ME338A
CONTINUUM MECHANICS

lecture notes 15

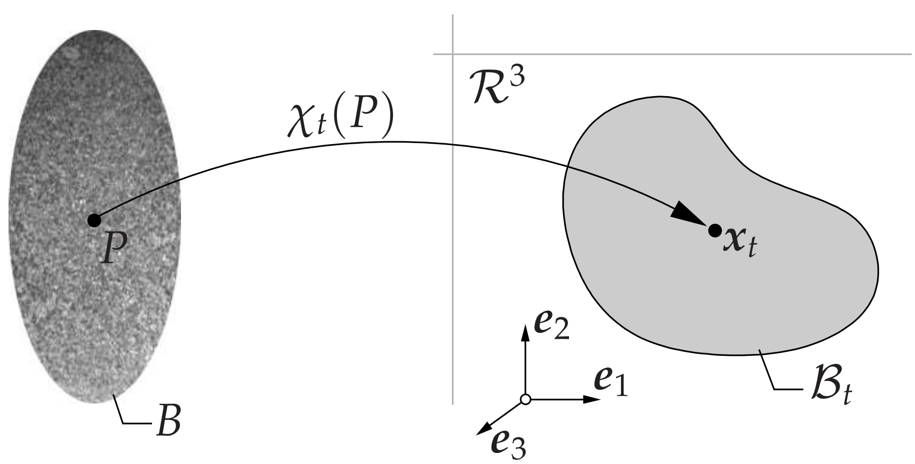
tuesday, march 2nd, 2010

5 Introduction to Nonlinear Continuum Mechanics

5.1 Kinematics

5.1.1 Motion and Configurations

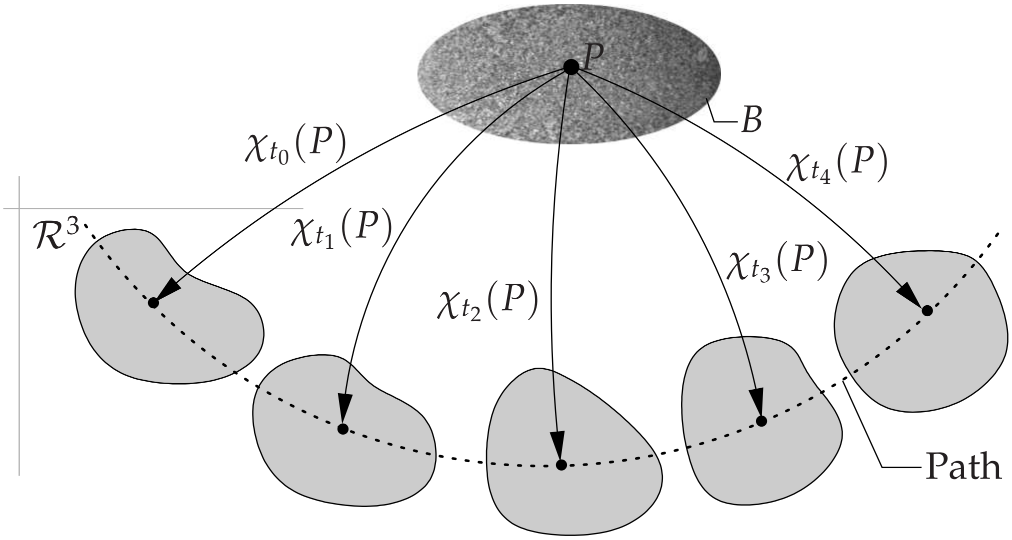
Consider a *material body* B composed of infinitely many *material points* $P \in B$ that are identified with geometrical points in the three-dimensional Euclidean space $\mathcal{R}^3$.



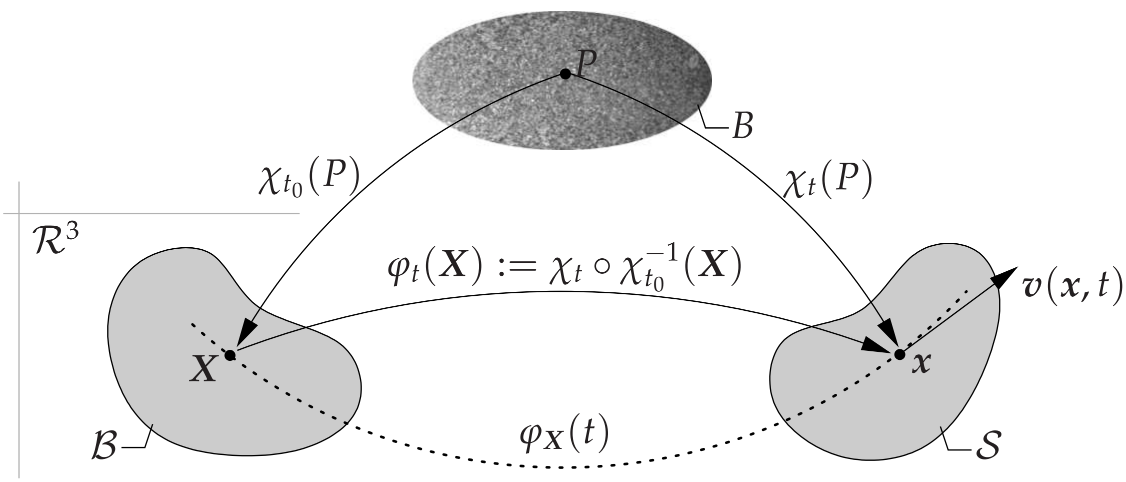
The *configuration* of the body B in $\mathcal{R}^3$ at time t is described by the one-to-one relation

$$\chi_t : \begin{cases} B \;\rightarrow\; \mathcal{B}_t \in \mathcal{R}^3 \,, \\ P \in B \;\mapsto\; \boldsymbol{x}_t = \chi_t(P) \in \mathcal{B}_t \,. \end{cases} \tag{5.1.1}$$

Having this definition at hand, we go on to describe the *motion* of the body as a family of configurations parameterized



by time in the Euclidean space $\mathcal{R}^3$. Throughout the motion, the material point P occupies a sequence of places in space, which forms the *path (trajectory)* of the material point.



While describing the motion of a solid body, it is common practice to name its placement at time t_0 as the *reference configuration* that generally possesses an undistorted stress-free state and is henceforth denoted as $\mathcal{B} \equiv \chi_{t_0}(B)$. Likewise, the configuration of the body at current time t is hereafter denoted as $\mathcal{S} \equiv \chi_t(B)$. The reference and the spatial positions occupied by a material point P within the Euclidean space are labeled by the *reference* coordinates $\boldsymbol{X} := \chi_{t_0}(P) \in \mathcal{B}$ and the *spatial* coordinates $\boldsymbol{x} := \chi_t(P) \in \mathcal{S}$, respectively. In order to describe the *motion* of the body in the Euclidean space, we introduce a non-linear *deformation map* $\varphi_t(\boldsymbol{X})$ between $\chi_{t_0}(P)$ and $\chi_t(P)$

$$\varphi_t(\boldsymbol{X}) : \begin{cases} \mathcal{B} \ \rightarrow \ \mathcal{S} \ , \\ \boldsymbol{X} \ \mapsto \ \boldsymbol{x} = \varphi_t(\boldsymbol{X}) := \chi_t \circ \chi_0^{-1}(\boldsymbol{X}) \end{cases} \tag{5.1.2}$$

that maps the material points $\boldsymbol{X} \in \mathcal{B}$ onto their deformed *spatial* positions $\boldsymbol{x} = \varphi_t(\boldsymbol{X}) \in \mathcal{S}$ at time $t \in \mathcal{R}_+$. Since the deformation map $\varphi_t(\boldsymbol{X})$ is bijective, it can be inverted uniquelly to obtain the *inverse deformation map*

$$\varphi_t^{-1}(\boldsymbol{x}) : \begin{cases} \mathcal{S} \ \rightarrow \ \mathcal{B} \ , \\ \boldsymbol{x} \ \mapsto \ \boldsymbol{X} = \varphi_t^{-1}(\boldsymbol{x}) := \chi_0 \circ \chi_t^{-1}(\boldsymbol{x}) \ . \end{cases} \tag{5.1.3}$$

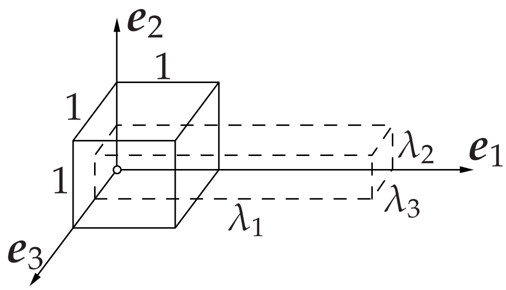
In indicial notation, we have

$$x_a = \varphi_a(X_1, X_2, X_3, t) \quad \text{and} \quad X_A = \varphi_A^{-1}(x_1, x_2, x_3, t) \ . \tag{5.1.4}$$

Although we use a common coordinate system to describe the motion of the body, it will be convenient to assign the lower-case indices to tensorial fields belonging to the current configuration and the upper-case indices for referential fields.

