ME338A
CONTINUUM MECHANICS

lecture notes 14

thursday, february 18th, 2010

volumetric - deviatoric decomposition

specific stored energy (quadratic in volumetric and deviatoric strains)

$$\begin{aligned} W &= W^{\text{vol}}(\boldsymbol{\epsilon}^{\text{vol}}) &+& \; W^{\text{dev}}(\boldsymbol{\epsilon}^{\text{dev}}) \\ &= \tfrac{1}{2}\,\kappa\,[\boldsymbol{\epsilon}^{\text{vol}} : \boldsymbol{I}]^2 &+& \; \mu\,[\boldsymbol{\epsilon}^{\text{dev}2} : \boldsymbol{I}] \end{aligned} \tag{4.3.16}$$

with

$$\begin{aligned} \boldsymbol{\epsilon}^{\text{vol}} &= \tfrac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\text{vol}} : \boldsymbol{\epsilon} \\ \boldsymbol{\epsilon}^{\text{dev}} &= \boldsymbol{\epsilon} - \tfrac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{\epsilon} \end{aligned} \tag{4.3.17}$$

stress tensor

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W = \mathbb{E} : \boldsymbol{\epsilon} = 3\,\kappa\,\boldsymbol{\epsilon}^{\text{vol}} + 2\,\mu\,\boldsymbol{\epsilon}^{\text{dev}} \tag{4.3.18}$$

matrix representation of coordinates

$$[\,\sigma_{ij}\,] = \begin{bmatrix} \kappa I_\epsilon + 2\,\mu\epsilon^{\text{dev}}_{11} & 2\,\mu\epsilon^{\text{dev}}_{12} & 2\,\mu\epsilon^{\text{dev}}_{13} \\ 2\,\mu\epsilon^{\text{dev}}_{21} & \kappa I_\epsilon + 2\,\mu\epsilon^{\text{dev}}_{22} & 2\,\mu\epsilon^{\text{dev}}_{23} \\ 2\,\mu\epsilon^{\text{dev}}_{31} & 2\,\mu\epsilon^{\text{dev}}_{32} & \kappa I_\epsilon + 2\,\mu\epsilon^{\text{dev}}_{33} \end{bmatrix} \tag{4.3.19}$$

linear elastic continuum tangent stiffness

$$\mathbb{E}^{\tan} = 3\,\kappa\mathbb{I}^{\text{vol}} + 2\,\mu\mathbb{I}^{\text{dev}} \qquad \mathrm{D}_t\boldsymbol{\sigma} = \mathbb{E}^{\tan} : \mathrm{D}_t\boldsymbol{\epsilon} \tag{4.3.20}$$

linear elastic continuum secant stiffness

$$\mathbb{E} = 3\,\kappa \mathbb{I}^{\text{vol}} + 2\,\mu \mathbb{I}^{\text{dev}} \qquad \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.3.21}$$

Voigt representation of stiffness tensor

$$\mathbb{E} = \begin{bmatrix} \kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & 0 & 0 \\ \kappa - \frac{2}{3}\mu & \kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & 0 & 0 \\ \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & \kappa + \frac{4}{3}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{4.3.22}$$

transformation with fourth order unit tensors

$$\begin{aligned} \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\,\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{dev}} &= \mathbb{I}^{\text{sym}} - \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{sym}} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \end{aligned} \tag{4.3.23}$$

thus

$$\begin{aligned} \mathbb{E} = \mathbb{E}^{\text{tan}} &= 3\,\kappa\,\mathbb{I}^{\text{vol}} + 2\,\mu\,\mathbb{I}^{\text{dev}} \\ &= 3\,[\,\kappa - \tfrac{2}{3}\mu]\,\mathbb{I}^{\text{vol}} + 2\,\mu\,\mathbb{I}^{\text{sym}} \\ &= 3\,\lambda\,\mathbb{I}^{\text{vol}} + 2\,\mu\,\mathbb{I}^{\text{sym}} \\ &= \lambda \boldsymbol{I} \otimes \boldsymbol{I} + 2\,\mu\,\mathbb{I}^{\text{sym}} \end{aligned} \tag{4.3.24}$$

