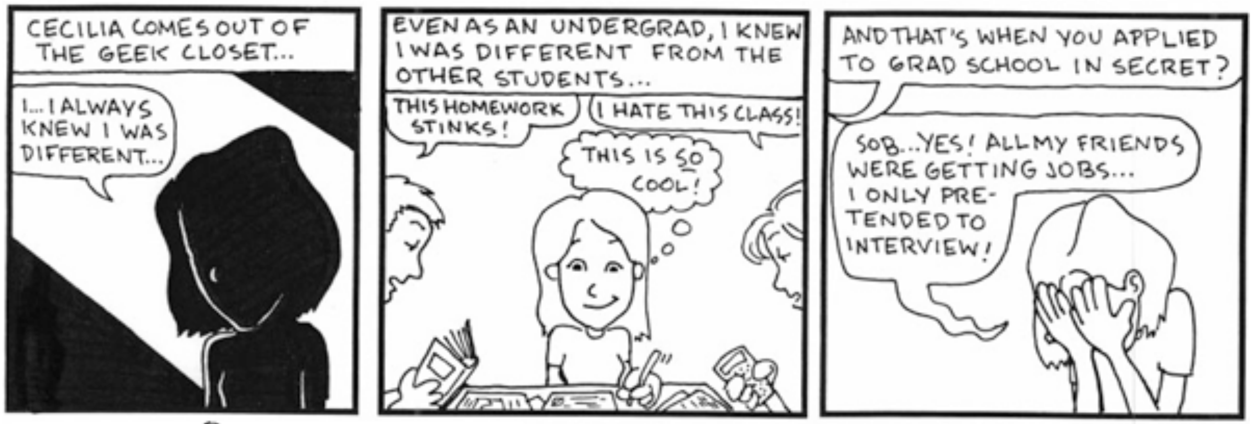
02 - tensor calculus - tensor algebra



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tensor calculus

tensor [ˈten.sor] the word tensor was introduced in 1846 by william rowan hamilton. it was used in its current meaning by woldemar voigt in 1899. tensor calculus was developed around 1890 by gregorio ricci-curbastro under the title absolute differential calculus. in the 20th century, the subject came to be known as tensor analysis, and achieved broader acceptance with the introduction of einsteins's theory of general relativity around 1915. tensors are used also in other fields such as continuum mechanics.

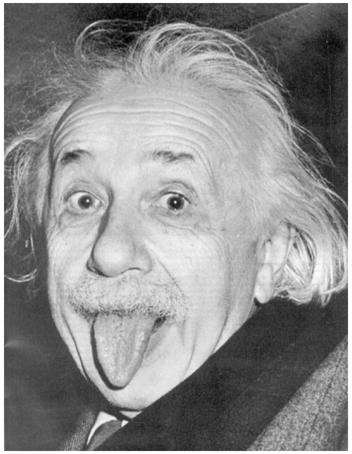
tensor calculus - repetition

- vector algebra
  notation, euklidian vector space, scalar product, vector product, scalar triple product
- tensor algebra
  notation, scalar products, dyadic product, invariants, trace, determinant, inverse, spectral decomposition, sym-skew decomposition, vol-dev decomposition, orthogonal tensor
- tensor analysis
  derivatives, gradient, divergence, laplace operator, integral transformations

vector algebra - notation

- einstein's summation convention

$$u_i = \sum_{j=1}^{3} A_{ij}\, x_j + b_i = A_{ij}\, x_j + b_i$$



- summation over any indices that appear twice in a term

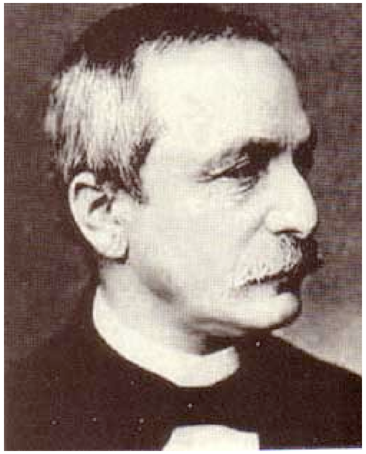
$$\begin{aligned} u_1 &= A_{11}\, x_1 + A_{12}\, x_2 + A_{13}\, x_3 + b_1 \\ u_2 &= A_{21}\, x_1 + A_{22}\, x_2 + A_{23}\, x_3 + b_2 \\ u_3 &= A_{31}\, x_1 + A_{32}\, x_2 + A_{33}\, x_3 + b_3 \end{aligned}$$