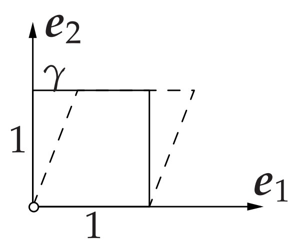
Examples:

Simple extension of cube



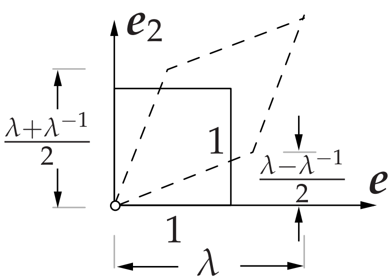
$$
\begin{aligned}
x_1 &= \lambda_1(t)X_1\\
x_2 &= \lambda_2(t)X_2\\
x_3 &= \lambda_3(t)X_3
\end{aligned}
$$

Simple shear



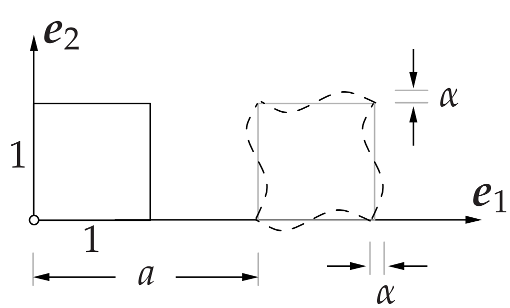
$$
\begin{aligned}
x_1 &= X_1 + \gamma(t)X_2\\
x_2 &= X_2\\
x_3 &= X_3
\end{aligned}
$$

Pure shear



$$
\begin{aligned}
x_1 &= \tfrac{\lambda+\lambda^{-1}}{2}X_1 + \tfrac{\lambda-\lambda^{-1}}{2}X_2\\
x_2 &= \tfrac{\lambda-\lambda^{-1}}{2}X_1 + \tfrac{\lambda+\lambda^{-1}}{2}X_2\\
x_3 &= X_3
\end{aligned}
$$

Periodic deformation



$$
\begin{aligned}
x_1 &= X_1 + \alpha(t)\sin(2\pi X_2) + a\\
x_2 &= X_2 - \alpha(t)\sin(2\pi X_1)\\
x_3 &= X_3
\end{aligned}
$$

5.1.2 Velocity and Acceleration

Recall the definition of motion in (5.1.2),

$$\boldsymbol{x} := \varphi(\boldsymbol{X}, t) = \varphi_t(\boldsymbol{X}) = \varphi_{\boldsymbol{X}}(t) \tag{5.1.5}$$

where $\varphi_t(\boldsymbol{X})$ denotes the configuration of $\mathcal{B}$ at time t and $\varphi_{\boldsymbol{X}}(t)$ is the path of the particle P.

5.1.2.1 Material Velocity and Acceleration

We can now define the *material velocity*

$$\boldsymbol{V}_t(\boldsymbol{X}) := \partial_t \varphi(\boldsymbol{X}, t) = \frac{d}{dt}\varphi_{\boldsymbol{X}}(t) \ , \tag{5.1.6}$$

and the *material acceleration*

$$\boldsymbol{A}_t(\boldsymbol{X}) := \partial_t \boldsymbol{V}(\boldsymbol{X}, t) = \frac{d}{dt}\boldsymbol{V}_{\boldsymbol{X}}(t) \tag{5.1.7}$$

of the motion. It is important to notice that both the material velocity $\boldsymbol{V}(\boldsymbol{X}, t)$ and the material acceleration $\boldsymbol{A}(\boldsymbol{X}, t)$ belong to the current configuration $\mathcal{S}$ but are functions of material coordinates $\boldsymbol{X}$ and time t.

5.1.2.2 Spatial Velocity and Acceleration

The spatial counterparts of the rates of material description of the motion can be obtained by reparameterizing the functional dependency of these fields in terms of spatial coordinates $\boldsymbol{x}$. The *spatial velocity* and the *spatial acceleration* are then defined as

$$\boldsymbol{v}(\boldsymbol{x}, t) := \boldsymbol{V}(\varphi_t^{-1}(\boldsymbol{x}), t) = \boldsymbol{V}_t(\boldsymbol{X}) \circ \varphi_t^{-1}(\boldsymbol{x}) \ , \tag{5.1.8}$$

and

$$\boldsymbol{a}(\boldsymbol{x}, t) := \boldsymbol{A}(\varphi_t^{-1}(\boldsymbol{x}), t) = \boldsymbol{A}_t(\boldsymbol{X}) \circ \varphi_t^{-1}(\boldsymbol{x}) \ , \tag{5.1.9}$$

respectively. The *path* $\varphi_{\boldsymbol{X}}(t)$ is called the *integral curve* of $\boldsymbol{v}$.

5.1.2.3 Material Time Derivative of Spatial Fields

Consider a spatial field

$$f(\boldsymbol{x}, t) : \varphi_t(\mathcal{B}) \times \mathcal{R}_+ \to \mathcal{R} \ .$$

The *material time derivative* of $f(\boldsymbol{x}, t)$ is defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} f(\boldsymbol{x}, t) = \dot{f}(\boldsymbol{x}, t) := \frac{\partial}{\partial t} f(\boldsymbol{x}, t) + \nabla_x f \cdot \boldsymbol{v} \tag{5.1.10}$$

For $F(\boldsymbol{X}, t) := f(\boldsymbol{x}, t) \circ \varphi_t(\boldsymbol{X})$, the following equality

$$\frac{\partial}{\partial t} F(\boldsymbol{X}, t) = \dot{f}(\boldsymbol{x}, t) \tag{5.1.11}$$

holds. An immediate example is the spatial acceleration $\boldsymbol{a}(\boldsymbol{x}, t)$ that is none other than the *material time derivative* of the spatial velocity

$$\boldsymbol{a}(\boldsymbol{x}, t) := \boldsymbol{A}_t(\boldsymbol{X}) \circ \varphi_t^{-1}(\boldsymbol{x}) = \partial_t \boldsymbol{v} + \nabla_x \boldsymbol{v} \cdot \boldsymbol{v} \ . \tag{5.1.12}$$

where $\nabla_x \boldsymbol{v}$ stands for the *spatial gradient of velocity*.

Example: Rigid Body Motion Chadwick (1976), p.52

The rigid body motion of a body is described by

$$\boldsymbol{x} = \varphi(\boldsymbol{X}, t) = \boldsymbol{c}(t) + \boldsymbol{Q}(t) \cdot \boldsymbol{X} ,$$

where $\boldsymbol{Q}$ is proper orthogonal, i.e. $\boldsymbol{Q} \cdot \boldsymbol{Q}^T = \mathbf{1}$ and $\det\boldsymbol{Q} = +1$. In *rigid* motion the distance between two points does not change. That is, for $\boldsymbol{T} := \boldsymbol{X}_B - \boldsymbol{X}_A$ and $\boldsymbol{t} := \boldsymbol{x}_B - \boldsymbol{x}_A = \boldsymbol{Q} \cdot \boldsymbol{T}$ with $\boldsymbol{x}_{\{A,B\}} = \varphi(\boldsymbol{X}_{\{A,B\}}, t)$, we end up with the identity

$$||\boldsymbol{t}|| := \sqrt{\boldsymbol{t} \cdot \boldsymbol{t}} = \sqrt{(\boldsymbol{Q} \cdot \boldsymbol{T}) \cdot (\boldsymbol{Q} \cdot \boldsymbol{T})} = \sqrt{\boldsymbol{T} \cdot (\boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \boldsymbol{T})} = ||\boldsymbol{T}|| .$$

The motion can be then inverted to obtain

$$\boldsymbol{X} = \varphi_t^{-1}(\boldsymbol{x}) = \boldsymbol{Q}^T \cdot (\boldsymbol{x} - \boldsymbol{c}) .$$

The material velocity and material acceleration

$$\boldsymbol{V}_t(\boldsymbol{X}) := \partial_t \varphi(\boldsymbol{X}, t) = \dot{\boldsymbol{c}} + \dot{\boldsymbol{Q}} \cdot \boldsymbol{X}$$

$$\boldsymbol{A}_t(\boldsymbol{X}) := \partial_t \boldsymbol{V}(\boldsymbol{X}, t) = \ddot{\boldsymbol{c}} + \ddot{\boldsymbol{Q}} \cdot \boldsymbol{X}$$