bulk modulus and shear modulus

$$\kappa = \lambda + \frac{2}{3}\,\mu \qquad \text{and} \qquad \mu \tag{4.3.25}$$

comparison of coordinates E_{1111} and E_{1122}

$$\begin{aligned} \kappa + \frac{4}{3}\mu &= \lambda + \frac{2}{3}\mu + \frac{4}{3}\mu = \lambda + 2\,\mu \\ \kappa - \frac{2}{3}\mu &= \lambda + \frac{2}{3}\mu - \frac{2}{3}\mu = \lambda \end{aligned} \tag{4.3.26}$$

restrictions to elastic constants from positive stored energy, i.e. positive definite elastic stiffness

$$\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon} > 0 \quad \forall \boldsymbol{\epsilon} \neq \mathbf{0} \quad \rightarrow \quad \kappa, \mu > 0, \lambda > -\frac{2}{3}\mu \tag{4.3.27}$$

4.3.2 Specific complementary energy

fourth order elasticity tensor

$$\mathbb{E} = 2\,\mu \mathbb{I}^{\text{sym}} + \lambda \boldsymbol{I} \otimes \boldsymbol{I} \qquad \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.3.28}$$

inversion by making use of Sherman-Morrison-Woodbury theorem

$$/\!\!\mathbb{A} = \mathbb{B} + \alpha \boldsymbol{C} \otimes \boldsymbol{D} \tag{4.3.29}$$

inverse of rank 1 modified fourth order tensor

$$/\!\!\mathbb{A}^{-1} = \mathbb{B}^{-1} - \alpha \, \frac{\mathbb{B}^{-1} : \boldsymbol{C} \otimes \boldsymbol{D} : \mathbb{B}^{-1}}{1 + \alpha \, \boldsymbol{D} : \mathbb{B}^{-1} : \boldsymbol{C}} \tag{4.3.30}$$

with $/\!\!\mathbb{A} = \mathbb{E}$, $/\!\!\mathbb{A}^{-1} = \mathbb{C}$, $\mathbb{B} = 2\,\mu \mathbb{I}^{\text{sym}}$, $\alpha = \lambda$, $\boldsymbol{C} = \boldsymbol{I}$ and $\boldsymbol{D} = \boldsymbol{I}$ we obtain the fourth order compliance tensor

$$\mathbb{C} = \frac{1}{2\,\mu}\mathbb{I}^{\text{sym}} - \frac{\lambda}{2\,\mu[2\,\mu + 3\,\lambda]}\boldsymbol{I} \otimes \boldsymbol{I} \qquad \boldsymbol{\epsilon} = \mathbb{C} : \boldsymbol{\sigma} \tag{4.3.31}$$

or rather

$$\mathbb{C} = \gamma \boldsymbol{I} \otimes \boldsymbol{I} + \frac{1}{2\,\mu}\mathbb{I}^{\text{sym}} \quad \text{with} \quad \gamma = -\frac{\lambda}{2\mu[2\mu + 3\lambda]} \tag{4.3.32}$$

check
with $\mathbb{I}^{\text{sym}} : \mathbb{I}^{\text{sym}} = \mathbb{I}^{\text{sym}}$, $\mathbb{I}^{\text{sym}} : [\boldsymbol{I} \otimes \boldsymbol{I}] = \boldsymbol{I} \otimes \boldsymbol{I}$ and $[\boldsymbol{I} \otimes \boldsymbol{I}] : [\boldsymbol{I} \otimes \boldsymbol{I}] = 3\,\boldsymbol{I} \otimes \boldsymbol{I}$

$$\begin{aligned} \mathbb{E} : \mathbb{E}^{-1} &= [2\,\mu \mathbb{I}^{\text{sym}} + \lambda \boldsymbol{I} \otimes \boldsymbol{I}] : [\frac{1}{2\,\mu}\mathbb{I}^{\text{sym}} - \frac{\lambda}{2\,\mu[2\,\mu + 3\,\lambda]}\boldsymbol{I} \otimes \boldsymbol{I}] \\ &= \mathbb{I}^{\text{sym}} + [\frac{\lambda}{2\mu} - \frac{2\mu\lambda}{2\mu[2\mu + 3\lambda]} - \frac{3\,\lambda^2}{2\mu[2\mu + 3\lambda]}]\boldsymbol{I} \otimes \boldsymbol{I} = \mathbb{I}^{\text{sym}} \end{aligned} \tag{4.3.33}$$

