ME338A
CONTINUUM MECHANICS

lecture notes 13

tuesday, february 16th, 2010

4.2 Hyperelasticity

4.2.1 Specific stored energy

a hyperelastic / Green elastic constitutive law can be represented in the following form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\epsilon}) \qquad \text{and} \qquad \mathcal{D}^{\text{loc}} = \mathcal{W} - \mathrm{D}_t\psi = 0 \tag{4.2.1}$$

• invertible relation between stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$ (and stress and strain rates $\mathrm{D}_t\boldsymbol{\sigma}$ and $\mathrm{D}_t\boldsymbol{\epsilon}$) based on a potential
• potential corresponds to elastically stored specific energy
• by construction no dissipation of energy in closed strain circles

stress power $\mathcal{W}$

$$\mathcal{W} = \boldsymbol{\sigma} : \mathrm{D}_t\boldsymbol{\epsilon} \doteq \mathrm{D}_t\psi \tag{4.2.2}$$

ensuring $\mathcal{D}^{\text{loc}} = \mathcal{W} - \mathrm{D}_t\psi = 0$ by construction, thus

$$\psi = \psi(\boldsymbol{\epsilon}) \qquad \text{and} \qquad \mathrm{D}_t\psi = \mathrm{D}_{\boldsymbol{\epsilon}}\psi : \mathrm{D}_t\boldsymbol{\epsilon} \tag{4.2.3}$$

specific stored energy W as path independent integral of stress power $\mathcal{W}$

$$W(\boldsymbol{\epsilon}) = \psi(\boldsymbol{\epsilon}) \tag{4.2.4}$$

with

$$W(\boldsymbol{\epsilon}_{t2}) - W(\boldsymbol{\epsilon}_{t1}) = \int_{t_1}^{t_2} \mathrm{D}_t W \mathrm{d}t = \int_{t_1}^{t_2} \mathcal{W} \mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\epsilon} \tag{4.2.5}$$

generic hyperelastic / Green elastic constitutive law

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W \qquad \text{with} \qquad W = W(\boldsymbol{\epsilon}) \tag{4.2.6}$$

- path independent $\qquad W(\boldsymbol{\epsilon}_{t2}) - W(\boldsymbol{\epsilon}_{t1}) = \int_{t_1}^{t_2} \mathrm{d}\, W$
- no dissipation $\qquad \oint \mathrm{d}\, W = 0$
- symmetric $\qquad \dfrac{\mathrm{D}^2 W}{\mathrm{D}\boldsymbol{\epsilon} \otimes \mathrm{D}\boldsymbol{\epsilon}}$

relation between stress rates and strain rates defines continuum tangent stiffness (fourth order tensor) $\mathbb{E}^{\tan}$

$$\mathrm{D}_t \boldsymbol{\sigma} = \mathbb{E}^{\tan} : \mathrm{D}_t \boldsymbol{\epsilon} \tag{4.2.7}$$

fourth order tangent stiffness / elastic material tangent

$$\mathbb{E}^{\tan} = \frac{\mathrm{D}^2 W}{\mathrm{D}\boldsymbol{\epsilon} \otimes \mathrm{D}\boldsymbol{\epsilon}} = E_{ijkl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{4.2.8}$$

minor and major symmetries: reduction from $3^4 = 81$ to $6^2 = 36$ to 21 coefficients

$$E_{ijkl} = E_{jikl} = E_{jilk} = E_{ijlk} \qquad \text{and} \qquad E_{ijkl} = E_{klij} \tag{4.2.9}$$

4.2.2 Specific complementary energy

specific stored energy

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W \qquad \text{with} \qquad W = W(\boldsymbol{\epsilon}) \tag{4.2.10}$$