The spatial velocity and spatial acceleration

$$\boldsymbol{v}(\boldsymbol{x}, t) := \boldsymbol{V}(\varphi_t^{-1}(\boldsymbol{x}), t) = \dot{\boldsymbol{c}} + \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T \cdot (\boldsymbol{x} - \boldsymbol{c})$$

$$\boldsymbol{a}(\boldsymbol{x}, t) := \boldsymbol{A}(\varphi_t^{-1}(\boldsymbol{x}), t) = \ddot{\boldsymbol{c}} + \ddot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T \cdot (\boldsymbol{x} - \boldsymbol{c})$$

Since $\boldsymbol{Q} \cdot \boldsymbol{Q}^T = \mathbf{1}$, we have

$$\dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T = -\boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^T = -(\dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T)^T$$

Therefore,

$$\Omega := \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T = -\Omega^T$$

is a skew-symmetric tensor, which has an associated *axial vector* $\omega \in \mathcal{R}^3$ defined by

$$\Omega \cdot \beta = \omega \times \beta \qquad \forall \beta \in \mathcal{R}^3 .$$

Furthermore, it can be shown that

$$\ddot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T = \dot{\Omega} - \dot{\boldsymbol{Q}} \cdot \dot{\boldsymbol{Q}}^T = \dot{\Omega} + \Omega^2 ,$$

$$\ddot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T \cdot \beta = \dot{\Omega} \cdot \beta + \Omega^2 \cdot \beta = \dot{\omega} \times \beta + \omega \times (\omega \times \beta) \quad \forall \beta \in \mathcal{R}^3 .$$

Incorparating these results in the spatial velocity and spatial acceleration expressions for $\beta \equiv (\boldsymbol{x} - \boldsymbol{c})$, we end up with

$$\boldsymbol{v}(\boldsymbol{x}, t) = \dot{\boldsymbol{c}} + \omega \times (\boldsymbol{x} - \boldsymbol{c})$$

$$\boldsymbol{a}(\boldsymbol{x}, t) = \ddot{\boldsymbol{c}} + \dot{\omega} \times (\boldsymbol{x} - \boldsymbol{c}) + \omega \times [\omega \times (\boldsymbol{x} - \boldsymbol{c})]$$

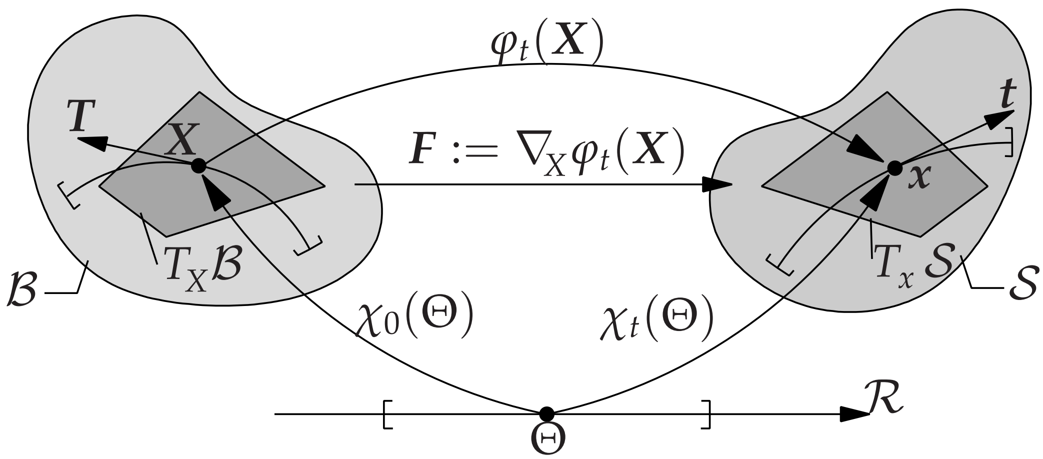
The latter can also be obtained by taking the material time derivative of the spatial velocity field as described in (5.1.12).

$$\begin{aligned}
\boldsymbol{a}(\boldsymbol{x}, t) &= \partial_t \boldsymbol{v} + \nabla_x \boldsymbol{v} \cdot \boldsymbol{v} \\
&= \partial_t [\dot{\boldsymbol{c}} + \omega \times (\boldsymbol{x} - \boldsymbol{c})] \\
&\quad + \nabla_x [\dot{\boldsymbol{c}} + \omega \times (\boldsymbol{x} - \boldsymbol{c})] \cdot [\dot{\boldsymbol{c}} + \omega \times (\boldsymbol{x} - \boldsymbol{c})] \\
&= \ddot{\boldsymbol{c}} + \dot{\omega} \times (\boldsymbol{x} - \boldsymbol{c}) - \omega \times \dot{\boldsymbol{c}} \\
&\quad + \omega \times [\dot{\boldsymbol{c}} + \omega \times (\boldsymbol{x} - \boldsymbol{c})] \\
&= \ddot{\boldsymbol{c}} + \dot{\omega} \times (\boldsymbol{x} - \boldsymbol{c}) + \omega \times [\omega \times (\boldsymbol{x} - \boldsymbol{c})]
\end{aligned}$$

5.1.3 Fundamental Geometric Maps

This section is devoted to the geometric mapping of basic *tangential, normal* and *volumetric* objects.

5.1.3.1 Deformation Gradient. Tangent Map



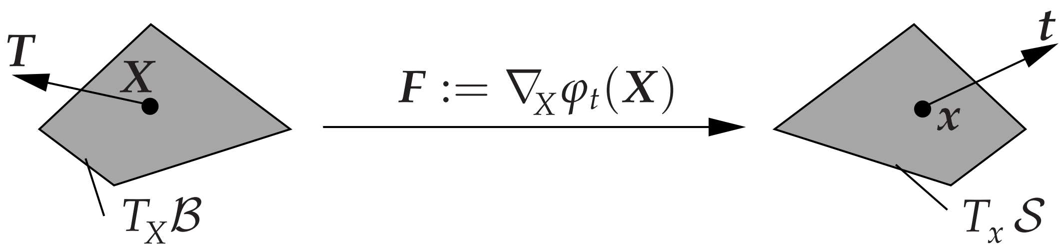
Probably the most fundamental deformation measure used in kinematics of finite deformation is the *deformation gradient*. It can be considered as a linear map of the referential *tangent vectors* onto the spatial counterparts. To this end, let $\chi_0(\Theta)$ and $\chi_t(\Theta)$ be the material and spatial curves parameterized by a common variable $\Theta \in \mathcal{R}$ on $\mathcal{B}$ and $\mathcal{S}$, respectively. It is important to observe that the spatial curve is related to the reference curve by the non-linear deformation map $\chi_t(\Theta) = \varphi_t(\chi_0(\Theta))$. Tangents of the curves belonging to the respective tangent spaces are defined as

$$\boldsymbol{T} := \frac{d\chi_0(\Theta)}{d\Theta} \in T_X\mathcal{B} \quad \text{and} \quad \boldsymbol{t} := \frac{d\chi_t(\Theta)}{d\Theta} \in T_x\mathcal{S} \tag{5.1.13}$$

Through the chain rule, we obtain

$$\boldsymbol{t} = \frac{d\chi_t(\Theta)}{d\Theta} = \nabla_X \varphi_t(\boldsymbol{X}) \cdot \frac{d\chi_0(\Theta)}{d\Theta} = \boldsymbol{F} \cdot \boldsymbol{T}, \tag{5.1.14}$$

where we introduced the *deformation gradient* $\boldsymbol{F} := \nabla_X \varphi_t(\boldsymbol{X})$ as the Fréchet derivative of the deformation.



The deformation gradient is also called the *tangent map*

$$\boldsymbol{F}(\boldsymbol{X}, t) : \begin{cases} T_X\mathcal{B} \;\rightarrow\; T_x\mathcal{S} \;, \\ \quad \boldsymbol{T} \;\mapsto\; \boldsymbol{t} = \boldsymbol{F} \cdot \boldsymbol{T} \;, \end{cases} \tag{5.1.15}$$

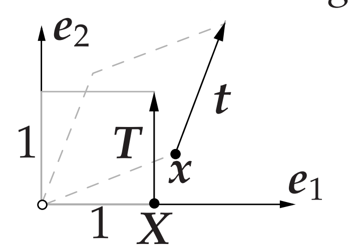
between the tangent spaces $T_X\mathcal{B}$ and $T_x\mathcal{S}$ of the configurations (manifolds) $\mathcal{B}$ and $\mathcal{S}$, respectively. Note that the deformation gradient is a *two-point* tensor that involves points from the both configurations; that is, in index notation

$$F_{aA} := \partial\varphi_a / \partial X_A \quad \text{and} \quad t_a = F_{aA} T_A \;.$$

The determinant of the deformation gradient $\det(\boldsymbol{F})$ is restricted to non-zero ($\det(\boldsymbol{F}) \neq 0$), positive ($\det(\boldsymbol{F}) > 0$) values due to the facts that $\varphi_t(\boldsymbol{X})$ is one-to-one and we rule out interpenetration of the material throughout the motion.

