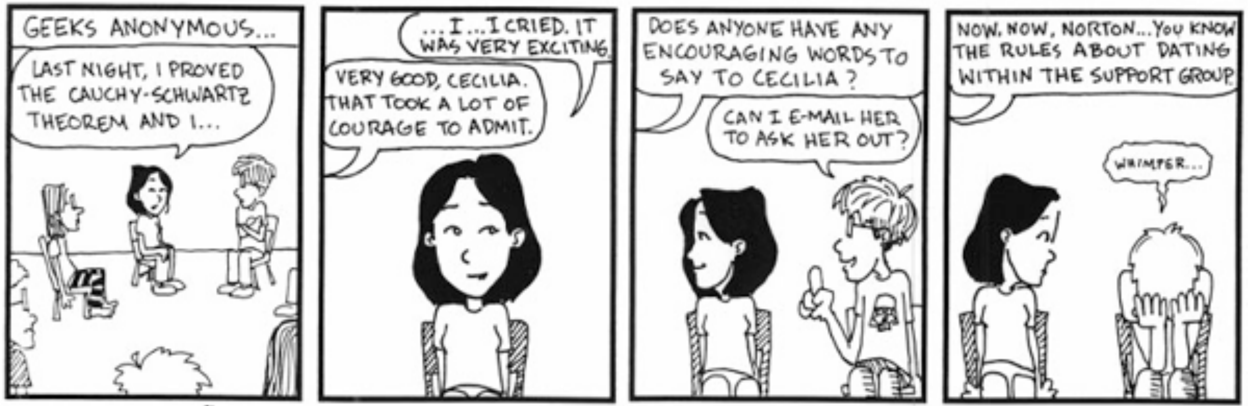
# 04 - tensor calculus - tensor analysis

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# tensor analysis - frechet derivative

- consider smooth differentiable scalar field $\Phi$ with

| | | |
|---|---|---|
| scalar argument | $\Phi : \mathcal{R} \rightarrow \mathcal{R};$ | $\Phi\,(x) = \alpha$ |
| vector argument | $\Phi : \mathcal{R}^3 \rightarrow \mathcal{R};$ | $\Phi\,(\boldsymbol{x}) = \alpha$ |
| tensor argument | $\Phi : \mathcal{R}^3 \times \mathcal{R}^3 \rightarrow \mathcal{R};$ | $\Phi(\boldsymbol{X}) = \alpha$ |

- frechet derivative (tensor notation)

scalar argument $$\mathrm{D}\,\Phi\,(x) = \frac{\partial \Phi(x)}{\partial x} = \partial_x \Phi\,(x)$$

vector argument $$\mathrm{D}\,\Phi\,(\boldsymbol{x}) = \frac{\partial \Phi(\boldsymbol{x})}{\partial \boldsymbol{x}} = \partial_{\boldsymbol{x}} \Phi\,(\boldsymbol{x})$$

tensor argument $$\mathrm{D}\,\Phi(\boldsymbol{X}) = \frac{\partial \Phi(\boldsymbol{X})}{\partial \boldsymbol{X}} = \partial_{\boldsymbol{X}} \Phi(\boldsymbol{X})$$

# tensor analysis - gateaux derivative

- consider smooth differentiable scalar field $\Phi$ with

scalar argument $\Phi : \mathcal{R} \rightarrow \mathcal{R}; \quad \Phi(x) = \alpha$

vector argument $\Phi : \mathcal{R}^3 \rightarrow \mathcal{R}; \quad \Phi(\boldsymbol{x}) = \alpha$

tensor argument $\Phi : \mathcal{R}^3 \times \mathcal{R}^3 \rightarrow \mathcal{R}; \quad \Phi(\boldsymbol{X}) = \alpha$

