ME338A
CONTINUUM MECHANICS

lecture notes 12

thursday, february 11th, 2010

4 Constitutive equations

motivation:
unknowns

• density ρ	1
• displacement $\boldsymbol{u}$	3
• temperature θ	1
• mass flux $\boldsymbol{r}$	3
• stress $\boldsymbol{\sigma}$	9
• heat flux $\boldsymbol{q}$	3
in total	20

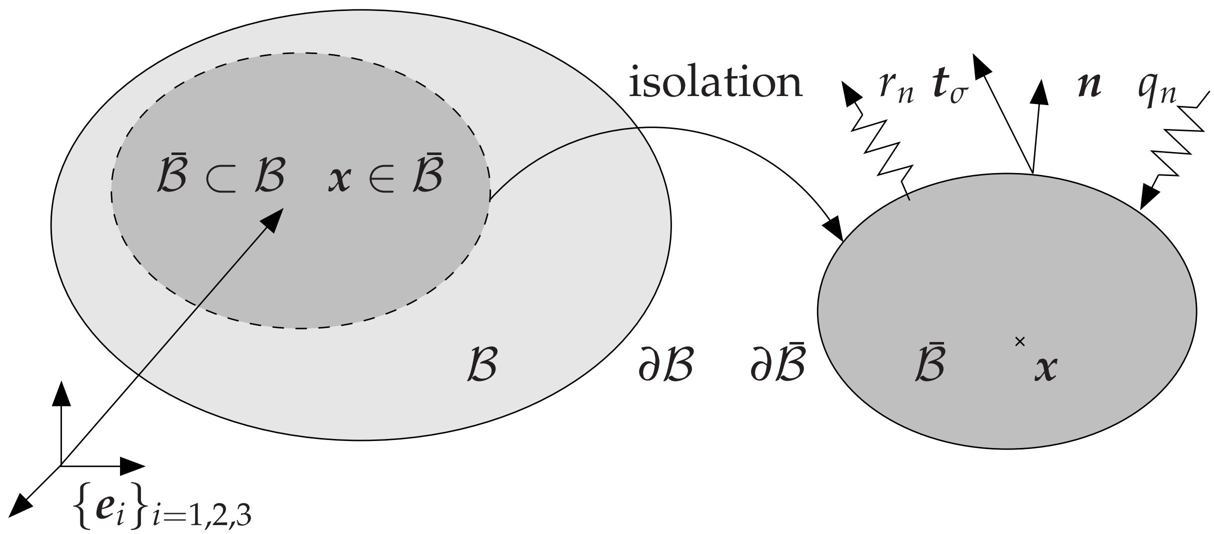
equations

• balance of mass	1
• balance of momentum	3
• balance of angular momentum	3
• balance of energy	1
in total	8

balance of unknowns vs. number of equations: 20-8=12 equations are missing, introduction of 12 material specific constitutive equations, i.e. equations for the mass flux $\boldsymbol{r}$ (3 eqns.), the symmetric stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$ (6 eqns.) and the heat flux $\boldsymbol{q}$ (3 eqns.)

4.1 Linear constitutive equations

- for the mass flux $\boldsymbol{r}$ with $r_n = \boldsymbol{r} \cdot \boldsymbol{n}$
- for the momentum flux / stress $\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{\sigma}$ with $\boldsymbol{t}_n = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}$
- for the heat flux $\boldsymbol{q}$ with $q_n = \boldsymbol{q} \cdot \boldsymbol{n}$



in the simplest case, we could introduce ad hoc definitions of the mass flux, the momentum flux and the heat flux in terms of the spatial gradients of the density, the deformation and the temperature