vector algebra - notation

- kronecker symbol

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

$$u_i = \delta_{ij}\, u_j$$



- permutation symbol

$$\overset{3}{e}_{ijk} = \begin{cases} 1 & \text{for} \quad \{i,j,k\} \quad \text{... even permutation} \\ -1 & \text{for} \quad \{i,j,k\} \quad \text{... odd permutation} \\ 0 & \quad\quad\quad\quad\quad\quad \text{... else} \end{cases}$$

vector algebra - euklidian vector space

- euklidian vector space $\mathcal{V}^3$

$$\begin{array}{lll} \alpha, \beta \in \mathcal{R} & \mathcal{R} & \dots \text{ real numbers} \\ \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}^3 & \mathcal{V}^3 & \dots \text{ linear vector space} \end{array}$$

- $\mathcal{V}^3$ is defined through the following axioms

$$\begin{aligned} \alpha\,(\boldsymbol{u} + \boldsymbol{v}) &= \alpha\,\boldsymbol{u} + \alpha\,\boldsymbol{v} \\ (\alpha + \beta)\,\boldsymbol{u} &= \alpha\,\boldsymbol{u} + \beta\,\boldsymbol{u} \\ (\alpha\,\beta)\,\boldsymbol{u} &= \alpha\,(\beta\,\boldsymbol{u}) \end{aligned}$$

- zero element and identity

$$0\,\boldsymbol{u} = \boldsymbol{0} \qquad 1\,\boldsymbol{u} = \boldsymbol{u}$$

- linear independence of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathcal{V}^3$ if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is the only (trivial) solution to $\alpha_i\,\boldsymbol{e}_i = 0$

vector algebra - euklidian vector space

- euklidian vector space $\mathcal{V}^3$ equipped with norm

$$n : \; \mathcal{V}^3 \rightarrow \mathcal{R} \qquad \dots \text{ norm}$$

- norm defined through the following axioms

$$n(\boldsymbol{u}) \geq 0 \qquad n(\boldsymbol{u}) = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$$

$$n(\alpha \, \boldsymbol{u}) = |\alpha| \, n(\boldsymbol{u})$$

$$n(\boldsymbol{u} + \boldsymbol{v}) \leq n(\boldsymbol{u}) + n(\boldsymbol{v})$$

$$n^2(\boldsymbol{u} + \boldsymbol{v}) + n^2(\boldsymbol{u} - \boldsymbol{v}) = 2 \left[n^2(\boldsymbol{u}) + n^2(\boldsymbol{v}) \right]$$

vector algebra - euklidian vector space

- euklidian vector space $\mathcal{E}^3$ equipped with euklidian norm

$$n : \ \mathcal{E}^3 \rightarrow \mathcal{R} \qquad \text{... euklidian norm}$$

$$n(\boldsymbol{u}) = ||\boldsymbol{u}|| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = [u_1^2 + u_2^2 + u_3^2]^{1/2}$$

- representation of 3d vector $\boldsymbol{u} \in \mathcal{E}^3$

$$\boldsymbol{u} = u_i \ \boldsymbol{e}_i = u_1 \ \boldsymbol{e}_1 + u_2 \ \boldsymbol{e}_2 + u_3 \ \boldsymbol{e}_3$$

with u_1, u_2, u_3 coordinates (components) of $\boldsymbol{u}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