Examples:

Pure shear (Homogeneous)

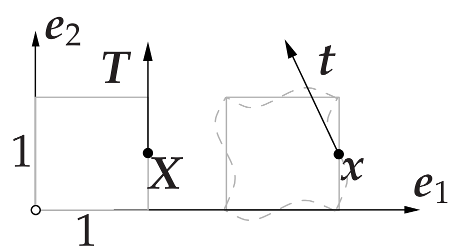


$$
\begin{aligned}
x_1 &= \tfrac{\lambda+\lambda^{-1}}{2}X_1 + \tfrac{\lambda-\lambda^{-1}}{2}X_2 \\
x_2 &= \tfrac{\lambda-\lambda^{-1}}{2}X_1 + \tfrac{\lambda+\lambda^{-1}}{2}X_2 \\
x_3 &= X_3
\end{aligned}
$$

$$
\boldsymbol{F} = \tfrac{1}{2}\begin{bmatrix} \lambda+\lambda^{-1} & \lambda-\lambda^{-1} & 0 \\ \lambda-\lambda^{-1} & \lambda+\lambda^{-1} & 0 \\ 0 & 0 & 2 \end{bmatrix}
$$

$$
\boldsymbol{T} = \boldsymbol{e}_2\,, \qquad \boldsymbol{t} = \boldsymbol{F}\cdot\boldsymbol{T} = \tfrac{1}{2}\left[(\lambda-\lambda^{-1})\boldsymbol{e}_1 + (\lambda+\lambda^{-1})\boldsymbol{e}_2\right]
$$

Periodic deformation (Inhomogeneous)

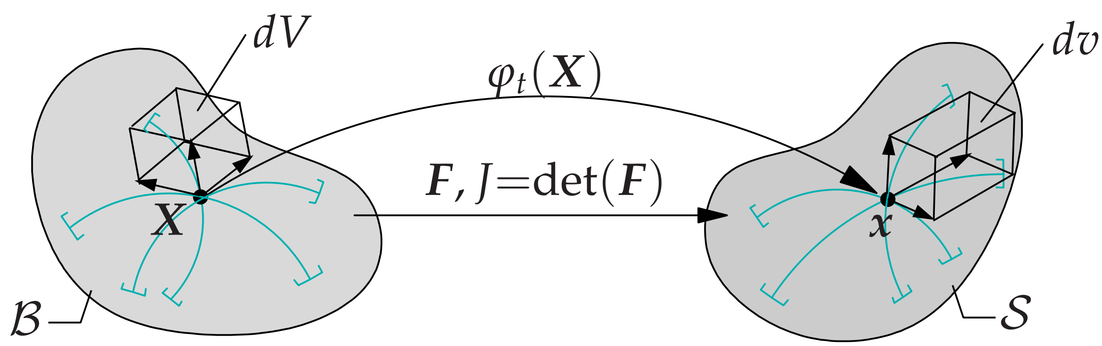


$$
\begin{aligned}
x_1 &= X_1 + \alpha(t)\sin(2\pi X_2) + a \\
x_2 &= X_2 - \alpha(t)\sin(2\pi X_1) \\
x_3 &= X_3
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F} &= \begin{bmatrix} 1 & 2\pi\alpha\cos(2\pi X_2) & 0 \\ -2\pi\alpha\cos(2\pi X_1) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\boldsymbol{X}=(1,0.5,0)} \\
&= \begin{bmatrix} 1 & -2\pi\alpha & 0 \\ -2\pi\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\boldsymbol{T} = \boldsymbol{e}_2\,, \qquad \boldsymbol{t} = \boldsymbol{F}\cdot\boldsymbol{T} = -2\pi\alpha\boldsymbol{e}_1 + \boldsymbol{e}_2
$$

5.1.3.2 Volume Transformation. Jacobi Map



Let dV and dv denote the infinitesimal volumes of parallelepipeds

$$dV := d\boldsymbol{X}_1 \cdot (d\boldsymbol{X}_2 \times d\boldsymbol{X}_3) \quad \text{and} \quad dv := d\boldsymbol{x}_1 \cdot (d\boldsymbol{x}_2 \times d\boldsymbol{x}_3) \tag{5.1.16}$$

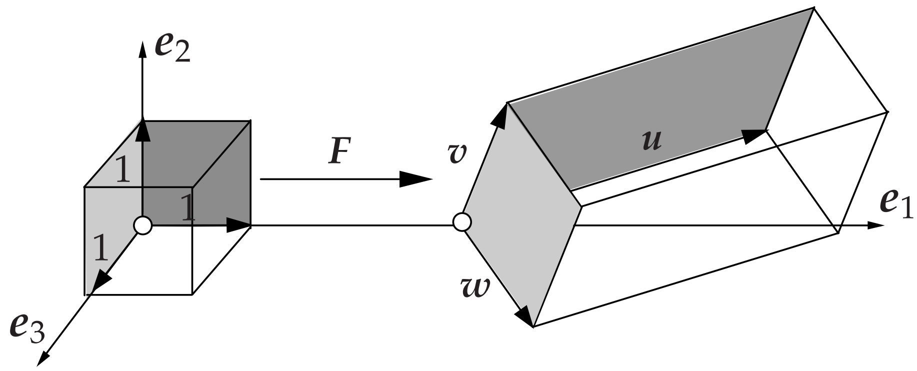
defined as the scalar triple product of vectors $d\boldsymbol{X}_{i=1,2,3} \in T_X\mathcal{B}$ and $d\boldsymbol{x}_{i=1,2,3} \in T_x\,\mathcal{S}$, respectively. Each spatial tangent vector $d\boldsymbol{x}_i$ is defined as a tangential map of its material counterpart, i.e. $d\boldsymbol{x}_i := \boldsymbol{F} \cdot d\boldsymbol{X}_i$ for $i = 1, 2, 3$. We define the *volume map*

$$\begin{aligned} dv &= (\boldsymbol{F} \cdot d\boldsymbol{X}_1) \cdot [(\boldsymbol{F} \cdot d\boldsymbol{X}_2) \times (\boldsymbol{F} \cdot d\boldsymbol{X}_3)] \\ &= \det(\boldsymbol{F})\; d\boldsymbol{X}_1 \cdot (d\boldsymbol{X}_2 \times d\boldsymbol{X}_3) = \det\boldsymbol{F}\, dV =: J\, dV \end{aligned} \tag{5.1.17}$$

through the conventional coordinate-free definition of the determinant of a second order tensor. Recall that the value of the Jacobian J is restricted to positive real numbers $\mathcal{R}_+$ in order to ensure the one-to-one relation between $\boldsymbol{x}$ and $\boldsymbol{X}$ and the impenetrability of a material. Then, we say that the volume map, det$\boldsymbol{F}$, maps the reference volume elements onto their spatial counterparts

$$J = \det \boldsymbol{F} := \begin{cases} \mathcal{R}_+ & \rightarrow & \mathcal{R}_+ \, , \\ dV & \mapsto & dv = \det \boldsymbol{F} dV \, . \end{cases} \tag{5.1.18}$$

Example:



Assume that vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are defined as

$$\boldsymbol{u} := \boldsymbol{F} \cdot \boldsymbol{e}_1, \, \boldsymbol{v} := \boldsymbol{F} \cdot \boldsymbol{e}_2 \quad \text{and} \quad \boldsymbol{w} := \boldsymbol{F} \cdot \boldsymbol{e}_3 \, .$$

Components of the deformation gradient are then

$$\begin{aligned} u_i &= F_{i1}, \\ v_i &= F_{i2}, \\ w_i &= F_{i3}, \end{aligned} \qquad \rightsquigarrow \boldsymbol{F} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

