ME338A
CONTINUUM MECHANICS

lecture notes 11

tuesday, february 09th, 2010

3.6 Balance of entropy

total entropy H of a body $\bar{\mathcal{B}}$

$$H := \int_{\bar{\mathcal{B}}} h \,\mathrm{d}V \tag{3.6.1}$$

entropy exchange of body $\bar{\mathcal{B}}$ with environment through H^{sur} and H^{vol}

$$H^{\text{sur}} := -\int_{\partial\bar{\mathcal{B}}} h_n \,\mathrm{d}\,A \qquad H^{\text{vol}} := \int_{\bar{\mathcal{B}}} \mathcal{H} \,\mathrm{d}\,V \tag{3.6.2}$$

with contact entropy flux $h_n = \boldsymbol{h} \cdot \boldsymbol{n}$ and entropy source $\mathcal{H}$

internal entropy production of body $\bar{\mathcal{B}}$ as H^{pro}

$$H^{\text{pro}} := \int_{\bar{\mathcal{B}}} \mathsf{h}^{\text{pro}} \,\mathrm{d}\,V \geq 0 \tag{3.6.3}$$

whereby H^{pro} is strictly non-negative

3.6.1 Global form of balance of entropy

"The time rate of change of the total entropy H of a body $\bar{\mathcal{B}}$ is balanced with the energy exchange due to contact energy flux H^{sur}, the at-a-distance entropy exchange H^{vol} and the non–negative entropy production H^{pro} with $H^{\text{pro}} \geq 0$".

$$\mathrm{D}_t H = H^{\text{sur}} + H^{\text{vol}} + H^{\text{pro}} \tag{3.6.4}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} h \mathrm{d}V = -\int_{\partial\bar{\mathcal{B}}} h_n \,\mathrm{d}A + \int_{\bar{\mathcal{B}}} \mathcal{H} \mathrm{d}V + \int_{\bar{\mathcal{B}}} \mathsf{h}^{\text{pro}} \mathrm{d}V \tag{3.6.5}$$

3.6.2 Local form of balance of entropy

modification of rate term $\mathrm{D}_t\, H$

$$\mathrm{D}_t\, H = \mathrm{D}_t \int_{\bar{\mathcal{B}}} h\, \mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t h\, \mathrm{d}V \tag{3.6.6}$$

modification of surface term H^{sur}

$$H^{\text{sur}} = -\int_{\partial\bar{\mathcal{B}}} h_n \mathrm{d}\, A \overset{\text{Cauchy}}{=} -\int_{\partial\bar{\mathcal{B}}} \boldsymbol{h}\cdot\boldsymbol{n} \mathrm{d}\, A \overset{\text{Gauss}}{=} -\int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{h}) \mathrm{d}\, V \tag{3.6.7}$$

and thus

$$\int_{\bar{\mathcal{B}}} \mathrm{D}_t h\, \mathrm{d}V = -\int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{h})\, \mathrm{d}\, V + \int_{\bar{\mathcal{B}}} \mathcal{H}\, \mathrm{d}\, V + \int_{\bar{\mathcal{B}}} \mathsf{h}^{\text{pro}}\, \mathrm{d}\, V \tag{3.6.8}$$

for arbitrary bodies $\bar{\mathcal{B}} \rightarrow$ local form of entropy balance

$$\mathrm{D}_t h = -\mathrm{div}\,(\boldsymbol{h}) + \mathcal{H} + \mathsf{h}^{\text{pro}} \tag{3.6.9}$$

3.6.3 Reduction with lower order balance equations

modification with the help of lower order balance equations
assumption of existience of absolute temperature $theta \geq 0$
which renders the following constitutive assumption

$$\boldsymbol{h} = \tfrac{1}{\theta}\, \boldsymbol{q} \qquad \mathcal{H} = \tfrac{1}{\theta}\, \mathcal{Q} \tag{3.6.10}$$

such that

$$\begin{aligned} \mathrm{D}_t h &= - \quad \mathrm{div}\,(\tfrac{1}{\theta}\, \boldsymbol{q}) \quad + \tfrac{1}{\theta}\mathcal{Q} + \mathsf{h}^{\text{pro}} \\ \theta\, \mathrm{D}_t h &= -\, \theta\, \tfrac{1}{\theta} \mathrm{div}\,(\boldsymbol{q}) - \boldsymbol{q}\cdot\theta\, \nabla(\tfrac{1}{\theta}) + \mathcal{Q} + \theta\, \mathsf{h}^{\text{pro}} \\ \mathrm{D}_t(\theta\, h) &= -\,\theta\, \tfrac{1}{\theta} \mathrm{div}\,(\boldsymbol{q}) - \boldsymbol{q}\cdot\nabla\, \ln(\theta) + \mathcal{Q} + \theta\, \mathsf{h}^{\text{pro}} - h\, \mathrm{D}_t\theta \end{aligned}$$