Voigt representation of compliance tensor

$$\mathbb{C} = \begin{bmatrix} \gamma + \frac{1}{2\mu} & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & \gamma + \frac{1}{2\mu} & \gamma & 0 & 0 & 0 \\ \gamma & \gamma & \gamma + \frac{1}{2\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4\mu} \end{bmatrix} \tag{4.3.34}$$

strain tensor (linear in stresses)

$$\boldsymbol{\epsilon} = \mathrm{D}_{\boldsymbol{\sigma}} W^* = \mathbb{C} : \boldsymbol{\sigma} = \gamma [\, \boldsymbol{\sigma} : \boldsymbol{I} \,] \, \boldsymbol{I} + \frac{1}{2\mu} \boldsymbol{\sigma} \tag{4.3.35}$$

matrix representation of coordinates

$$[\, \epsilon_{ij} \,] = \begin{bmatrix} \gamma I_\sigma + \frac{1}{2\mu}\sigma_{11} & \frac{1}{2\mu}\sigma_{12} & \frac{1}{2\mu}\sigma_{13} \\ \frac{1}{2\mu}\sigma_{21} & \gamma I_\sigma + \frac{1}{2\mu}\sigma_{22} & \frac{1}{2\mu}\sigma_{23} \\ \frac{1}{2\mu}\sigma_{31} & \frac{1}{2\mu}\sigma_{32} & \gamma I_\sigma + \frac{1}{2\mu}\sigma_{33} \end{bmatrix} \tag{4.3.36}$$

specific complementary energy (quadratic in stresses)

$$W^* = \frac{1}{2} \boldsymbol{\sigma} : \mathbb{C} : \boldsymbol{\sigma} = \frac{1}{2} \gamma \, [\boldsymbol{\sigma} : \boldsymbol{I}]^2 + \frac{1}{4\mu} [\boldsymbol{\sigma}^2 : \boldsymbol{I}] \tag{4.3.37}$$

volumetric - deviatoric decomposition

fourth order elasticity tensor

$$\mathbb{E} = 2\,\mu \mathbb{I}^{\text{dev}} + 3\,\kappa \mathbb{I}^{\text{vol}} \qquad \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.3.38}$$

inversion by making use of orthogonality of $\mathbb{I}^{\text{vol}}$ and $\mathbb{I}^{\text{dev}}$ yields fourth order compliance tensor

$$\mathbb{C} = \frac{1}{2\,\mu}\mathbb{I}^{\text{dev}} + \frac{1}{3\,\kappa}\mathbb{I}^{\text{vol}} \qquad \boldsymbol{\epsilon} = \mathbb{C} : \boldsymbol{\sigma} \tag{4.3.39}$$

check

with $\mathbb{I}^{\text{dev}} : \mathbb{I}^{\text{dev}} = \mathbb{I}^{\text{dev}}$, $\mathbb{I}^{\text{vol}} : \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{vol}}$ and $\mathbb{I}^{\text{dev}} : \mathbb{I}^{\text{vol}} = \mathbb{O}$

$$\begin{aligned} \mathbb{E} : \mathbb{E}^{-1} &= [2\,\mu \mathbb{I}^{\text{dev}} + 3\,\kappa\, \mathbb{I}^{\text{vol}}] : [\frac{1}{2\,\mu}\mathbb{I}^{\text{dev}} + \frac{1}{3\,\kappa}\mathbb{I}^{\text{vol}}] \\ &= \mathbb{I}^{\text{dev}} + \mathbb{I}^{\text{vol}} = \mathbb{I} \end{aligned} \tag{4.3.40}$$