$$\boldsymbol{u} = [\, u_1, u_2, u_3 \,]^{\mathrm{t}}$$

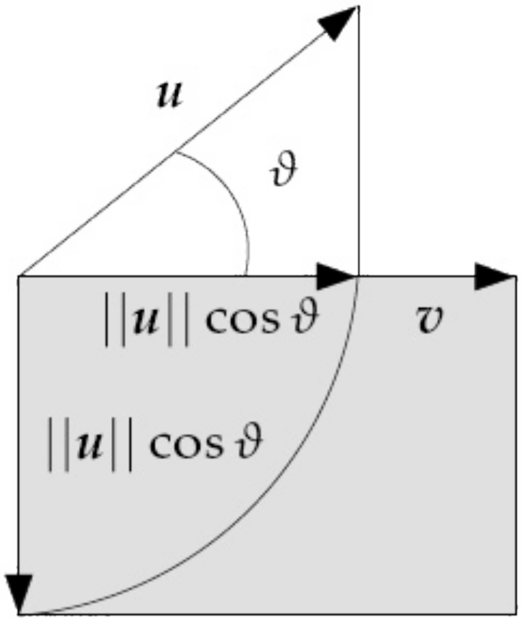
vector algebra - scalar product

- euklidian norm enables definition of scalar (inner) product

$$\boldsymbol{u}\cdot\boldsymbol{v} = \alpha \qquad \alpha \in \mathcal{R}$$

$$\boldsymbol{u}\cdot\boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}||\,\cos\vartheta$$

$$||\boldsymbol{u}\cdot\boldsymbol{v}|| \le ||\boldsymbol{u}||\,||\boldsymbol{v}||$$



- properties of scalar product

$$\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u}$$

$$(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v})\cdot\boldsymbol{w} = \alpha\,(\boldsymbol{u}\cdot\boldsymbol{w}) + \beta\,(\boldsymbol{v}\cdot\boldsymbol{w})$$

$$\boldsymbol{w}\cdot(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v}) = \alpha\,(\boldsymbol{w}\cdot\boldsymbol{u}) + \beta\,(\boldsymbol{w}\cdot\boldsymbol{v})$$

- positive definiteness $\quad \boldsymbol{u}\cdot\boldsymbol{u} \ge 0, \quad \boldsymbol{u}\cdot\boldsymbol{u} = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$
- orthogonality $\quad \boldsymbol{u}\cdot\boldsymbol{v} = 0 \quad \Leftrightarrow \quad \boldsymbol{u}\perp\boldsymbol{v}$

vector algebra - vector product

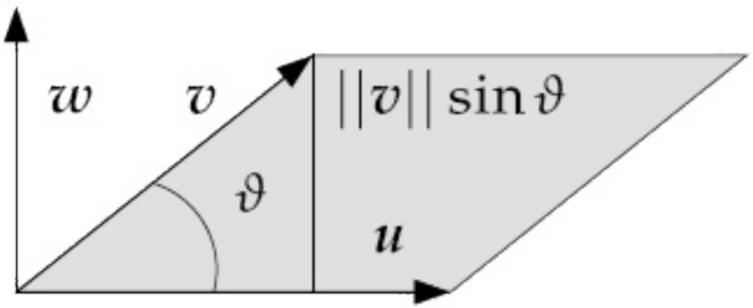
- vector product

$$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w} \qquad \boldsymbol{w} \in \mathcal{E}^3$$

$$\boldsymbol{u} \times \boldsymbol{v} = ||\boldsymbol{u}||\, ||\boldsymbol{v}|| \, \sin\vartheta \, \boldsymbol{n}$$

$$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0} \qquad \Leftrightarrow \qquad \boldsymbol{u} \parallel \boldsymbol{v}$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_2\, v_3 & - & u_3\, v_2 \\ u_3\, v_1 & - & u_1\, v_3 \\ u_1\, v_2 & - & u_2\, v_1 \end{bmatrix}$$



- properties of vector product

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$$

$$(\alpha\, \boldsymbol{u} + \beta\, \boldsymbol{v}) \times \boldsymbol{w} = \alpha\, (\boldsymbol{u} \times \boldsymbol{w}) + \beta\, (\boldsymbol{v} \times \boldsymbol{w})$$

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0$$

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{u} \cdot \boldsymbol{u})\, (\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})^2$$