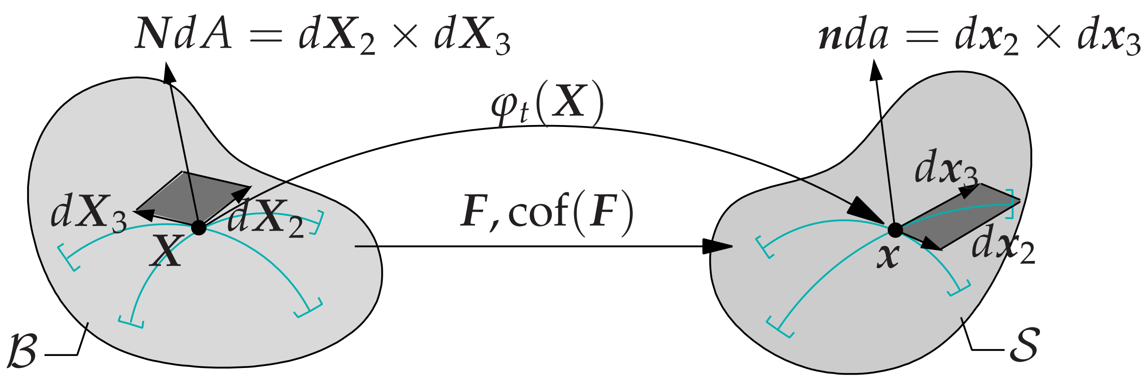
Volume of the deformed parallelpiped is identical to def$(\boldsymbol{F})$:

$$\begin{aligned} dv &:= \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{F} \cdot \boldsymbol{e}_1) \cdot [(\boldsymbol{F} \cdot \boldsymbol{e}_2) \times (\boldsymbol{F} \cdot \boldsymbol{e}_3)] \\ &= \det(\boldsymbol{F}) \boldsymbol{e}_1 \cdot [\boldsymbol{e}_2 \times \boldsymbol{e}_3] \\ &= \det(\boldsymbol{F}) \end{aligned}$$

5.1.3.3 Adjoint Transformation. Normal Map

The reference and spatial *area normals* are defined as

$$\boldsymbol{N} dA := d\boldsymbol{X}_2 \times d\boldsymbol{X}_3 \quad \text{and} \quad \boldsymbol{n} da := d\boldsymbol{x}_2 \times d\boldsymbol{x}_3 \,.$$



With these definitions at hand, we can recast (5.1.17) into the following from

$$d\boldsymbol{x}_1 \cdot \boldsymbol{n} da = J d\boldsymbol{X}_1 \cdot \boldsymbol{N} dA \,. \tag{5.1.19}$$

If we incorporate the identity $d\boldsymbol{x}_1 = \boldsymbol{F} d\boldsymbol{X}_1$ in (5.1.19)

$$(\boldsymbol{F} \cdot d\boldsymbol{X}_1) \cdot \boldsymbol{n} da = d\boldsymbol{X}_1 \cdot (\boldsymbol{F}^T \cdot \boldsymbol{n} da) = J d\boldsymbol{X}_1 \cdot \boldsymbol{N} dA \,,$$

and solve this equality for $\boldsymbol{n} da$ for an arbitrary tangent vector $d\boldsymbol{X}_1$, we end up with the *area (normal) map*

$$\boldsymbol{n} da = \text{cof}(\boldsymbol{F}) \cdot \boldsymbol{N} dA \quad \text{with} \quad \text{cof}(\boldsymbol{F}) := J \boldsymbol{F}^{-T} \,. \tag{5.1.20}$$

It transforms the normals of material surfaces onto the normal vectors of spatial surfaces. Furthermore, we observe that the tensorial quantity carrying out the mapping operation in (5.1.20) is none other than $\boldsymbol{F}^{-T}$. Thus, we consider $\boldsymbol{F}^{-T}$ as the *normal map* transforming the reference normals $\boldsymbol{N}$ onto the spatial normals (co-vectors) $\boldsymbol{n}$ belonging to the respective *co-tangent* spaces $T_X^*\mathcal{B}$ and $T_x^*\mathcal{S}$. The *normal map* is then defined as

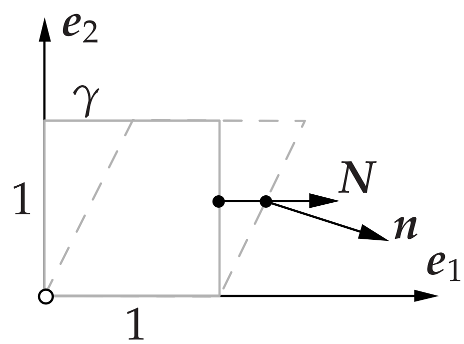
$$\boldsymbol{F}^{-T} := \begin{cases} T_X^*\mathcal{B} & \rightarrow \quad T_x^*\mathcal{S}\,, \\ \boldsymbol{N} & \mapsto \quad \boldsymbol{n} = \boldsymbol{F}^{-T} \cdot \boldsymbol{N}\,. \end{cases} \tag{5.1.21}$$

The *co-factor* of the deformation gradient $\mathrm{cof}(\boldsymbol{F})$ is also defined as the derivative of the volume map $J := \det\boldsymbol{F}$ with respect to the deformation gradient $\boldsymbol{F}$

$$\mathrm{cof}\,\boldsymbol{F} := \partial_{\boldsymbol{F}}\det\boldsymbol{F} = \det(\boldsymbol{F})\boldsymbol{F}^{-T}\,. \tag{5.1.22}$$

Example:

Simple shear



$$\begin{aligned} x_1 &= X_1 + \gamma(t) X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned}$$

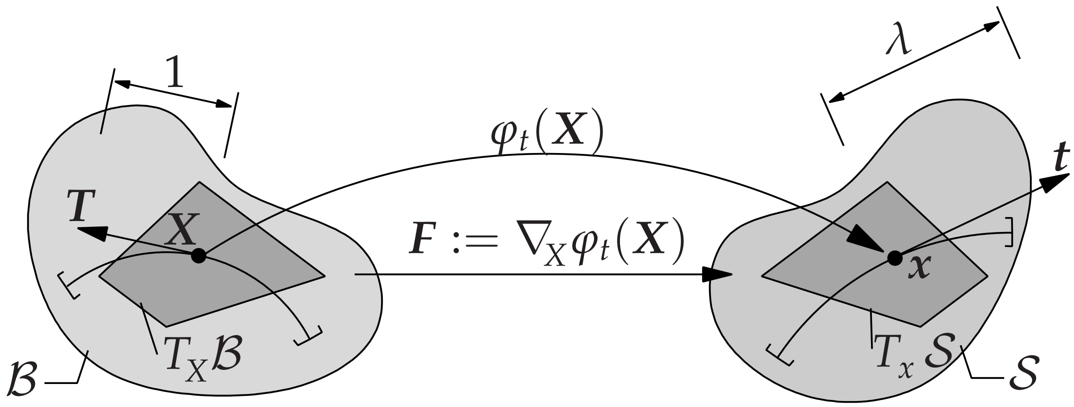
$$\boldsymbol{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{F}^{-T} = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{aligned} \boldsymbol{N} &= \boldsymbol{e}_1 \\ \boldsymbol{n} &= \boldsymbol{F}^{-T} \cdot \boldsymbol{N} \\ &= \boldsymbol{e}_1 - \gamma \boldsymbol{e}_2 \end{aligned}$$

Observe that the area change of the side surface is given by $||\boldsymbol{n}|| = \sqrt{1+\gamma^2}$.

5.1.4 Deformation and Strain Tensors

This section outlines the fundamental deformation and strain tensors that often enter the strain energy functions.

5.1.4.1 Stretch Vector and Stretch



Let $\boldsymbol{T} \in T_X \mathcal{B}$ be a material tangent vector emanating from $\boldsymbol{X} \in \mathcal{B}$. We define the associated *stretch vector* $\boldsymbol{t} \in T_x \mathcal{S}$ as the Gateaux derivative of the deformation $\varphi_t(\boldsymbol{X})$ in the direction $\boldsymbol{T}$

$$\begin{aligned} \boldsymbol{t} &:= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_t(\boldsymbol{X} + \epsilon \boldsymbol{T}) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} [\varphi_t(\boldsymbol{X}) + \epsilon \boldsymbol{F} \cdot \boldsymbol{T} + O(\epsilon^2)] \\ &= \boldsymbol{F} \cdot \boldsymbol{T} \, . \end{aligned} \tag{5.1.23}$$

The ratio of the length of the spatial tangent vector $\boldsymbol{t}$ to the length of the corresponding reference tangent vector $\boldsymbol{T}$ is called the *stretch*

$$\lambda := \frac{||\boldsymbol{t}||}{||\boldsymbol{T}||} = \frac{\sqrt{\boldsymbol{t} \cdot \boldsymbol{t}}}{\sqrt{\boldsymbol{T} \cdot \boldsymbol{T}}} > 0 \, . \tag{5.1.24}$$