- gateaux derivative,i.e.,frechet wrt direction (tensor notation)

scalar argument $\mathrm{D}\,\Phi(x)\; u = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(x + \epsilon\, u)\,|_{\epsilon=0} \quad \forall\, u \in \mathcal{R}$

vector argument $\mathrm{D}\,\Phi(\boldsymbol{x}) \cdot \boldsymbol{u} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(\boldsymbol{x} + \epsilon\, \boldsymbol{u})\,|_{\epsilon=0} \quad \forall\, \boldsymbol{u} \in \mathcal{R}^3$

tensor argument $\mathrm{D}\,\Phi(\boldsymbol{X}) : \boldsymbol{U} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(\boldsymbol{X} + \epsilon\, \boldsymbol{U})|_{\epsilon=0} \quad \forall\, \boldsymbol{U} \in \mathcal{R}^3 \otimes \mathcal{R}^3$

# tensor analysis - gradient

- consider scalar- and vector field in domain $\mathcal{B} \in \mathcal{R}^3$

$$f : \mathcal{B} \to \mathcal{R} \qquad f : \boldsymbol{x} \to f \; (\boldsymbol{x})$$
$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f} \; (\boldsymbol{x})$$

- gradient of scalar- and vector field

$$\nabla f \,(\boldsymbol{x}) = \frac{\partial f \,(\boldsymbol{x})}{\partial x_i} = f_{,i}(\boldsymbol{x}) \; \boldsymbol{e}_i \qquad \nabla f \,(\boldsymbol{x}) = \begin{bmatrix} f_{,1} \\ f_{,2} \\ f_{,3} \end{bmatrix}$$

$$\nabla \boldsymbol{f} \,(\boldsymbol{x}) = \frac{\partial f_i \,(\boldsymbol{x})}{\partial x_j} = f_{i,j}(\boldsymbol{x}) \; \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \nabla \boldsymbol{f} \,(\boldsymbol{x}) = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix}$$

renders vector- and 2nd order tensor field

# tensor analysis - divergence

- consider vector- and 2nd order tensor field in domain $\mathcal{B}$

$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x})$$

$$\boldsymbol{F} : \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 \qquad \boldsymbol{F} : \boldsymbol{x} \to \boldsymbol{F}(\boldsymbol{x})$$

- divergence of vector- and 2nd order tensor field

$$\mathrm{div}(\boldsymbol{f}(\boldsymbol{x})) = \mathrm{tr}(\nabla \boldsymbol{f}(\boldsymbol{x})) = \nabla \boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{I}$$

$$\mathrm{div}(\boldsymbol{f}(\boldsymbol{x})) = f_{i,i}(\boldsymbol{x}) = f_{1,1} + f_{2,2} + f_{3,3}$$

$$\mathrm{div}(\boldsymbol{F}(\boldsymbol{x})) = \mathrm{tr}(\nabla \boldsymbol{F}(\boldsymbol{x})) = \nabla \boldsymbol{F}(\boldsymbol{x}) : \boldsymbol{I}$$

$$\mathrm{div}(\boldsymbol{F}(\boldsymbol{x})) = F_{ij,j}(\boldsymbol{x}) = \begin{bmatrix} F_{11,1} + F_{12,2} + F_{13,3} \\ F_{21,1} + F_{22,2} + F_{23,3} \\ F_{31,1} + F_{32,2} + F_{33,3} \end{bmatrix}$$

renders scalar- and vector field

# tensor analysis - laplace operator

- consider scalar- and vector field in domain $\mathcal{B} \in \mathcal{R}^3$

$$f : \mathcal{B} \to \mathcal{R} \qquad f : \boldsymbol{x} \to f(\boldsymbol{x})$$
$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x})$$

- laplace operator acting on scalar- and vector field

$$\Delta f(\boldsymbol{x}) = \operatorname{div}(\nabla(f(\boldsymbol{x}))) \qquad \Delta f(\boldsymbol{x}) = f_{,ii} = f_{,11} + f_{,22} + f_{,33}$$

$$\Delta \boldsymbol{f}(\boldsymbol{x}) = \operatorname{div}(\nabla(\boldsymbol{f}(\boldsymbol{x}))) \qquad \Delta \boldsymbol{f}(\boldsymbol{x}) = f_{i,jj} = \begin{bmatrix} f_{1,11} + f_{1,22} + f_{1,33} \\ f_{2,11} + f_{2,22} + f_{2,33} \\ f_{3,11} + f_{3,22} + f_{3,33} \end{bmatrix}$$