Legendre-Fenchel tranform $\boldsymbol{\sigma} \to \boldsymbol{\epsilon}$

$$W^*(\boldsymbol{\sigma}) = \sup_{\boldsymbol{\epsilon}} (\boldsymbol{\sigma} : \boldsymbol{\epsilon} - W(\boldsymbol{\epsilon})) \tag{4.2.11}$$

specific complementary stored energy

$$W^* = W^*(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\boldsymbol{\sigma}) - W(\boldsymbol{\epsilon}(\boldsymbol{\sigma})) \tag{4.2.12}$$

general hyperelastic constitutive law

$$\boldsymbol{\epsilon} = \mathrm{D}_{\boldsymbol{\sigma}} W^* \qquad \text{with} \qquad W^* = W^*(\boldsymbol{\sigma}) \tag{4.2.13}$$

relation between strain rates and stress rates defines continuum tangent compliance (fourth order tensor)$\mathbb{C}^{\tan}$

$$\mathrm{D}_t \boldsymbol{\epsilon} = \mathbb{C}^{\tan} : \mathrm{D}_t \boldsymbol{\sigma} \qquad \text{with} \qquad \mathbb{C}^{\tan} = \mathbb{E}^{\tan\,-1} \tag{4.2.14}$$

fourth order tangent compliance

$$\mathbb{C}^{\tan} = \frac{\mathrm{D}^2 W}{\mathrm{D}\boldsymbol{\sigma} \otimes \mathrm{D}\boldsymbol{\sigma}} = C_{ijkl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{4.2.15}$$

minor and major symmetries: reduction from $3^4 = 81$ to $6^2 = 36$ to 21 coefficients

$$C_{ijkl} = C_{jikl} = C_{jilk} = C_{ijlk} \qquad \text{and} \qquad C_{ijkl} = C_{klij} \tag{4.2.16}$$

4.3 Isotropic hyperelasticity

4.3.1 Specific stored energy

isotropy: identical eigenbasis of stress and strain

$$\boldsymbol{\sigma} = \sum_{i_1}^{3} \lambda_{\sigma i} \boldsymbol{n}_{\sigma i} \otimes \boldsymbol{n}_{\sigma i} \qquad \boldsymbol{\epsilon} = \sum_{i_1}^{3} \lambda_{\epsilon i} \boldsymbol{n}_{\epsilon i} \otimes \boldsymbol{n}_{\epsilon i} \tag{4.3.1}$$

representation theorem for isotropic tensor–valued tensor functions

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon}) = f_1 \boldsymbol{I} + f_2 \boldsymbol{\epsilon} + f_3 \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} \tag{4.3.2}$$

with $f_i = f_i(I_\epsilon, II_\epsilon, III_\epsilon)$ function of strain invariants

$$\begin{aligned} I_\epsilon &= \operatorname{tr}(\boldsymbol{\epsilon}) \\ II_\epsilon &= \tfrac{1}{2}[\operatorname{tr}^2(\boldsymbol{\epsilon}) - \operatorname{tr}(\boldsymbol{\epsilon}^2)] \\ III_\epsilon &= \det(\boldsymbol{\epsilon}) \end{aligned} \tag{4.3.3}$$

representation theorem for isotropic scalar–valued tensor functions

$$W(\boldsymbol{\epsilon}) = W(I_\epsilon, II_\epsilon, III_\epsilon) \tag{4.3.4}$$

with $W(\boldsymbol{\epsilon}) = W(\boldsymbol{Q} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{Q}^{\mathrm{t}}) \; \forall \, \boldsymbol{Q} \in SO(3)$

stress for hyperelastic material

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W = \frac{\mathrm{D}W}{\mathrm{D}I_\epsilon} \frac{\mathrm{D}I_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}II_\epsilon} \frac{\mathrm{D}II_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}III_\epsilon} \frac{\mathrm{D}III_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} \tag{4.3.5}$$

with derivatives of invariants I_ϵ, II_ϵ, III_ϵ with respect to second order tensor $\boldsymbol{\epsilon}$