By choosing $||\boldsymbol{T}|| = 1$ as the *reference* value, the stretch can be expressed as

$$\begin{aligned} \lambda &= \sqrt{\boldsymbol{t} \cdot \boldsymbol{t}} = \sqrt{(\boldsymbol{F} \cdot \boldsymbol{T}) \cdot (\boldsymbol{F} \cdot \boldsymbol{T})} \\ &= \sqrt{\boldsymbol{T} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F}) \cdot \boldsymbol{T}} = \sqrt{\boldsymbol{T} \cdot \boldsymbol{C} \cdot \boldsymbol{T}} =: ||\boldsymbol{T}||_C \,, \end{aligned} \tag{5.1.25}$$

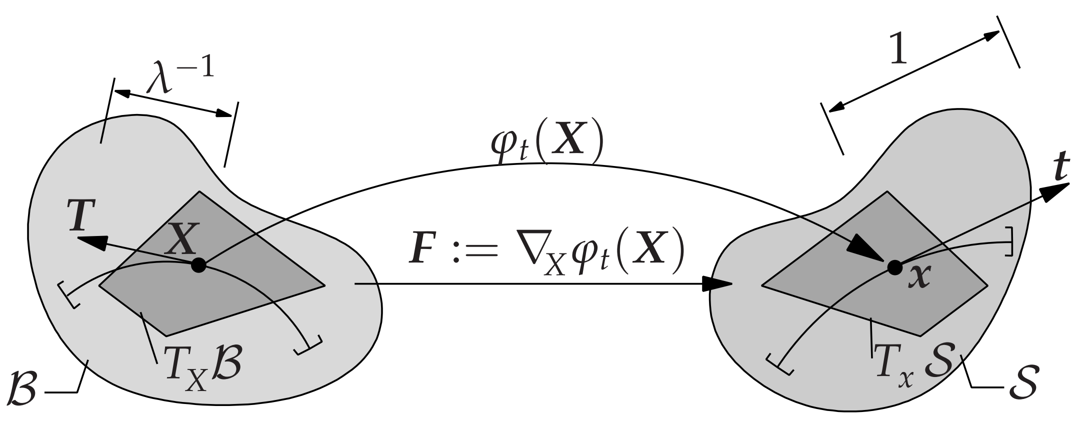
where we already introduced the *right Cauchy-Green tensor*

$$\boldsymbol{C} := \boldsymbol{F}^T \cdot \boldsymbol{F} \,, \qquad C_{AB} = F_{aA} F_{aB} \,. \tag{5.1.26}$$

It is important to observe that the right Cauchy-Green tensor is symmetric and positive-definite; that is,

$$\boldsymbol{C} = \boldsymbol{C}^T = (\boldsymbol{F}^T \cdot \boldsymbol{F})^T = \boldsymbol{F}^T \cdot \boldsymbol{F} \quad \text{and} \quad \boldsymbol{T} \cdot \boldsymbol{C} \cdot \boldsymbol{T} \geq 0 \quad \forall \boldsymbol{T} \in \mathcal{R}^3 \,. \tag{5.1.27}$$

Thus, its eigenvalues $\lambda^2_{\alpha=1,2,3}$ are positive and real numbers. The stretch formulae $\lambda = ||\boldsymbol{t}||_{\boldsymbol{1}}$ and $\lambda = ||\boldsymbol{T}||_C$ are often referred to as the Eulerian and Lagrangean descriptions, respectively. Observe that the latter allows us to compute the stretch in terms of material quantities.



Now, let us consider a dual Eulerian (spatial) approach by setting $||\boldsymbol{t}||_{\boldsymbol{1}} = 1$ as a reference value. We then express the inverse stretch

$$\begin{aligned}\lambda^{-1} &= \sqrt{\boldsymbol{T}\cdot\boldsymbol{T}} = \sqrt{(\boldsymbol{F}^{-1}\cdot\boldsymbol{t})\cdot(\boldsymbol{F}^{-1}\cdot\boldsymbol{t})} \\ &= \sqrt{\boldsymbol{t}\cdot(\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1})\cdot\boldsymbol{t}} = \sqrt{\boldsymbol{t}\cdot\boldsymbol{b}^{-1}\boldsymbol{t}} =: ||\boldsymbol{t}||_{\boldsymbol{b}^{-1}}\end{aligned} \tag{5.1.28}$$

in terms of the *inverse left Cauchy-Green tensor* $\boldsymbol{b}^{-1}$

$$\boldsymbol{b}^{-1} := \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\,, \qquad b^{-1}_{ab} = F^{-1}_{Aa}F^{-1}_{Ab}\,. \tag{5.1.29}$$

The *left Cauchy-Green tensor* $\boldsymbol{b}$ is then defined as

$$\boldsymbol{b} := \boldsymbol{F}\cdot\boldsymbol{F}^{T}\,, \qquad b_{ab} = F_{aA}F_{bA}\,. \tag{5.1.30}$$

From this definition, we readily note that the left Cauchy-Green (Finger) tensor is also symmetric and positive-definite, i.e.

$$\boldsymbol{b} = \boldsymbol{b}^{T} = (\boldsymbol{F}\cdot\boldsymbol{F}^{T})^{T} = \boldsymbol{F}\cdot\boldsymbol{F}^{T} \quad \text{and} \quad \boldsymbol{t}\cdot\boldsymbol{b}\cdot\boldsymbol{t} \geq 0 \quad \forall\boldsymbol{t}\in\mathcal{R}^{3}\,. \tag{5.1.31}$$

In (5.1.25) and (5.1.28), we observe that $\boldsymbol{C}$ and $\boldsymbol{b}^{-1}$ act as metric tensors in the respective Lagrangean and Eulerian description of the length deformation.

5.1.4.2 Strain Tensors

Having the stretch defined in (5.1.25), we are now in a position to define the *Green-Lagrange strain tensor*. The Green-Lagrange strain measure ε_{GL}, Lagrangean strain measure, compares the squared lengths of the spatial vector $||\boldsymbol{t}||^2_{\boldsymbol{1}} = ||\boldsymbol{T}||^2_{\boldsymbol{C}}$ and the reference vector $||\boldsymbol{T}||^2_{\boldsymbol{1}} = 1$ in an *additive* manner

$$\varepsilon_{GL} := \tfrac{1}{2}\left[\lambda^2 - 1\right]\,. \tag{5.1.32}$$

Insertion of the above definitions yields

$$\begin{aligned} \varepsilon_{GL} &:= \tfrac{1}{2}\,[\,||\boldsymbol{T}||^2_{\boldsymbol{C}} - ||\boldsymbol{T}||^2_{\boldsymbol{1}}] \\ &= \tfrac{1}{2}\,[\boldsymbol{T}\cdot\boldsymbol{C}\cdot\boldsymbol{T} - \boldsymbol{T}\cdot\boldsymbol{1}\cdot\boldsymbol{T}] \\ &= \boldsymbol{T}\cdot\tfrac{1}{2}\,[\boldsymbol{C}-\boldsymbol{1}]\cdot\boldsymbol{T} =: \boldsymbol{T}\cdot\boldsymbol{E}\cdot\boldsymbol{T}\,. \end{aligned} \tag{5.1.33}$$

The Green-Lagrange strain tensor $\boldsymbol{E}$ is then defined as

$$\boldsymbol{E} := \tfrac{1}{2}\,[\boldsymbol{C}-\boldsymbol{1}]\,, \qquad E_{AB} = \tfrac{1}{2}\,[C_{AB}-\delta_{AB}]\,. \tag{5.1.34}$$

Analogous to the dual approach we considered for the inverse stretch, the Eulerian strain measure, the so-called *Almansi strain* measure is defined as

$$\varepsilon_A := \tfrac{1}{2}\,[1-\lambda^{-2}]\,, \tag{5.1.35}$$

where $||\boldsymbol{t}||^2_{\boldsymbol{1}} = 1$ and $||\boldsymbol{t}||^2_{\boldsymbol{b}^{-1}} = \lambda^{-2}$. Incoporating these definitions, we have

$$\begin{aligned} \varepsilon_{A} &:= \tfrac{1}{2}\,[\,||\boldsymbol{t}||^2_{\boldsymbol{1}} - ||\boldsymbol{t}||^2_{\boldsymbol{b}^{-1}}] \\ &= \tfrac{1}{2}\,[\boldsymbol{t}\cdot\boldsymbol{1}\cdot\boldsymbol{t} - \boldsymbol{t}\cdot\boldsymbol{b}^{-1}\cdot\boldsymbol{t}] \\ &= \boldsymbol{t}\cdot\tfrac{1}{2}\,[\boldsymbol{1}-\boldsymbol{b}^{-1}]\cdot\boldsymbol{t} =: \boldsymbol{t}\cdot\boldsymbol{e}\cdot\boldsymbol{t}\,. \end{aligned} \tag{5.1.36}$$

with

$$\boldsymbol{e} := \tfrac{1}{2}\,[\boldsymbol{1}-\boldsymbol{b}^{-1}]\,, \qquad e_{ab} = \tfrac{1}{2}\,[\delta_{ab}-b^{-1}_{ab}]\,. \tag{5.1.37}$$

denoting the Eulerian *Almansi strain tensor*.

