol{F})) + \frac{1}{2}\,\mu[\,\boldsymbol{F}^{\text{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$$

- definition of stress

$$\boldsymbol{P}^{\text{neo}} = \rho_0 \mathrm{D}_F \psi = \mu\,\boldsymbol{F} + [\,\lambda\ln(\det(\boldsymbol{F})) - \mu\,]\boldsymbol{F}^{\text{-t}}$$

- remember! mashing potatoes is not an elastic process!

neo hooke'ian elasticity



- free energy $$\psi_0^{\text{neo}} = \tfrac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F})) + \tfrac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\rm t}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\rm dim} - 2\,\ln(\det(\boldsymbol{F}))\,]$$
- large strain - lamé parameters and bulk modulus $$\lambda = \frac{E\,\nu}{[1+\nu][1-2\,\nu]} \qquad \mu = \frac{E}{2\,[1+\nu]} \qquad \kappa = \frac{E}{3\,[1-2\,\nu]}$$
- small strain - young's modulus and poisson's ratio $$E = 3\,\kappa\,[1-2\,\nu] \qquad \nu = \frac{3\,\kappa - 2\,\mu}{2\,[3\,\kappa + \mu]}$$