(3.6.11)

with balance of internal energy

$$\mathrm{D}_t\, i = \mathsf{q}^{\text{ext}} + \mathsf{p}^{\text{int}} = -\operatorname{div}(\boldsymbol{q}) + \mathcal{Q} + \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} \tag{3.6.12}$$

combination of balance of entropy and internal energy

$$\mathrm{D}_t\,(\, i - \theta\, h\,) = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} - h\,\mathrm{D}_t\theta - \boldsymbol{q}\cdot\nabla\ln(\theta) - \theta\,\mathsf{h}^{\text{pro}} \tag{3.6.13}$$

Legendre transform: free Helmholz energy

$$\psi = i - \theta\, h \tag{3.6.14}$$

and thus

$$\mathrm{D}_t\,\psi = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} - h\,\mathrm{D}_t\theta - \boldsymbol{q}\cdot\nabla\ln(\theta) - \theta\,\mathsf{h}^{\text{pro}} \tag{3.6.15}$$

3.6.4 Second law of thermodynamics

”The production of entropy h^{pro} is non-negative.”

$$\mathsf{h}^{\text{pro}} \geq 0 \tag{3.6.16}$$

entropy is a measure of microscopic disorder of a system, positive entropy production gives a preferred direction to thermodynamic processes

Clausius: ’heat never flows from a colder to a warmer system’

introduction of dissipation (-rate) $\mathcal{D}$

$$\mathcal{D} := \theta\,\mathsf{h}^{\text{pro}} \geq 0 \tag{3.6.17}$$

dissipation inequality

$$\mathcal{D} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} - h\,\mathrm{D}_t\theta - \mathrm{D}_t\,\psi - \boldsymbol{q}\cdot\nabla\ln(\theta) \geq 0 \tag{3.6.18}$$

the above equation is referred to as 'Clausius–Duhem inequality', Clausius [1822-1888]

$$\begin{aligned} &\mathcal{D} = 0 \quad \text{... reversible process} \\ &\mathcal{D} > 0 \quad \text{... irreversible process} \end{aligned} \tag{3.6.19}$$

conjugate pairs

$$\begin{aligned} &\text{stress} \quad \boldsymbol{\sigma} \quad \text{vs.} \quad \text{strain} \quad \boldsymbol{\epsilon} \\ &\text{entropy} \quad h \quad \text{vs.} \quad \text{temperature} \quad \theta \end{aligned} \tag{3.6.20}$$

decomposition into local and conductive part

$$\mathcal{D} = \mathcal{D}^{\text{loc}} + \mathcal{D}^{\text{con}} \geq 0 \tag{3.6.21}$$

local part / Clausius–Planck inequality

$$\mathcal{D}^{\text{loc}} = \boldsymbol{\sigma}^{\text{t}} : \text{D}_t\boldsymbol{\epsilon} - h\,\text{D}_t\theta - \text{D}_t\,\psi \geq 0 \tag{3.6.22}$$

conductive part / Fourier inequality

$$\mathcal{D}^{\text{con}} = -\boldsymbol{q} \cdot \nabla \ln(\theta) \geq 0 \tag{3.6.23}$$

example: Fourier law of heat conduction

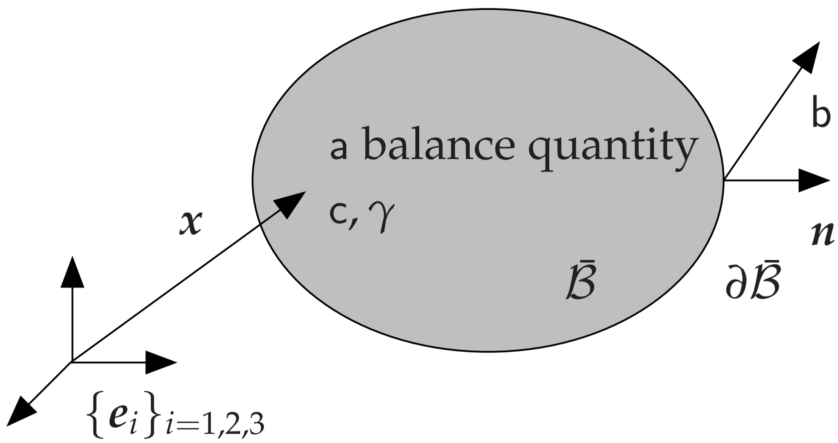
$$\boldsymbol{q} = -\boldsymbol{\kappa} \cdot \nabla\theta \qquad \text{isotropic} \qquad \boldsymbol{\kappa} = \kappa\,\boldsymbol{I} \tag{3.6.24}$$