4.1.1 Mass flux – Fick's law

linear relation between mass flux $\boldsymbol{r}$ (vector) and density gradient $\nabla\rho$ (vector) in terms of mass conduction coefficient $\boldsymbol{R}$ (second order tensor)

$$\boldsymbol{r} = \boldsymbol{R} \cdot \nabla\rho \tag{4.1.1}$$

index representation

$$r_i\,\boldsymbol{e}_i = [\,R_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j\,] \cdot [\,\rho_{,k}\boldsymbol{e}_k\,] = R_{ij}\boldsymbol{e}_i\delta_{jk}\rho_{,k} = R_{ij}\rho_{,j}\boldsymbol{e}_i \tag{4.1.2}$$

matrix representation of coordinates

$$[r_i] = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} R_{11}\,\rho_{,1} & + & R_{12}\,\rho_{,2} & + & R_{13}\,\rho_{,3} \\ R_{21}\,\rho_{,1} & + & R_{22}\,\rho_{,2} & + & R_{23}\,\rho_{,3} \\ R_{31}\,\rho_{,1} & + & R_{32}\,\rho_{,2} & + & R_{33}\,\rho_{,3} \end{bmatrix} \tag{4.1.3}$$

special case of isotropie

$$\boldsymbol{R} = R\,\boldsymbol{I} \qquad \boldsymbol{r} = R\nabla\rho \qquad [r_i] = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = R \begin{bmatrix} \rho_{,1} \\ \rho_{,2} \\ \rho_{,3} \end{bmatrix} \tag{4.1.4}$$

a linear relation between the flux of matter $\boldsymbol{r}$ and the gradient of concentrations $\nabla\rho$ is referred to as Fick's law

4.1.2 Momentum flux – Hook's law

linear relation between momentum flux $\boldsymbol{\sigma}$ (second order tensor) and displacement gradient $\nabla \boldsymbol{u}$ or rather $\boldsymbol{\epsilon} = \nabla^{\text{sym}} \boldsymbol{u}$ (second order tensor) in terms of elasticity tensor $\mathbb{E}$ (fourth order tensor)

$$\boldsymbol{\sigma} = \mathbb{E} : \nabla^{\text{sym}} \boldsymbol{u} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.1.5}$$

index representation

$$\begin{aligned} \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j &= \left[\, E_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \,\right] \cdot \left[\, \epsilon_{mn} \boldsymbol{e}_m \otimes \boldsymbol{e}_n \,\right] \\ &= E_{ijkl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \delta_{km} \delta_{ln} \epsilon_{mn} = E_{ijkl} \epsilon_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{aligned} \tag{4.1.6}$$

special case of isotropie
i.e. identical Eigenbasis of stress & strain

$$\boldsymbol{\sigma} = \sum_{i_1}^{3} \lambda_{\sigma i} \boldsymbol{n}_{\sigma i} \otimes \boldsymbol{n}_{\sigma i} \qquad \boldsymbol{\epsilon} = \sum_{i_1}^{3} \lambda_{\epsilon i} \boldsymbol{n}_{\epsilon i} \otimes \boldsymbol{n}_{\epsilon i} \tag{4.1.7}$$

representation theorem for isotropic tensor–valued tensor-functions

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon}) = f_1 \boldsymbol{I} + f_2 \boldsymbol{\epsilon} + f_3 \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} \tag{4.1.8}$$

with $f_i = f_i(I_\epsilon, II_\epsilon, III_\epsilon)$ function of strain invariants

$$\begin{aligned} I_\epsilon &= \operatorname{tr}(\boldsymbol{\epsilon}) &&= \lambda_{\epsilon 1} + \lambda_{\epsilon 2} + \lambda_{\epsilon 3} \\ II_\epsilon &= \tfrac{1}{2}[\operatorname{tr}^2(\boldsymbol{\epsilon}) - \operatorname{tr}(\boldsymbol{\epsilon}^2)] &&= \lambda_{\epsilon 2}\lambda_{\epsilon 3} + \lambda_{\epsilon 3}\lambda_{\epsilon 1} + \lambda_{\epsilon 1}\lambda_{\epsilon 2} \\ III_\epsilon &= \det(\boldsymbol{\epsilon}) &&= \lambda_{\epsilon 1}\, \lambda_{\epsilon 2}\, \lambda_{\epsilon 3} \end{aligned} \tag{4.1.9}$$

a linear relation between the momentum flux $\boldsymbol{\sigma}$ and the strains $\boldsymbol{\epsilon}$ represents the generalized form of Hook's law

Hypoelasticity / Cauchy Elasticity

a hypoelastic / Cauchy elastic constitutive law can be represented in the following form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\epsilon}) \tag{4.1.10}$$