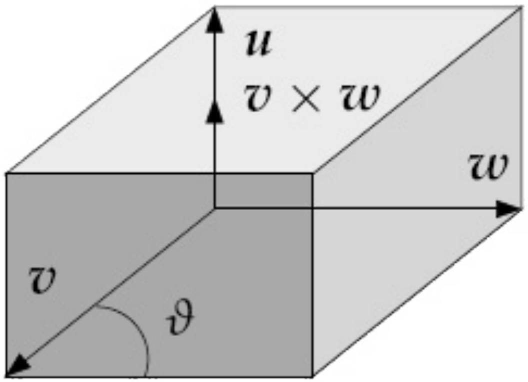
vector algebra - scalar triple product

- scalar triple product

$$[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] = \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = \alpha \quad \alpha \in \mathcal{R}$$

$$\boldsymbol{u}\times\boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}||\,\sin\vartheta\,\boldsymbol{n}$$ area

$$[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] = \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})$$ volume



$$\alpha = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_3w_1)$$

- properties of scalar triple product

$$[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] = [\boldsymbol{v},\boldsymbol{w},\boldsymbol{u}] = [\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}]$$

$$= -\,[\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}] = -\,[\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}] = -\,[\boldsymbol{w},\boldsymbol{v},\boldsymbol{u}]$$

$$[\alpha\boldsymbol{u}+\beta\boldsymbol{v},\boldsymbol{w},\boldsymbol{d}] = \alpha\,[\boldsymbol{u},\boldsymbol{w},\boldsymbol{d}] + \beta\,[\boldsymbol{v},\boldsymbol{w},\boldsymbol{d}]$$

- linear independency $[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \neq 0$

tensor algebra - second order tensors

- second order tensor

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} \qquad\qquad \boldsymbol{u} = u_i \, \boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{v} = v_j \, \boldsymbol{e}_j$$

$$\boldsymbol{A} = A_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad\qquad [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

with $A_{ij} = u_i \, v_j$ coordinates (components) of $\boldsymbol{A}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- transpose of second order tensor

$$\boldsymbol{A}^{\mathrm{t}} = (\boldsymbol{u} \otimes \boldsymbol{v})^{\mathrm{t}} = \boldsymbol{v} \otimes \boldsymbol{u}$$

$$\boldsymbol{A}^{\mathrm{t}} = A_{ji} \, \boldsymbol{e}_j \otimes \boldsymbol{e}_i \qquad\qquad [A_{ji}] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

tensor algebra - second order tensors

- second order unit tensor in terms of kronecker symbol

$$\boldsymbol{I} = \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

with δ_{ij} coordinates (components) of $\boldsymbol{I}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- matrix representation of coordinates

$$[\delta_{ji}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- identity

$$\boldsymbol{I} \cdot \boldsymbol{u} = \boldsymbol{u} \qquad \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot u_j\, \boldsymbol{e}_j = u_i\, \boldsymbol{e}_i$$

tensor algebra - third order tensors

- third order tensor

$$\overset{3}{\boldsymbol{a}} = \boldsymbol{A} \otimes \boldsymbol{v} \qquad \boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \text{ and } \boldsymbol{v} = v_k\, \boldsymbol{e}_k$$

$$\overset{3}{\boldsymbol{a}} = a_{ijk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$$

with $a_{ijk} = A_{ij}\, v_k$ coordinates (components) of $\boldsymbol{A}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- third order permutation tensor in terms of permutation symbol $\overset{3}{e}_{ijk}$

$$\overset{3}{\boldsymbol{e}} = e_{ijk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$$

tensor algebra - fourth order tensors

- fourth order tensor

$$\boldsymbol{\mathsf{A}} = \boldsymbol{A} \otimes \boldsymbol{B} \qquad \boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \text{ and } \boldsymbol{B} = B_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

$$\boldsymbol{\mathsf{A}} = A_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

with $A_{ijkl} = A_{ij}\, B_{kl}$ coordinates (components) of $\boldsymbol{\mathsf{A}}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- fourth order unit tensor

$$\boldsymbol{\mathsf{I}} = \delta_{ik}\, \delta_{jl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \boldsymbol{\mathsf{I}} : \boldsymbol{A} = \boldsymbol{A}$$