Voigt representation of compliance

$$\mathbb{C} = \begin{bmatrix} \frac{1}{9\kappa} + \frac{2}{6\mu} & \frac{1}{9\kappa} - \frac{1}{6\mu} & \frac{1}{9\kappa} - \frac{1}{6\mu} & 0 & 0 & 0 \\ \frac{1}{9\kappa} - \frac{1}{6\mu} & \frac{1}{9\kappa} + \frac{2}{6\mu} & \frac{1}{9\kappa} - \frac{1}{6\mu} & 0 & 0 & 0 \\ \frac{1}{9\kappa} - \frac{1}{6\mu} & \frac{1}{9\kappa} - \frac{1}{6\mu} & \frac{1}{9\kappa} + \frac{2}{6\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4\mu} \end{bmatrix} \tag{4.3.41}$$

comparison of coordinates C_{1111} and C_{1122}

$$\begin{aligned}
\frac{1}{9\kappa}+\frac{2}{6\mu} &= \frac{1}{9\lambda+6\mu}+\frac{2}{6\mu}=\frac{1}{3[3\lambda+2\mu]}-\frac{1}{3[2\mu]}+\frac{3}{3[2\mu]}\\
&= \frac{2\mu-3\lambda-2\mu}{3[2\mu][3\lambda+2\mu]}+\frac{1}{2\mu}\\
&= \frac{-\lambda}{2\mu[2\mu+3\lambda]}+\frac{1}{2\mu}=\gamma+\frac{1}{2\mu}\\
\frac{1}{9\kappa}-\frac{1}{6\mu} &= \frac{1}{9\lambda+6\mu}-\frac{1}{6\mu}=\frac{1}{3[3\lambda+2\mu]}-\frac{1}{3[2\mu]}\\
&= \frac{2\mu-3\lambda-2\mu}{3[2\mu][3\lambda+2\mu]}=-\frac{\lambda}{2\mu[2\mu+3\lambda]}=\gamma
\end{aligned} \tag{4.3.42}$$

strain tensor (linear in stresses)

$$\boldsymbol{\epsilon} = \mathrm{D}_{\boldsymbol{\sigma}} W^* = \mathbb{C} : \boldsymbol{\sigma} = \kappa[\,\boldsymbol{\sigma} : \boldsymbol{I}\,]\,\boldsymbol{I} + \frac{1}{4\mu}\,\boldsymbol{\sigma}^{\text{dev}} \tag{4.3.43}$$

matrix representation of coordinates

$$[\,\epsilon_{ij}\,] = \begin{bmatrix} \kappa I_\epsilon + 2\,\mu\epsilon_{11}^{\text{dev}} & 2\,\mu\epsilon_{12}^{\text{dev}} & 2\,\mu\epsilon_{13}^{\text{dev}} \\ 2\,\mu\epsilon_{21}^{\text{dev}} & \kappa I_\epsilon + 2\,\mu\epsilon_{22}^{\text{dev}} & 2\,\mu\epsilon_{23}^{\text{dev}} \\ 2\,\mu\epsilon_{31}^{\text{dev}} & 2\,\mu\epsilon_{32}^{\text{dev}} & \kappa I_\epsilon + 2\,\mu\epsilon_{33}^{\text{dev}} \end{bmatrix} \tag{4.3.44}$$

specific stored energy (quadratic in volumetric and deviatoric stresses)

$$\begin{aligned}
W^* &= W^{*\text{vol}}(\boldsymbol{\sigma}^{\text{vol}}) + W^{*\text{dev}}(\boldsymbol{\sigma}^{\text{dev}})\\
&= \tfrac{1}{2}\,\kappa\,\epsilon^{\text{vol}2} + \mu\,[\boldsymbol{\epsilon}^{\text{dev}2} : \boldsymbol{I}]
\end{aligned} \tag{4.3.45}$$

