# 03 - tensor calculus - tensor analysis

# tensor algebra - invariants

- (principal) invariants of second order tensor

$$
\begin{aligned}
I_A &= \operatorname{tr}(\boldsymbol{A}) \\
II_A &= \tfrac{1}{2}\left[\operatorname{tr}^2(\boldsymbol{A}) - \operatorname{tr}(\boldsymbol{A}^2)\right] \\
III_A &= \det(\boldsymbol{A})
\end{aligned}
$$

- derivatives of invariants wrt second order tensor

$$
\begin{aligned}
\partial_{\boldsymbol{A}}\ I_A &= \boldsymbol{I} \\
\partial_{\boldsymbol{A}}\ II_A &= I_A\,\boldsymbol{I} - \boldsymbol{A} \\
\partial_{\boldsymbol{A}}\ III_A &= III_A\,\boldsymbol{A}^{-\mathrm{t}}
\end{aligned}
$$

# tensor algebra - trace

- trace of second order tensor

$$\mathrm{tr}\,(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$$

$$\begin{aligned} \mathrm{tr}(\boldsymbol{A}) &= \mathrm{tr}(A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \\ &= A_{ij}\, \mathrm{tr}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = A_{ij}\, \boldsymbol{e}_i \cdot \boldsymbol{e}_j \\ &= A_{ij}\, \delta_{ij} = A_{ii} = A_{11} + A_{22} + A_{33} \end{aligned}$$

- properties of traces of second order tensors

$$\begin{aligned} &\mathrm{tr}\,(\boldsymbol{I}) = 3 \\ &\mathrm{tr}\,(\boldsymbol{A}^{\mathrm{t}}) = \mathrm{tr}\,(\boldsymbol{A}) \\ &\mathrm{tr}\,(\boldsymbol{A} \cdot \boldsymbol{B}) = \mathrm{tr}\,(\boldsymbol{B} \cdot \boldsymbol{A}) \\ &\mathrm{tr}\,(\alpha\, \boldsymbol{A} + \beta\, \boldsymbol{B}) = \alpha\, \mathrm{tr}(\boldsymbol{A}) + \beta\, \mathrm{tr}(\boldsymbol{B}) \\ &\mathrm{tr}\,(\boldsymbol{A} \cdot \boldsymbol{B}^{\mathrm{t}}) = \boldsymbol{A} : \boldsymbol{B} \\ &\mathrm{tr}\,(\boldsymbol{A}) = \mathrm{tr}\,(\boldsymbol{A} \cdot \boldsymbol{I}) = \boldsymbol{A} : \boldsymbol{I} \end{aligned}$$

# tensor algebra - determinant

- determinant of second order tensor $III_A = \det(\boldsymbol{A})$

$$\begin{aligned}\det(\boldsymbol{A}) &= \det(A_{ij}) = \tfrac{1}{6}\, e_{ijk}\, e_{abc}\, A_{ia}\, A_{jb}\, A_{kc} \\ &= A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} \\ &- A_{11}A_{23}A_{32} - A_{22}A_{31}A_{13} - A_{33}A_{12}A_{21}\end{aligned}$$

- properties of determinants of second order tensors

$$\begin{aligned}&\det(\boldsymbol{I}) = 1 \\ &\det(\boldsymbol{A}^{\mathrm{t}}) = \det(\boldsymbol{A}) \\ &\det(\alpha\,\boldsymbol{A}) = \alpha^3 \det(\boldsymbol{A}) \\ &\det(\boldsymbol{A}\cdot\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) \\ &\det(\boldsymbol{u}\otimes\boldsymbol{v}) = 0\end{aligned}$$

# tensor algebra - determinant

- determinant defining vector product

$$\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} u_1 & v_1 & \boldsymbol{e}_1 \\ u_2 & v_2 & \boldsymbol{e}_2 \\ u_3 & v_3 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 & - & u_3 v_2 \\ u_3 v_1 & - & u_1 v_3 \\ u_1 v_2 & - & u_2 v_1 \end{bmatrix}$$

- determinant defining scalar triple product

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

# tensor algebra - inverse

- inverse of second order tensor

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I} \quad \text{in particular } \boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{u} \quad \boldsymbol{A}^{-1} \cdot \boldsymbol{v} = \boldsymbol{u}$$