- invertible relation between stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$ (rates)
- possible dissipation of energy in closed strain circles

$$\oint \mathcal{D}^{\text{loc}}\,\text{d}t = \oint \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}\,\text{d}t - \oint \text{D}_t\psi\,\text{d}t = \oint \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}\,\text{d}t \tag{4.1.11}$$

homogeneous strain path from $\boldsymbol{\epsilon}_{t_1}$ to $\boldsymbol{\epsilon}_{t_2}$

$$\boldsymbol{\epsilon}(\alpha) = [\,1-\alpha\,]\,\boldsymbol{\epsilon}_{t_1} + \alpha\,\boldsymbol{\epsilon}_{t_2} \qquad \text{d}\boldsymbol{\epsilon} = [\,\boldsymbol{\epsilon}_{t_2} - \boldsymbol{\epsilon}_{t_1}\,]\text{d}\alpha \tag{4.1.12}$$

stress work for linear elastic material

$$\begin{aligned} \int_{t_1}^{t_2} \boldsymbol{\sigma} : \text{d}\boldsymbol{\epsilon} &= \int_0^1 \boldsymbol{\epsilon}(\alpha) : \mathbb{E} : [\boldsymbol{\epsilon}_{t_2} - \boldsymbol{\epsilon}_{t_1}]\text{d}\alpha \\ &= \frac{1}{2}[\boldsymbol{\epsilon}_{t_2} + \boldsymbol{\epsilon}_{t_1}] : \mathbb{E} : [\boldsymbol{\epsilon}_{t_2} - \boldsymbol{\epsilon}_{t_1}] \end{aligned} \tag{4.1.13}$$

dissipation in isothermal closed strain cycle $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_1$

$$\oint \mathcal{D}^{\text{loc}}\,\text{d}t = \boldsymbol{\epsilon}_{t_1} : \mathbb{E}^{\text{skw}} : \boldsymbol{\epsilon}_{t_2} + \boldsymbol{\epsilon}_{t_2} : \mathbb{E}^{\text{skw}} : \boldsymbol{\epsilon}_{t_3} + \boldsymbol{\epsilon}_{t_3} : \mathbb{E}^{\text{skw}} : \boldsymbol{\epsilon}_{t_1} \neq 0 \tag{4.1.14}$$

4.1.3 Heat flux – Fourier's law

linear relation between heat flux $\boldsymbol{q}$ (vector) and temperature gradient $\nabla\theta$ (vector) in terms of heat conduction coefficient $\boldsymbol{\kappa}$ (second order tensor)

$$\boldsymbol{q} = \boldsymbol{\kappa} \cdot \nabla\theta \tag{4.1.15}$$

index representation

$$q_i\,\boldsymbol{e}_i = [\,\kappa_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j\,] \cdot [\,\theta_{,k}\boldsymbol{e}_k\,] = \kappa_{ij}\boldsymbol{e}_i\delta_{jk}\theta_{,k} = \kappa_{ij}\theta_{,j}\boldsymbol{e}_i \tag{4.1.16}$$

matrix representation of coordinates

$$[q_i] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \kappa_{11}\,\theta_{,1} + \kappa_{12}\,\theta_{,2} + \kappa_{13}\,\theta_{,3} \\ \kappa_{21}\,\theta_{,1} + \kappa_{22}\,\theta_{,2} + \kappa_{23}\,\theta_{,3} \\ \kappa_{31}\,\theta_{,1} + \kappa_{32}\,\theta_{,2} + \kappa_{33}\,\theta_{,3} \end{bmatrix} \tag{4.1.17}$$

special case of isotropie

$$\boldsymbol{\kappa} = \kappa\,\boldsymbol{I} \qquad \boldsymbol{q} = \kappa\nabla\theta \qquad [q_i] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \kappa \begin{bmatrix} \theta_{,1} \\ \theta_{,2} \\ \theta_{,3} \end{bmatrix} \tag{4.1.18}$$

a linear relation between the heat flux vector $\boldsymbol{q}$ and the temperature gradient $\nabla\theta$ is referred to as Fourier's law which goes back to Fourier [1822]