4.3.3 Elastic constants

isotropic linear elasticity can be characterized by only two elastic constants

	E, ν	E, μ	λ, μ	κ, μ
E	E	E	$\dfrac{\mu\,[3\lambda+2\mu]}{\lambda+\mu}$	$\dfrac{9\kappa\mu}{3\kappa+\mu}$
ν	ν	$\dfrac{E-2\mu}{2\mu}$	$\dfrac{\lambda}{2[\lambda+\mu]}$	$\dfrac{3\kappa-2\mu}{6\kappa+2\mu}$
μ	$\dfrac{E}{2[1+\nu]}$	μ	μ	μ
λ	$\dfrac{E\nu}{[1+\nu][1-2\nu]}$	$\dfrac{\mu[E-2\mu]}{3\mu-E}$	λ	$\kappa-\dfrac{2}{3}\mu$
κ	$\dfrac{E}{3[1-2\nu]}$	$\dfrac{E\mu}{3[3\mu-E]}$	$\lambda+\dfrac{2}{3}\mu$	κ

Table 4.1: relations betweeb elastic constants

4.4 Transversely isotropic hyperelasticity

fiber direction $\boldsymbol{n}$ and structural tensor $\boldsymbol{N}$

$$\boldsymbol{N} = \boldsymbol{n} \otimes \boldsymbol{n} \qquad \text{with} \qquad |\boldsymbol{n}| = 1 \tag{4.4.1}$$

specific stored energy:
isotropic tensor function with two arguments

$$\begin{aligned} W &= W(\boldsymbol{\epsilon}, \boldsymbol{n}) = W(\boldsymbol{\epsilon}, -\boldsymbol{n}) = W(\boldsymbol{\epsilon}, \boldsymbol{N}) \\ &= W(\boldsymbol{Q} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{Q}^{\mathrm{t}}, \boldsymbol{Q} \cdot \boldsymbol{N} \cdot \boldsymbol{Q}^{\mathrm{t}}) \qquad \forall\, \boldsymbol{Q} \in SO(3) \end{aligned} \tag{4.4.2}$$

representation theorem for isotropic tensor functions with two arguments

$$W = W(\boldsymbol{\epsilon}, \boldsymbol{N}) = W(i_{\epsilon}, i_N, i_{\epsilon N},) \tag{4.4.3}$$

irreducible set of ten invariants, integrity basis

$$\begin{aligned} \bar{i}_{\epsilon} &= \{\boldsymbol{\epsilon} : \boldsymbol{I}, \boldsymbol{\epsilon}^2 : \boldsymbol{I}, \boldsymbol{\epsilon}^3 : \boldsymbol{I}\} \\ i_N &= \{\boldsymbol{N} : \boldsymbol{I}, \boldsymbol{N}^2 : \boldsymbol{I}, \boldsymbol{N}^3 : \boldsymbol{I}\} \\ i_{\epsilon,N} &= \{\boldsymbol{\epsilon} : \boldsymbol{N}, \boldsymbol{\epsilon} : \boldsymbol{N}^2, \boldsymbol{\epsilon}^2 : \boldsymbol{N}, \boldsymbol{\epsilon}^2 : \boldsymbol{N}^2\} \end{aligned} \tag{4.4.4}$$

recall different representation of set of three invariants

$$\begin{aligned} \bar{i}_{\epsilon} &= \{\boldsymbol{\epsilon} : \boldsymbol{I}, \boldsymbol{\epsilon}^2 : \boldsymbol{I}, \boldsymbol{\epsilon}^3 : \boldsymbol{I}\} && \text{basic invariants} \\ i_{\epsilon} &= \{\mathrm{tr}(\boldsymbol{\epsilon}), \tfrac{1}{2}[\mathrm{tr}^2(\boldsymbol{\epsilon}) - \mathrm{tr}(\boldsymbol{\epsilon}^2)], \det(\boldsymbol{\epsilon})\} && \text{principal invariants} \\ i_{\epsilon} &= \{(\lambda_{\epsilon 1} + \lambda_{\epsilon 2} + \lambda_{\epsilon 3}), (\lambda_{\epsilon 2}\lambda_{\epsilon 3} + \lambda_{\epsilon 3}\lambda_{\epsilon 1} + \lambda_{\epsilon 1}\lambda_{\epsilon 3}), (\lambda_{\epsilon 1}\lambda_{\epsilon 2}\lambda_{\epsilon 3})\} \\ & && \text{eigenvalue representation of principal invariants} \end{aligned} \tag{4.4.5}$$