- adjoint and cofactor

$$\boldsymbol{A}^{\mathrm{adj}} = \det(\boldsymbol{A})\, \boldsymbol{A}^{-1} \qquad \boldsymbol{A}^{\mathrm{cof}} = \det(\boldsymbol{A})\, \boldsymbol{A}^{-\mathrm{t}} = (\boldsymbol{A}^{\mathrm{adj}})^{\mathrm{t}}$$

$$\partial_{\boldsymbol{A}} \det(\boldsymbol{A}) = \det(\boldsymbol{A})\, \boldsymbol{A}^{-\mathrm{t}} = III_A\, \boldsymbol{A}^{-\mathrm{t}} = \boldsymbol{A}^{\mathrm{cof}}$$

- properties of inverse

$$\det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A})$$

$$\begin{aligned} (\boldsymbol{A}^{-1})^{-1} &= \boldsymbol{A} \\ (\alpha\, \boldsymbol{A}^{-1})^{-1} &= \alpha^{-1}\, \boldsymbol{A} \\ (\boldsymbol{A} \cdot \boldsymbol{B})^{-1} &= \boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1} \end{aligned}$$

# tensor algebra - spectral decomposition

- eigenvalue problem of second order tensor

$$\boldsymbol{A}\cdot\boldsymbol{n}_A = \lambda_A\,\boldsymbol{n}_A \qquad [\,\boldsymbol{A} - \lambda_A\,\boldsymbol{I}\,]\cdot\boldsymbol{n}_A = \boldsymbol{0}$$

- solution $\det(\boldsymbol{A} - \lambda_A\,\boldsymbol{I}) = 0$ in terms of scalar triple product

$$[\boldsymbol{A}\cdot\boldsymbol{u} - \lambda_A\boldsymbol{u},\, \boldsymbol{A}\cdot\boldsymbol{v} - \lambda_A\boldsymbol{v},\, \boldsymbol{A}\cdot\boldsymbol{w} - \lambda_A\boldsymbol{w}] = 0$$

- characteristic equation

$$\lambda_A^3 - I_A\,\lambda_A^2 + II_A\,\lambda_A - III_A = 0$$

$$\begin{aligned} I_A &= \mathrm{tr}\,(\boldsymbol{A}) \\ II_A &= \tfrac{1}{2}\,[\mathrm{tr}^2\,(\boldsymbol{A}) - \mathrm{tr}\,(\boldsymbol{A}^2)] \\ III_A &= \det\,(\boldsymbol{A}) \end{aligned}$$

- spectral decomposition

$$\boldsymbol{A} = \textstyle\sum_{i=1}^{3}\lambda_{Ai}\,\boldsymbol{n}_{Ai}\otimes\boldsymbol{n}_{Ai}$$

- cayleigh hamilton theorem

$$\boldsymbol{A}^3 - I_A\,\boldsymbol{A}^2 + II_A\,\boldsymbol{A} - III_A\,\boldsymbol{I} = \boldsymbol{0}$$

# tensor algebra - sym/skw decomposition

- symmetric - skew-symmetric decomposition

$$\boldsymbol{A} = \tfrac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\mathrm{t}}] + \tfrac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\mathrm{t}}] = \boldsymbol{A}^{\mathrm{sym}} + \boldsymbol{A}^{\mathrm{skw}}$$

- symmetric and skew-symmetric tensor

$$\boldsymbol{A}^{\mathrm{sym}} = (\boldsymbol{A}^{\mathrm{sym}})^{\mathrm{t}} \qquad \boldsymbol{A}^{\mathrm{skw}} = -(\boldsymbol{A}^{\mathrm{skw}})^{\mathrm{t}}$$

- symmetric tensor

$$\boldsymbol{A}^{\mathrm{sym}} = \tfrac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\mathrm{t}}] = \mathsf{I}^{\mathrm{sym}} : \boldsymbol{A}$$

- skew-symmetric tensor

$$\boldsymbol{A}^{\mathrm{skw}} = \tfrac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\mathrm{t}}] = \mathsf{I}^{\mathrm{skw}} : \boldsymbol{A}$$

# tensor algebra - symmetric tensor

- symmetric second order tensor

$$\boldsymbol{A}^{\text{sym}} = \tfrac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\text{t}}] \qquad \boldsymbol{A}^{\text{sym}} = (\boldsymbol{A}^{\text{sym}})^{\text{t}} \qquad \boldsymbol{A}^{\text{sym}} = \boldsymbol{S}$$