It is important to note that linearization of the both strain tensors, $\boldsymbol{E}$ and $\boldsymbol{e}$ about the undeformed state leads to the strain tensor ε=sym$(\nabla_x \boldsymbol{u})$ in the geometrically linear theory.

$$\varepsilon = \mathrm{sym}(\nabla_x \boldsymbol{u}) = \mathrm{Lin}|_{\boldsymbol{F}=\boldsymbol{1}}\boldsymbol{E} = \mathrm{Lin}|_{\boldsymbol{F}=\boldsymbol{1}}\boldsymbol{e} \tag{5.1.38}$$

5.1.5 Material and Spatial Velocity Gradients

Recall that the stretch vector $\boldsymbol{t} = \boldsymbol{F} \cdot \boldsymbol{T} \;\; \in T_x \mathcal{S}$ was defined as a tangent map of the material tangent $\boldsymbol{T} \in T_X \mathcal{B}$. The total time derivative of $\boldsymbol{t}$ is then given by

$$\dot{\boldsymbol{t}} = \dot{\boldsymbol{F}} \cdot \boldsymbol{T} =: \boldsymbol{L} \cdot \boldsymbol{T} \,, \tag{5.1.39}$$

where the time derivative of the deformation gradient can be expressed as

$$\boldsymbol{L} := \dot{\boldsymbol{F}} = \frac{\partial}{\partial t} \left(\nabla_X \varphi(\boldsymbol{X}, t) \right) = \nabla_X \left(\frac{\partial}{\partial t} \varphi(\boldsymbol{X}, t) \right) = \nabla_X \boldsymbol{V}(\boldsymbol{X}, t) \tag{5.1.40}$$

the gradient of the material velocity with respect to the material coordinates. Therefore, we call $\boldsymbol{L}$ the *material velocity gradient* that maps the material tangent vector $\boldsymbol{T}$ on the total time derivative of its spatial counter part as shown in (5.1.39).

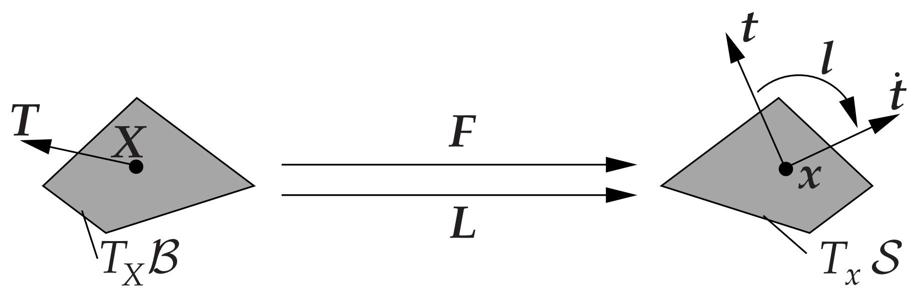
The material vector $\boldsymbol{T}$ in (5.1.39) can be expressed as $\boldsymbol{T} = \boldsymbol{F}^{-1} \cdot \boldsymbol{t}$. Here, the inverse of the deformation gradient can be obtained thorugh $\boldsymbol{F}^{-1} = \nabla_x \boldsymbol{X}$. Then, the rate of the stretch vector assumes the form

$$\dot{\boldsymbol{t}} = (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}) \cdot \boldsymbol{t} =: \boldsymbol{l} \cdot \boldsymbol{t} \tag{5.1.41}$$

where we introduced the *spatial velocity gradient*

$$\boldsymbol{l} := \boldsymbol{L} \cdot \boldsymbol{F}^{-1} = \nabla_X \boldsymbol{V}(\boldsymbol{X}, t) \cdot \nabla_x \boldsymbol{X} = \nabla_x \boldsymbol{v}(\boldsymbol{x}, t) \tag{5.1.42}$$

by using the relation $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{V}(\boldsymbol{X}, t) \circ \varphi_t^{-1}(\boldsymbol{X})$.



Furthermore, we note that the material and spatial velocity gradients can be conceived as the maps

$$\boldsymbol{L}(\boldsymbol{X},t) : \begin{cases} T_X\mathcal{B} \;\rightarrow\; T_x\,\mathcal{S}\;, \\ \quad \boldsymbol{T} \;\mapsto\; \dot{\boldsymbol{t}} = \boldsymbol{L}\cdot\boldsymbol{T}\;, \end{cases} \tag{5.1.43}$$

and

$$\boldsymbol{l}(\boldsymbol{x},t) : \begin{cases} T_x\,\mathcal{S} \;\rightarrow\; T_x\,\mathcal{S}\;, \\ \quad \boldsymbol{t} \;\mapsto\; \dot{\boldsymbol{t}} = \boldsymbol{l}\cdot\boldsymbol{t}\;. \end{cases} \tag{5.1.44}$$

5.1.6 Rate of Deformation and Spin Tensors

Having the spatial velocity gradient $\boldsymbol{l}$ defined, we are now in a position to define the *rate of deformation tensor* $\boldsymbol{d}$ and the *spin tensor* $\boldsymbol{w}$ by additively splitting

$$\boldsymbol{l} = \boldsymbol{d} + \boldsymbol{w} \tag{5.1.45}$$

into its symmetric and skew-symmetric parts. That is, the *rate of deformation tensor* is defined by

$$\boldsymbol{d} := \mathrm{sym}(\boldsymbol{l}) = \tfrac{1}{2}\,(\boldsymbol{l} + \boldsymbol{l}^T)\;, \tag{5.1.46}$$

and

$$\boldsymbol{w} := \mathrm{skew}(\boldsymbol{l}) = \tfrac{1}{2}\,(\boldsymbol{l} - \boldsymbol{l}^T)\;, \tag{5.1.47}$$

denotes the *spin tensor*. Therefore the map (5.1.41) can also be additively decomposed into

$$\dot{\boldsymbol{t}} = \boldsymbol{l} \cdot \boldsymbol{t} = \boldsymbol{d} \cdot \boldsymbol{t} + \boldsymbol{w} \cdot \boldsymbol{t} \,, \tag{5.1.48}$$

in which the part $\boldsymbol{d} \cdot \boldsymbol{t}$ related with the rate of stretching and the part $\boldsymbol{w} \cdot \boldsymbol{t}$ associated with rotation.

In order to interpret the geometrical meaning of the rate of deformation tensor, we consider the time rate of square of the stretch by following its material description in (5.1.25)

$$\frac{d}{dt}(\lambda^2) = 2\lambda\dot{\lambda} = \boldsymbol{T} \cdot \dot{\boldsymbol{C}} \cdot \boldsymbol{T} \,. \tag{5.1.49}$$

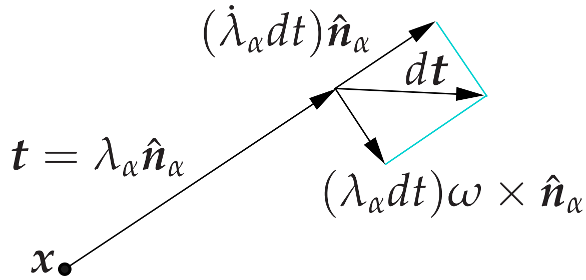
Since $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{t} = \boldsymbol{F} \cdot \boldsymbol{T}$, we have $\dot{\boldsymbol{C}} = \dot{\boldsymbol{F}}^T \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}$ and $\boldsymbol{T} = \boldsymbol{F}^{-1} \cdot \boldsymbol{t}$. Insertion of these results leads us to

$$\begin{aligned} 2\lambda\dot{\lambda} &= \boldsymbol{T} \cdot \dot{\boldsymbol{C}} \cdot \boldsymbol{T} \\ &= (\boldsymbol{F}^{-1} \cdot \boldsymbol{t}) \cdot (\dot{\boldsymbol{F}}^T \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}) \cdot (\boldsymbol{F}^{-1} \cdot \boldsymbol{t}) \\ &= \boldsymbol{t} \cdot (\boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^T + \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}) \cdot \boldsymbol{t} \\ &= \boldsymbol{t} \cdot (\boldsymbol{l}^T + \boldsymbol{l}) \cdot \boldsymbol{t} = \boldsymbol{t} \cdot 2\boldsymbol{d} \cdot \boldsymbol{t} \end{aligned} \tag{5.1.50}$$