until now: principal invariants i_{ϵ}, now: basic invariants $\bar{i}_{\epsilon}$

with properties of structural tensor, idempotence & normalization

$$\begin{aligned} \boldsymbol{N}^2 &= \boldsymbol{N}\cdot\boldsymbol{N} = [\boldsymbol{n}\otimes\boldsymbol{n}]\cdot[\boldsymbol{n}\otimes\boldsymbol{n}] = \boldsymbol{n}\otimes\boldsymbol{n} = \boldsymbol{N} \\ \boldsymbol{N}:\boldsymbol{I} &= \boldsymbol{n}\cdot\boldsymbol{n} = |\boldsymbol{n}|^2 = 1 \end{aligned} \tag{4.4.6}$$

reduced representation with five invariants

$$W = W(\boldsymbol{\epsilon},\boldsymbol{N}) = W(\bar{I}_\epsilon, \bar{II}_\epsilon, \bar{III}_\epsilon, \bar{IV}_\epsilon, \bar{V}_\epsilon) \tag{4.4.7}$$

representation theorem for isotropic tensor functions with two arguments

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon},\boldsymbol{N}) = f_1\boldsymbol{I} + f_2\boldsymbol{\epsilon} + f_3\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon} + f_4\boldsymbol{N} + f_5[\boldsymbol{\epsilon}\cdot\boldsymbol{N} + \boldsymbol{N}\cdot\boldsymbol{\epsilon}] \tag{4.4.8}$$

stress for transversaly isotropic hyperelastic material

$$\begin{aligned} \boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}}W &= \frac{\mathrm{D}W}{\mathrm{D}\bar{I}_\epsilon}\frac{\mathrm{D}\bar{I}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}\bar{II}_\epsilon}\frac{\mathrm{D}\bar{II}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}\bar{III}_\epsilon}\frac{\mathrm{D}\bar{III}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} \\ &= \frac{\mathrm{D}W}{\mathrm{D}\bar{IV}_\epsilon}\frac{\mathrm{D}\bar{IV}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}\bar{V}_\epsilon}\frac{\mathrm{D}\bar{V}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} \end{aligned} \tag{4.4.9}$$

with derivatives of invariants $\bar{I}_\epsilon$, $\bar{II}_\epsilon$, $\bar{III}_\epsilon$, $\bar{IV}_\epsilon$,$\bar{V}_\epsilon$ with respect to second order tensor $\boldsymbol{\epsilon}$

$$\begin{aligned} \bar{I}_\epsilon &= \boldsymbol{\epsilon} : \boldsymbol{I} && \text{linear} && \mathrm{D}_{\boldsymbol{\epsilon}}\bar{I}_\epsilon = \boldsymbol{I} \\ \bar{II}_\epsilon &= \boldsymbol{\epsilon}^2 : \boldsymbol{I} && \text{quadr.} && \mathrm{D}_{\boldsymbol{\epsilon}}\bar{II}_\epsilon = 2\,\boldsymbol{\epsilon} \\ \bar{III}_\epsilon &= \boldsymbol{\epsilon}^3 : \boldsymbol{I} && \text{cubic} && \mathrm{D}_{\boldsymbol{\epsilon}}\bar{III}_\epsilon = 3\,\boldsymbol{\epsilon}^2 \\ \bar{IV}_\epsilon &= \boldsymbol{\epsilon} : \boldsymbol{N} && \text{linear} && \mathrm{D}_{\boldsymbol{\epsilon}}\bar{IV}_\epsilon = \boldsymbol{N} \\ \bar{V}_\epsilon &= \boldsymbol{\epsilon}^2 : \boldsymbol{N} && \text{quadr.} && \mathrm{D}_{\boldsymbol{\epsilon}}\bar{V}_\epsilon = \boldsymbol{\epsilon}\cdot\boldsymbol{N} + \boldsymbol{N}\cdot\boldsymbol{\epsilon} \end{aligned} \tag{4.4.10}$$