renders scalar- and vector field

# tensor analysis - transformation formulae

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$
\begin{aligned}
\alpha &: \mathcal{B} \to \mathcal{R} & \alpha &: \boldsymbol{x} \to \alpha \; (\boldsymbol{x}) \\
\boldsymbol{u} &: \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{u} &: \boldsymbol{x} \to \boldsymbol{u} \; (\boldsymbol{x}) \\
\boldsymbol{v} &: \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{v} &: \boldsymbol{x} \to \boldsymbol{v} \; (\boldsymbol{x}) \\
\boldsymbol{A} &: \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{A} &: \boldsymbol{x} \to \boldsymbol{A} \; (\boldsymbol{x})
\end{aligned}
$$

- useful transformation formulae (tensor notation)

$$
\begin{aligned}
\nabla \; (\alpha \, \boldsymbol{u}) &= \boldsymbol{u} \otimes \nabla \alpha + \alpha \, \nabla \boldsymbol{u} \\
\nabla \; (\boldsymbol{u} \cdot \boldsymbol{v}) &= \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} \\
\operatorname{div} \; (\alpha \, \boldsymbol{u}) &= \alpha \operatorname{div}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla \alpha \\
\operatorname{div} \; (\alpha \, \boldsymbol{A}) &= \alpha \operatorname{div}(\boldsymbol{A}) + \boldsymbol{A} \cdot \nabla \alpha \\
\operatorname{div} \; (\boldsymbol{u} \cdot \boldsymbol{A}) &= \boldsymbol{u} \cdot \operatorname{div}(\boldsymbol{A}) + \boldsymbol{A} : \nabla \boldsymbol{u} \\
\operatorname{div} \; (\boldsymbol{u} \otimes \boldsymbol{v}) &= \boldsymbol{u} \operatorname{div}(\boldsymbol{v}) + \boldsymbol{v} \cdot \nabla \boldsymbol{u}^{\mathrm{t}}
\end{aligned}
$$

# tensor analysis - transformation formulae

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$
\begin{array}{llll}
\alpha : & \mathcal{B} \to \mathcal{R} & \alpha : & x_k \to \alpha \quad (x_k) \\
u_i : & \mathcal{B} \to \mathcal{R}^3 & u_i : & x_k \to u_i \quad (x_k) \\
v_i : & \mathcal{B} \to \mathcal{R}^3 & v_i : & x_k \to v_i \quad (x_k) \\
A_{ij} : & \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & A_{ij} : & x_k \to A_{ij} \quad (x_k)
\end{array}
$$

- useful transformation formulae (index notation)

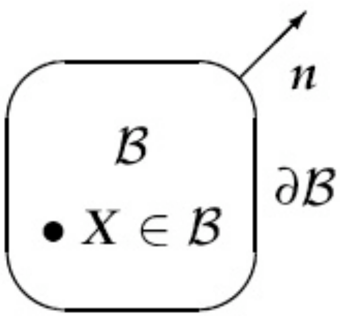
$$
\begin{aligned}
(\alpha\, u_i)_{,j} &= u_i\, \alpha_{,j} + \alpha\, u_{i,j} \\
(u_i\, v_i)_{,j} &= u_i\, v_{i,j} + v_i\, u_{i,j} \\
(\alpha\, u_i)_{,i} &= \alpha\, u_{i,i} + u_i\, \alpha_{,i} \\
(\alpha\, A_{ij})_{,j} &= \alpha\, A_{ij,j} + A_{ij}\, \alpha_{,j} \\
(u_i\, A_{ij})_{,j} &= u_i\, A_{ij,j} + A_{ij}\, u_{i,j} \\
(u_i\, v_j)_{,j} &= u_i\, v_{j,j} + v_j\, u_{i,j}
\end{aligned}
$$

# tensor analysis - integral theorems

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$
\begin{aligned}
\alpha &: \mathcal{B} \to \mathcal{R} & \alpha &: \boldsymbol{x} \to \alpha\ (\boldsymbol{x}) \\
\boldsymbol{u} &: \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{u} &: \boldsymbol{x} \to \boldsymbol{u}\ (\boldsymbol{x}) \\
\boldsymbol{A} &: \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{A} &: \boldsymbol{x} \to \boldsymbol{A}\ (\boldsymbol{x})
\end{aligned}
$$