Introducing a unit vector $\boldsymbol{g}$, i.e. $||\boldsymbol{g}||$=1, parallel to $\boldsymbol{t} = \lambda \boldsymbol{g}$, we end up with the time derivative of the *logarithmic strain* $\varepsilon_{\text{ln}} := \ln(\lambda)$ in the direction $\boldsymbol{g}$

$$\dot{\varepsilon}_{\text{ln}} = \dot{\overline{\ln \lambda}} = \dot{\lambda}/\lambda = \boldsymbol{d} : (\boldsymbol{g} \otimes \boldsymbol{g}) = \boldsymbol{g} \cdot \boldsymbol{d} \cdot \boldsymbol{g}$$

identical to the component $d_{gg} = \boldsymbol{g} \cdot \boldsymbol{d} \cdot \boldsymbol{g}$ of the rate of deformation tensor. This means, if we had $\boldsymbol{g} = \hat{\boldsymbol{n}}_\alpha$ with $\hat{\boldsymbol{n}}_\alpha$ denoting an eigenvector of $\boldsymbol{d} = \sum_{\alpha=1}^{3} d_\alpha \hat{\boldsymbol{n}}_\alpha \otimes \hat{\boldsymbol{n}}_\alpha$ then the corresponding eigenvalue d_α would be identical to rate of the logarithmic strain in that principal direction, i.e. $d_\alpha = \dot{\lambda}_\alpha / \lambda_\alpha$.



We can now exploit this result to elucidate the geometrical meaning of the spin tensor $\boldsymbol{w}$. For this purpose, we consider a spatial vector $\boldsymbol{t} = \lambda_\alpha \hat{\boldsymbol{n}}_\alpha$ chosen to be parallel to a principal (eigen) vector of $\boldsymbol{d}$. Its time derivative can then be written as

$$\dot{\boldsymbol{t}} = \dot{\lambda}_\alpha \hat{\boldsymbol{n}}_\alpha + \lambda_\alpha \dot{\hat{\boldsymbol{n}}}_\alpha \,. \tag{5.1.51}$$

From (5.1.48), we also know that

$$\dot{\boldsymbol{t}} = \boldsymbol{l} \cdot \boldsymbol{t} = \lambda_\alpha \boldsymbol{d} \cdot \hat{\boldsymbol{n}}_\alpha + \lambda_\alpha \boldsymbol{w} \cdot \hat{\boldsymbol{n}}_\alpha = \dot{\lambda}_\alpha \hat{\boldsymbol{n}}_\alpha + \lambda_\alpha \boldsymbol{w} \cdot \hat{\boldsymbol{n}}_\alpha \tag{5.1.52}$$

where we incorporated the result $\boldsymbol{d} \cdot \hat{\boldsymbol{n}}_\alpha = d_\alpha \hat{\boldsymbol{n}}_\alpha = \dot{\lambda}_\alpha / \lambda_\alpha \hat{\boldsymbol{n}}_\alpha$. Comparison of the expressions in (5.1.51) and (5.1.52) yields the identity

$$\dot{\hat{\boldsymbol{n}}}_\alpha = \boldsymbol{w} \cdot \hat{\boldsymbol{n}}_\alpha = \omega \times \hat{\boldsymbol{n}}_\alpha \tag{5.1.53}$$

where ω stands for the axial vector of $\boldsymbol{w}$. From this result, it can be concluded that the spin tensor $\boldsymbol{w}$ governs the infinitesimal rotation of the eigenvectors of the rate of deformation tensor. Combining the both results for this particular vector $\boldsymbol{t} = \lambda_\alpha \hat{\boldsymbol{n}}_\alpha$, we end up with

$$\dot{\boldsymbol{t}} = \dot{\lambda}_\alpha \hat{\boldsymbol{n}}_\alpha + \lambda_\alpha \omega \times \hat{\boldsymbol{n}}_\alpha \,. \tag{5.1.54}$$

5.1.7 Push-Forward and Pull-Back Relations

Lagrangean and Eulerian fields are related through the *push-forward* and *pull-back* operations. For the rate of deformation tensors, for example, we observe in (5.1.50) that

$$\boldsymbol{d} = \boldsymbol{F}^{-T} \cdot \tfrac{1}{2}\dot{\boldsymbol{C}} \cdot \boldsymbol{F}^{-1} \tag{5.1.55}$$

holds. We then say that the rate of deformation tensor $\boldsymbol{d}$ is the *push-forward* of the time derivative of the Green-Lagrange strain tensor $\dot{\boldsymbol{E}} \equiv \frac{1}{2}\dot{\boldsymbol{C}}$ and denote the push-forward operation

$$\boldsymbol{d} = \varphi_*(\dot{\boldsymbol{E}}) = \varphi_*(\tfrac{1}{2}\dot{\boldsymbol{C}}) := \boldsymbol{F}^{-T} \cdot \left(\tfrac{1}{2}\dot{\boldsymbol{C}}\right) \cdot \boldsymbol{F}^{-1} \, , \tag{5.1.56}$$

or in indicial notation

$$d_{ab} = F^{-1}_{Aa}\,\tfrac{1}{2}\,\dot{C}_{AB}F^{-1}_{Bb} \, .$$

Analogously, we have the identity

$$\tfrac{1}{2}\dot{\boldsymbol{C}} = \dot{\boldsymbol{E}} = \boldsymbol{F}^{T} \cdot \boldsymbol{d} \cdot \boldsymbol{F} \tag{5.1.57}$$

where $\frac{1}{2}\dot{\boldsymbol{C}} = \dot{\boldsymbol{E}}$ is the *pull-back* of the rate of deformation tensor $\boldsymbol{d}$

$$\dot{\boldsymbol{E}} = \tfrac{1}{2}\dot{\boldsymbol{C}} = \varphi^*(\boldsymbol{d}) = \boldsymbol{F}^{T} \cdot \boldsymbol{d} \cdot \boldsymbol{F} \, , \tag{5.1.58}$$

or in indicial notation

$$\dot{E}_{AB} = \tfrac{1}{2}\,\dot{C}_{AB} = F_{aA}d_{ab}F_{bB} \, .$$

The Almansi strain tensor and the Green-Lagrange strain tensor are also related through the pull-back and push-forward maps, i.e. $\boldsymbol{E} = \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F}$. The Almansi strain tensor $\boldsymbol{e}$ is the *push-forward* of the Green-Lagrange strain tensor $\boldsymbol{E}$

$$\boldsymbol{e} = \varphi_*(\boldsymbol{E}) := \boldsymbol{F}^{-T} \cdot \boldsymbol{E} \cdot \boldsymbol{F}^{-1} \, , \quad e_{ab} = F^{-1}_{Aa}E_{AB}F^{-1}_{Bb} \, . \tag{5.1.59}$$

Analogously, $\boldsymbol{E}$ is the *pull-back* of the Almansi strain tensor $\boldsymbol{e}$

$$\boldsymbol{E} = \varphi^*(\boldsymbol{e}) = \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F} \ , \quad E_{AB} = F_{aA} e_{ab} F_{bB} \ . \tag{5.1.60}$$

Similar pull-back and push-forward relations exist also among the different stress measures that will be discussed in the next chapter.

5.1.8 Lie Derivative of Spatial Fields

The Lie derivative of a spatial field, say $f(\boldsymbol{x}, t)$, is defined as

$$\pounds_v f(\boldsymbol{x}, t) := \varphi_* \left\{ \frac{d}{dt} \left[\varphi^*(f(\boldsymbol{x}, t)) \right] \right\} \ , \tag{5.1.61}$$

which is none other than the push-forward of the time derivative of the pull-back of the spatial field $f(\boldsymbol{x}, t)$.

For instance, it can be readily shown that the rate of deformation tensor $\boldsymbol{d}$ is the Lie derivative of the Almansi strain tensor $\boldsymbol{e}$. Since we have shown that the rate of deformation tensor $\boldsymbol{d}$ is push-forward of $\dot{\boldsymbol{E}}$, $\boldsymbol{d} = \varphi_*(\dot{\boldsymbol{E}})$ and the Green-Lagrange strain tensor is pull-back of the Almansi strain tensor $\boldsymbol{E} = \varphi^*(\boldsymbol{e})$, we have

$$\pounds_v \boldsymbol{e} := \varphi_* \left\{ \frac{d}{dt} \left[\varphi^*(\boldsymbol{e}) \right] \right\} = \varphi_* \left\{ \frac{d}{dt} \left[\boldsymbol{E} \right] \right\} = \varphi_*(\dot{\boldsymbol{E}}) = \boldsymbol{d} \ . \tag{5.1.62}$$

The Lie (Oldroyd) derivatives constitute the basic class of objective time derivatives of Eulerian objects.