with $\bar{V}_\epsilon = \boldsymbol{\epsilon}^2 : \boldsymbol{N} = [\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}] : \boldsymbol{N} = \boldsymbol{\epsilon} : [\boldsymbol{\epsilon}\cdot\boldsymbol{N}] = \boldsymbol{\epsilon} : [\boldsymbol{N}\cdot\boldsymbol{\epsilon}]$

alternatively
$I\bar{V}_\epsilon = \boldsymbol{\epsilon} : \boldsymbol{N} = \boldsymbol{n} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{n}$ and $\bar{V}_\epsilon = \boldsymbol{\epsilon}^2 : \boldsymbol{N} = \boldsymbol{n} \cdot \boldsymbol{\epsilon}^2 \cdot \boldsymbol{n}$
general represenation of stress

$$\begin{aligned} \boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W &= \mathrm{D}_{\bar{I}_\epsilon} W \, \boldsymbol{I} + 2\,\mathrm{D}_{\bar{II}_\epsilon} W \, \boldsymbol{\epsilon} + 3\,\mathrm{D}_{I\bar{I}I_\epsilon} W \, \boldsymbol{\epsilon}^2 \\ &= \mathrm{D}_{I\bar{V}_\epsilon} W \, \boldsymbol{N} + \mathrm{D}_{\bar{V}_\epsilon} W \, [\boldsymbol{\epsilon} \cdot \boldsymbol{N} + \boldsymbol{N} \cdot \boldsymbol{\epsilon}] \end{aligned} \tag{4.4.11}$$

comparison of coefficients

assumption of linearity (quadratic term vanishes), six material parameters $f_{11}, f_{14}, f_2, f_{41}, f_{44}, f_5$

$$\begin{aligned} f_1 &= \ \mathrm{D}_{\bar{I}_\epsilon} W &&= f_{11} \bar{I}_\epsilon + f_{14} I\bar{V}_\epsilon \\ f_2 &= 2\ \mathrm{D}_{\bar{II}_\epsilon} W &&= \text{const.} \\ f_3 &= 3\ \mathrm{D}_{I\bar{I}I_\epsilon} W &&= 0 \\ f_4 &= \ \mathrm{D}_{I\bar{V}_\epsilon} W &&= f_{41} \bar{I}_\epsilon + f_{44} I\bar{V}_\epsilon \\ f_5 &= \ \mathrm{D}_{\bar{V}_\epsilon} W &&= \text{const.} \end{aligned} \tag{4.4.12}$$

specific stored energy (quadratic in strains)

$$\begin{aligned} W = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\epsilon} &= \frac{1}{2} f_{11} \, \bar{I}_\epsilon^2 + \frac{1}{2} f_{14} \, I\bar{V}_\epsilon \bar{I}_\epsilon + \frac{1}{2} f_2 \, \bar{II}_\epsilon \\ &+ \frac{1}{2} f_{41} \, \bar{I}_\epsilon I\bar{V}_\epsilon + \frac{1}{2} f_{44} \, I\bar{V}_\epsilon^2 + f_5 \, \bar{V}_\epsilon^2 \end{aligned} \tag{4.4.13}$$

stress tensor (linear in strains)

$$\begin{aligned} \boldsymbol{\sigma} = \mathrm{D}_\epsilon W = [f_{11} \bar{I}_\epsilon + f_{14} I\bar{V}_\epsilon] \, \boldsymbol{I} \ + f_2 \, \boldsymbol{\epsilon} \\ + [f_{41} \bar{I}_\epsilon + f_{44} I\bar{V}_\epsilon] \, \boldsymbol{N} + f_5 \, [\boldsymbol{\epsilon} \cdot \boldsymbol{N} + \boldsymbol{N} \cdot \boldsymbol{\epsilon}] \end{aligned} \tag{4.4.14}$$

linear elastic continuum tangent stiffness (constant in strains)
$\mathbb{E}^{\text{tan}} = \mathrm{D}_{\boldsymbol{\epsilon}} \boldsymbol{\sigma}$