$$\begin{aligned} \mathrm{D}_{\boldsymbol{\epsilon}}\ I_\epsilon &= \boldsymbol{I} \\ \mathrm{D}_{\boldsymbol{\epsilon}}\ II_\epsilon &= -\boldsymbol{\epsilon} + I_\epsilon\,\boldsymbol{I} \\ \mathrm{D}_{\boldsymbol{\epsilon}}\ III_\epsilon &= = III_\epsilon\,\boldsymbol{\epsilon}^{-\mathrm{t}} = \boldsymbol{\epsilon}^2 - I_\epsilon\boldsymbol{\epsilon} + II_\epsilon\boldsymbol{I} \end{aligned} \tag{4.3.6}$$

general representation of stress

$$\boldsymbol{\sigma} = \mathrm{D}_{I_\epsilon}W\ \boldsymbol{I} + \mathrm{D}_{II_\epsilon}W\ [-\boldsymbol{\epsilon} + I_\epsilon\,\boldsymbol{I}] + \mathrm{D}_{III_\epsilon}W\ [\boldsymbol{\epsilon}^2 - I_\epsilon\boldsymbol{\epsilon} + II_\epsilon\boldsymbol{I}] \tag{4.3.7}$$

comparison of coefficients

$$\begin{aligned} f_1 &= \quad \mathrm{D}_{I_\epsilon}W + I_\epsilon\,\mathrm{D}_{II_\epsilon}W + II_\epsilon\,\mathrm{D}_{III_\epsilon}W \\ f_2 &= -\ \mathrm{D}_{II_\epsilon}W - I_\epsilon\,\mathrm{D}_{III_\epsilon}W \\ f_3 &= \quad \mathrm{D}_{III_\epsilon}W \end{aligned} \tag{4.3.8}$$

assumption of linearity (quadratic term vanishes), two Lamé constants λ and μ

$$f_1 = I_\epsilon\lambda = [\boldsymbol{\epsilon} : \boldsymbol{I}]\lambda \qquad f_2 = 2\,\mu \qquad f_3 = 0 \tag{4.3.9}$$

specific stored energy (quadratic in strains)

$$W = \frac{1}{2}\,\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon} = \frac{1}{2}\lambda\,[\boldsymbol{\epsilon} : \boldsymbol{I}]^2 + \mu[\boldsymbol{\epsilon}^2 : \boldsymbol{I}] \tag{4.3.10}$$

stress tensor (linear in strains)

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}}W = \mathbb{E} : \boldsymbol{\epsilon} = f_1\boldsymbol{I} + f_2\boldsymbol{\epsilon} = \lambda\,[\,\boldsymbol{\epsilon} : \boldsymbol{I}\,]\,\boldsymbol{I} + 2\,\mu\,\boldsymbol{\epsilon} \tag{4.3.11}$$

matrix representation of coordinates

$$[\,\sigma_{ij}\,] = \begin{bmatrix} \lambda I_\epsilon + 2\,\mu\epsilon_{11} & 2\,\mu\epsilon_{12} & 2\,\mu\epsilon_{13} \\ 2\,\mu\epsilon_{21} & \lambda I_\epsilon + 2\,\mu\epsilon_{22} & 2\,\mu\epsilon_{23} \\ 2\,\mu\epsilon_{31} & 2\,\mu\epsilon_{32} & \lambda I_\epsilon + 2\,\mu\epsilon_{33} \end{bmatrix} \tag{4.3.12}$$

linear elastic continuum tangent stiffness (constant in strains)

$$\mathbb{E}^{\text{tan}} = \lambda \boldsymbol{I} \otimes \boldsymbol{I} + 2\,\mu \mathbb{I}^{\text{sym}} \qquad \mathrm{D}_t\boldsymbol{\sigma} = \mathbb{E}^{\text{tan}} : \mathrm{D}_t\boldsymbol{\epsilon} \tag{4.3.13}$$

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linear elastic continuum secant stiffness

$$\mathbb{E} = \lambda \boldsymbol{I} \otimes \boldsymbol{I} + 2\,\mu \mathbb{I}^{\text{sym}} \qquad \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.3.14}$$

Voigt representation of stiffness tensor

$$\mathbb{E} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{4.3.15}$$