- transpose of fourth order unit tensor

$$\boldsymbol{\mathsf{I}}^{\mathrm{t}} = \delta_{il}\, \delta_{jk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \boldsymbol{\mathsf{I}}^{\mathrm{t}} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{t}}$$

tensor algebra - fourth order tensors

- symmetric fourth order unit tensor

$$\mathsf{I}^{\text{sym}} = \frac{1}{2}\left[\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\text{sym}} : \boldsymbol{A} = \boldsymbol{A}^{\text{sym}}$$

- screw-symmetric fourth order unit tensor

$$\mathsf{I}^{\text{skw}} = \frac{1}{2}\left[\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\text{skw}} : \boldsymbol{A} = \boldsymbol{A}^{\text{skw}}$$

- volumetric fourth order unit tensor

$$\mathsf{I}^{\text{vol}} = \frac{1}{3}\,\delta_{ij}\,\delta_{kl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\text{vol}} : \boldsymbol{A} = \boldsymbol{A}^{\text{vol}}$$

- deviatoric fourth order unit tensor

$$\mathsf{I}^{\text{dev}} : \boldsymbol{A} = \boldsymbol{A}^{\text{dev}}$$

$$\mathsf{I}^{\text{dev}} = \left[-\frac{1}{3}\,\delta_{ij}\,\delta_{kl} + \frac{1}{2}\,\delta_{ik}\,\delta_{jl} + \frac{1}{2}\,\delta_{il}\,\delta_{jk}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

tensor algebra - scalar product

- scalar (inner) product

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{u} &= (A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot(u_k\,\boldsymbol{e}_k) \\ &= A_{ij}\,u_k\,\delta_{jk}\,\boldsymbol{e}_i = A_{ij}\,u_j\,\boldsymbol{e}_i = v_i\boldsymbol{e}_i = \boldsymbol{v} \end{aligned}$$

of second order tensor $\boldsymbol{A}$ and vector $\boldsymbol{u}$

- zero and identity $\quad \boldsymbol{0}\cdot\boldsymbol{u}=\boldsymbol{0} \qquad \boldsymbol{I}\cdot\boldsymbol{u}=\boldsymbol{u}$
- positive definiteness $\quad \boldsymbol{a}\cdot\boldsymbol{A}\cdot\boldsymbol{a}>0$
- properties of scalar product

$$\boldsymbol{A}\cdot(\alpha\boldsymbol{a}+\beta\boldsymbol{b}) = \alpha\,(\boldsymbol{A}\cdot\boldsymbol{a})+\beta\,(\boldsymbol{A}\cdot\boldsymbol{b})$$
$$(\boldsymbol{A}+\boldsymbol{B})\cdot\boldsymbol{a} = \boldsymbol{A}\cdot\boldsymbol{a}+\boldsymbol{B}\cdot\boldsymbol{a}$$
$$(\alpha\,\boldsymbol{A})\cdot\boldsymbol{a} = \alpha\,(\boldsymbol{A}\cdot\boldsymbol{a})$$

tensor algebra - scalar product

- scalar (inner) product

$$
\begin{aligned}
\boldsymbol{A} \cdot \boldsymbol{B} &= (A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (B_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l) \\
&= A_{ij}\, B_{kl}\, \delta_{ik} \boldsymbol{e}_i \otimes \boldsymbol{e}_l \\
&= A_{ij}\, B_{jl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_l = C_{il} \boldsymbol{e}_i \otimes \boldsymbol{e}_l = \boldsymbol{C}
\end{aligned}
$$

  of two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$

- zero and identity $\quad \boldsymbol{0} \cdot \boldsymbol{A} = \boldsymbol{A} \quad\quad \boldsymbol{I} \cdot \boldsymbol{A} = \boldsymbol{A}$