- integral theorems (tensor notation)



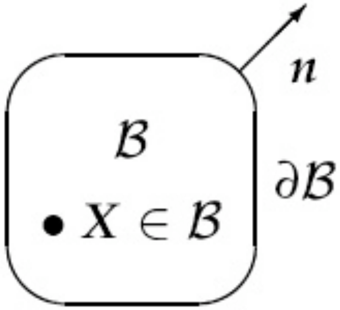

$$
\begin{aligned}
\int_{\partial\mathcal{B}} \alpha\, \boldsymbol{n}\, \mathrm{d}A &= \int_{\mathcal{B}} \nabla\alpha\, \mathrm{d}V & &\text{green} \\
\int_{\partial\mathcal{B}} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}A &= \int_{\mathcal{B}} \mathrm{div}(\boldsymbol{u})\, \mathrm{d}V & &\text{gauss} \\
\int_{\partial\mathcal{B}} \boldsymbol{A} \cdot \boldsymbol{n}\, \mathrm{d}A &= \int_{\mathcal{B}} \mathrm{div}(\boldsymbol{A})\, \mathrm{d}V & &\text{gauss}
\end{aligned}
$$

# tensor analysis - integral theorems

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{aligned} \alpha &: \quad \mathcal{B} \to \mathcal{R} & \alpha &: \quad x_k \to \alpha \quad (x_k) \\ u_i &: \quad \mathcal{B} \to \mathcal{R}^3 & u_i &: \quad x_k \to u_i \quad (x_k) \\ A_{ij} &: \quad \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & A_{ij} &: \quad x_k \to A_{ij} \quad (x_k) \end{aligned}$$

- integral theorems (tensor notation)




$$\int_{\partial\mathcal{B}} \alpha\, n_i \,\mathrm{d}A = \int_{\mathcal{B}} \alpha_{,i} \,\mathrm{d}V \qquad \text{green}$$

$$\int_{\partial\mathcal{B}} u_i\, n_i \,\mathrm{d}A = \int_{\mathcal{B}} u_{i,i} \,\mathrm{d}V \qquad \text{gauss}$$

$$\int_{\partial\mathcal{B}} A_{ij}\, n_j \,\mathrm{d}A = \int_{\mathcal{B}} A_{ij,j} \,\mathrm{d}V \qquad \text{gauss}$$

# voigt / matrix vector notation

- strain tensors as vectors in voigt notation

$$E_{ij} = \begin{bmatrix} E_{11} & E_{12} & E_{31} \\ E_{12} & E_{22} & E_{23} \\ E_{31} & E_{23} & E_{33} \end{bmatrix}$$

$$E^{\text{voigt}} = [E_{11}, E_{22}, E_{33}, 2\,E_{12}, 2\,E_{23}, 2\,E_{31}]^{\text{t}}$$

- stress tensors as vectors in voigt notation

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{31} \\ S_{12} & S_{22} & S_{23} \\ S_{31} & S_{23} & S_{33} \end{bmatrix}$$

$$S^{\text{voigt}} = [S_{11}, S_{22}, S_{33}, S_{12}, S_{23}, S_{31}]^{\text{t}}$$

- why are strain & stress different? check energy expression!

$$\psi = \frac{1}{2}\,\boldsymbol{E} : \boldsymbol{S} \qquad \psi = \frac{1}{2}\,\boldsymbol{E}^{\text{voigt}} : \boldsymbol{S}^{\text{voigt}}$$

# voigt / matrix vector notation

- fourth order material operators as matrix in voigt notation

$$
C^{\text{voigt}} = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\
C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\
C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\
C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\
C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\
C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3331}
\end{bmatrix}
$$

- why are strain & stress different? check these expressions!

$$\boldsymbol{S} = \mathbf{C} : \boldsymbol{E} \qquad \boldsymbol{S}^{\text{voigt}} = \boldsymbol{C}^{\text{voigt}} \cdot \boldsymbol{E}^{\text{voigt}}$$