$$\mathbb{E}^{\text{tan}} = f_{11} \boldsymbol{I} \otimes \boldsymbol{I} + f_{14} \boldsymbol{I} \otimes \boldsymbol{N} + f_{41} \boldsymbol{N} \otimes \boldsymbol{I} + f_{44} \boldsymbol{N} \otimes \boldsymbol{N} + f_2 \mathbb{I}^{\text{sym}} + f_5 \not{\mathbb{A}} \tag{4.4.15}$$

whereby $\not{\mathbb{A}} = 2\, \mathrm{D}_{\boldsymbol{\epsilon}} [\boldsymbol{\epsilon} \cdot \boldsymbol{N} + \boldsymbol{N} \cdot \boldsymbol{\epsilon}]$

due to symmetry reduction to five material parameters $f_{14} = f_{41}$

$$\mathbb{E}^{\text{tan}} = f_{11} \boldsymbol{I} \otimes \boldsymbol{I} + f_{14} [\boldsymbol{I} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{I}] + f_{44} \boldsymbol{N} \otimes \boldsymbol{N} + f_2 \mathbb{I}^{\text{sym}} + f_5 \not{\mathbb{A}} \tag{4.4.16}$$

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linear elastic continuum secant stiffness

$$\mathbb{E} = f_{11} \boldsymbol{I} \otimes \boldsymbol{I} + f_{14} [\boldsymbol{I} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{I}] + f_{44} \boldsymbol{N} \otimes \boldsymbol{N} + f_2 \mathbb{I}^{\text{sym}} + f_5 \not{\mathbb{A}} \tag{4.4.17}$$

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physical interpretation of parameters

interpretation of f_{11}, $f_{14} = f_{41}$, f_{44}, f_2 and f_5 Spencer [1984]

$$W = \frac{1}{2}\, \lambda\, \bar{I}_{\epsilon}^2 + \alpha\, \bar{I}_{\epsilon} I\bar{V}_{\epsilon} + \frac{1}{2}\, \beta\, I\bar{V}_{\epsilon}^2 + \mu_{\perp}\, \bar{I}I_{\epsilon} + 2[\mu_{\parallel} - \mu_{\perp}] \bar{V}_{\epsilon}^2 \tag{4.4.18}$$

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stress tensor (linear in strains)

$$\boldsymbol{\sigma} = \mathrm{D}_{\epsilon} W = [\lambda \, \bar{I}_{\epsilon} + \alpha \, I\bar{V}_{\epsilon}] \, \boldsymbol{I} \; + 2 \, \mu_{\perp} \, \boldsymbol{\epsilon} \\ + [\alpha \, \bar{I}_{\epsilon} + \beta \, I\bar{V}_{\epsilon}] \, \boldsymbol{N} + 2 \, [\mu_{\parallel} - \mu_{\perp}] \, [\boldsymbol{\epsilon} \cdot \boldsymbol{N} + \boldsymbol{N} \cdot \boldsymbol{\epsilon}] \tag{4.4.19}$$

Voigt representation of stiffness tensor for $\boldsymbol{n} = [\,1, 0, 0\,]^{\mathrm{t}}$, i.e. transverse isotropy with respect to the $\boldsymbol{e}_1$ axis

$$\mathbb{E} = \begin{bmatrix} \begin{matrix} \lambda + 2\alpha + 4\mu_{\parallel} \\ -2\mu_{\perp} + \beta \end{matrix} & \lambda + \alpha & \lambda + \alpha & 0 & 0 & 0 \\ \lambda + \alpha & \lambda + 2\mu_{\perp} & \lambda & 0 & 0 & 0 \\ \lambda + \alpha & \lambda & \lambda + 2\mu_{\perp} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_{\parallel} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{\perp} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{\parallel} \end{bmatrix} \tag{4.4.20}$$

comparison of coefficients

$$f_{11} = \lambda \quad f_2 = 2\,\mu_{\perp} \quad f_{14} = \alpha = f_{41} \quad f_{44} = \beta \quad f_5 = 2\,[\mu_{\parallel} - \mu_{\perp}] \tag{4.4.21}$$

with shear moduli $\mu_{\parallel}$ for shear in fiber direction and $\mu_{\perp}$ for shear parallel to the fiber direction