- properties of scalar product $\quad (\boldsymbol{A} \cdot \boldsymbol{B})^{\mathrm{t}} = \boldsymbol{B}^{\mathrm{t}} \cdot \boldsymbol{A}^{\mathrm{t}}$

$$
\begin{aligned}
&\alpha\, (\boldsymbol{A} \cdot \boldsymbol{B}) = (\alpha\, \boldsymbol{A}) \cdot \boldsymbol{B} = \boldsymbol{A} \cdot (\alpha\, \boldsymbol{B}) \\
&\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C} \\
&(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}
\end{aligned}
$$

tensor algebra - scalar product

- scalar (inner) product

$$\begin{aligned} \boldsymbol{A} : \boldsymbol{B} &= (A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (B_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l) \\ &= A_{ij}\, B_{kl}\, \delta_{ik}\, \delta_{jl} = A_{ij}\, B_{ij} = \alpha \end{aligned}$$

of two second order tensors $\boldsymbol{A}, \boldsymbol{B}$

- scalar (inner) product

$$\begin{aligned} \mathsf{A} : \boldsymbol{B} &= (A_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l) : (B_{mn}\, \boldsymbol{e}_m \otimes \boldsymbol{e}_n) \\ &= A_{ijkl}\, B_{mn}\, \delta_{km}\, \delta_{ln}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= A_{ijkl}\, B_{kl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{A} \end{aligned}$$

of fourth order tensors A and second order tensor $\boldsymbol{B}$

- zero and identity $\quad \boldsymbol{0} : \boldsymbol{A} = \boldsymbol{0} \quad\quad \mathsf{I} : \boldsymbol{A} = \boldsymbol{A}$

tensor algebra - dyadic product

- dyadic (outer) product

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} = u_i \, \boldsymbol{e}_i \otimes v_j \boldsymbol{e}_j = u_i \, v_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

of two vectors $\boldsymbol{u}, \boldsymbol{v}$ introduces second order tensor $\boldsymbol{A}$

- properties of dyadic product (tensor notation)

$$(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w}) \, \boldsymbol{u}$$
$$(\alpha \, \boldsymbol{u} + \beta \, \boldsymbol{v}) \otimes \boldsymbol{w} = \alpha \, (\boldsymbol{u} \otimes \boldsymbol{w}) + \beta \, (\boldsymbol{v} \otimes \boldsymbol{w})$$
$$\boldsymbol{u} \otimes (\alpha \, \boldsymbol{v} + \beta \, \boldsymbol{w}) = \alpha \, (\boldsymbol{u} \otimes \boldsymbol{v}) + \beta \, (\boldsymbol{u} \otimes \boldsymbol{w})$$
$$(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot (\boldsymbol{w} \otimes \boldsymbol{x}) = (\boldsymbol{v} \cdot \boldsymbol{w}) \, (\boldsymbol{u} \otimes \boldsymbol{x})$$
$$\boldsymbol{A} \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{A} \cdot \boldsymbol{u}) \otimes \boldsymbol{v}$$
$$(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{A} = \boldsymbol{u} \otimes (\boldsymbol{A}^{\mathrm{t}} \cdot \boldsymbol{v})$$

tensor algebra - dyadic product

- dyadic (outer) product

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} = u_i\, \boldsymbol{e}_i \otimes v_j \boldsymbol{e}_j = u_i\, v_j\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

of two vectors $\boldsymbol{u}, \boldsymbol{v}$ introduces second order tensor $\boldsymbol{A}$

- properties of dyadic product (index notation)

$$(u_i\, v_j)\, w_j = (v_j\, w_j)\, u_i$$
$$(\alpha\, u_i + \beta\, v_i) w_j = \alpha\, (u_i\, w_j) + \beta\, (v_i\, w_j)$$
$$u_i(\alpha\, v_j + \beta\, w_j) = \alpha\, (u_i\, v_j) + \beta\, (u_i\, w_j)$$
$$(u_i\, v_j)(w_j\, x_k) = (v_j\, w_j)\, (u_i x_k)$$
$$A_{ij}(u_j\, v_k) = (A_{ij}\, u_i) v_k$$
$$(u_i\, v_j) A_{jk} = u_i(A_{kj}\, v_j)$$