- processes three real eigenvalues and corresp.eigenvectors

$$\boldsymbol{S} = \textstyle\sum_{i=1}^{3} \lambda_{Si} \, (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \qquad \begin{aligned} I_S &= \lambda_{S1} + \lambda_{S2} + \lambda_{S3} \\ II_S &= \lambda_{S2}\,\lambda_{S3} + \lambda_{S3}\,\lambda_{S1} + \lambda_{S1}\,\lambda_{S2} \\ III_S &= \lambda_{S1}\,\lambda_{S2}\,\lambda_{S3} \end{aligned}$$

- square root, inverse, exponent and log

$$\begin{aligned} \sqrt{\boldsymbol{S}} &= \textstyle\sum_{i=1}^{3} \sqrt{\lambda_{Si}} \, (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \boldsymbol{S}^{-1} &= \textstyle\sum_{i=1}^{3} \lambda_{Si}^{-1} \, (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \exp(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3} \exp(\lambda_{Si}) \, (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \ln(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3} \ln(\lambda_{Si}) \, (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \end{aligned}$$

# tensor algebra - skew-symmetric tensor

- skew-symmetric second order tensor

$$\boldsymbol{A}^{\text{skw}} = \tfrac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\text{t}}] \qquad \boldsymbol{A}^{\text{skw}} = -(\boldsymbol{A}^{\text{skw}})^{\text{t}} \qquad \boldsymbol{A}^{\text{skw}} = \boldsymbol{W}$$

- processes three independent entries defining axial vector

$$\boldsymbol{w} = -\tfrac{1}{2} \overset{3}{\boldsymbol{e}} : \boldsymbol{W} \qquad \boldsymbol{w} = -\overset{3}{\boldsymbol{e}} \cdot \boldsymbol{w} \quad \text{such that} \quad \boldsymbol{W} \cdot \boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{v}$$

- invariants of skew-symmetric tensor

$$\begin{aligned} I_W &= \operatorname{tr}(\boldsymbol{W}) &&= 0 \\ II_W &= \boldsymbol{w} \cdot \boldsymbol{w} && \\ III_W &= \det(\boldsymbol{W}) &&= 0 \end{aligned}$$

# tensor algebra - vol/dev decomposition

- volumetric - deviatoric decomposition

$$\boldsymbol{A} = \boldsymbol{A}^{\text{vol}} + \boldsymbol{A}^{\text{dev}}$$

- volumetric and deviatoric tensor

$$\text{tr}(\boldsymbol{A}^{\text{vol}}) = \text{tr}(\boldsymbol{A}) \qquad \text{tr}(\boldsymbol{A}^{\text{dev}}) = 0$$

- volumetric tensor

$$\boldsymbol{A}^{\text{vol}} = \tfrac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\,\boldsymbol{I} = \mathsf{I}^{\text{vol}} : \boldsymbol{A}$$

- deviatoric tensor

$$\boldsymbol{A}^{\text{dev}} = \boldsymbol{A} - \tfrac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\,\boldsymbol{I} = \mathsf{I}^{\text{dev}} : \boldsymbol{A}$$

# tensor algebra - orthogonal tensor

- orthogonal second order tensor $\boldsymbol{Q} \in \mathrm{S0}(3)$

$$\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathrm{t}} \quad \Leftrightarrow \quad \boldsymbol{Q}^{\mathrm{t}} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathrm{t}} = \boldsymbol{I}$$

- decomposition of second order tensor

$$\boldsymbol{A} = \boldsymbol{Q} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{Q}$$

such that $\boldsymbol{a} \cdot \boldsymbol{U} \cdot \boldsymbol{a} \geq 0$ and $\boldsymbol{a} \cdot \boldsymbol{V} \cdot \boldsymbol{a} \geq 0$

- proper orthogonal tensor $\boldsymbol{Q} \in \mathrm{S0}(3)$ has eigenvalue $\lambda_Q = 1$

$$\boldsymbol{Q} \cdot \boldsymbol{n}_Q = \boldsymbol{n}_Q \qquad \text{with} \qquad [Q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\cos\varphi & +\sin\varphi \\ 0 & -\sin\varphi & +\cos\varphi \end{bmatrix}$$

interpretation: finite rotation around axis $\boldsymbol{n}_Q$