Fourier inequality a priori statisfied by construction

$$\nabla\theta \cdot \boldsymbol{\kappa} \cdot \nabla\theta \geq 0 \tag{3.6.25}$$

3.7 Generic balance equation

any balance law can be expressed a general format



3.7.1 Global / integral format

$$\mathrm{D}_t\,\mathsf{A} = \mathsf{B} + \mathsf{C} + \Gamma \tag{3.7.1}$$

or alternatively

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} \mathsf{a}\,\mathrm{d}V = \int_{\partial\bar{\mathcal{B}}} \mathsf{b}\cdot\boldsymbol{n}\,\mathrm{d}A + \int_{\bar{\mathcal{B}}} \mathsf{c}\,\mathrm{d}V + \int_{\bar{\mathcal{B}}} \gamma\,\mathrm{d}V \tag{3.7.2}$$

whereby

$$\begin{aligned} &\mathsf{A} \text{ ... balance quantity} \\ &\mathsf{B} \text{ ... surface transport through } \partial\bar{\mathcal{B}} \\ &\mathsf{C} \text{ ... volume source in } \bar{\mathcal{B}} \\ &\Gamma \text{ ... production in } \bar{\mathcal{B}} \end{aligned} \tag{3.7.3}$$

3.7.2 Local / differential format

local format of balance law follows from

- application of Gauss' theorem
- localization theorem, i.e. $\bar{\mathcal{B}}$ arbitrarily small

$$\mathrm{D}_t\,\mathsf{a} = \mathrm{div}\,(\mathsf{b}) + \mathsf{c} + \gamma \tag{3.7.4}$$

	quantity a	flux b	source c	production γ
mass	ρ	$\boldsymbol{r}$	$\mathcal{R}$	–
lin. mom.	$\rho\,\boldsymbol{v}$	$\boldsymbol{\sigma}^{\mathrm{t}}$	$\boldsymbol{b}$	–
ang. mom.	$\boldsymbol{x} \times \rho\,\boldsymbol{v}$	$\boldsymbol{x} \times \boldsymbol{\sigma}^{\mathrm{t}}$	$\boldsymbol{x} \times \boldsymbol{b}$	–
energy	e	$\boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}} - \boldsymbol{q}$	$\boldsymbol{v} \cdot \boldsymbol{b} + \mathcal{Q}$	–
kin. energy	k	$\boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}}$	$\boldsymbol{v} \cdot \boldsymbol{b}$	$-\boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon}$
int. energy	i	$-\boldsymbol{q}$	$\mathcal{Q}$	$\boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon}$
entropy	h	$-\boldsymbol{h}$	$\mathcal{H}$	$\mathsf{h}^{\mathrm{pro}}$

Table 3.1: generic balance law

mass, linear momentum, angular momentum and total energy are conservation properties, while kinetic energy, internal energy and entropy are not, they possess a production term

3.8 Thermodynamic potentials

internal energy

$$i = i\,(\boldsymbol{\epsilon}, h, ...) \quad \rightarrow \quad \boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} i \quad \text{and} \quad \theta = \mathrm{D}_h i \tag{3.8.1}$$

Legendre-Fenchel transform $h \rightarrow \theta$

$$\psi\,(\boldsymbol{\epsilon}, \theta) = \inf_h \left(i\,(\boldsymbol{\epsilon}, h) - \theta\, h \right) = i\,(\boldsymbol{\epsilon}, h(\theta)) - \theta\, h(\theta) \tag{3.8.2}$$

Helmholtz free energy

$$\psi = \psi\,(\boldsymbol{\epsilon}, \theta, ...) \quad \rightarrow \quad \boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} \psi \quad \text{and} \quad h = -\mathrm{D}_\theta \psi \tag{3.8.3}$$

Legendre-Fenchel transform $\boldsymbol{\epsilon} \rightarrow \boldsymbol{\sigma}$

$$g\,(\boldsymbol{\sigma}, \theta) = \sup_{\boldsymbol{\epsilon}} \left(\boldsymbol{\sigma} : \boldsymbol{\epsilon} - \psi\,(\boldsymbol{\epsilon}, \theta) \right) = \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\boldsymbol{\sigma}) - \psi\,(\boldsymbol{\epsilon}(\boldsymbol{\sigma}), \theta) \tag{3.8.4}$$

Gibbs free energy

$$g = g\,(\boldsymbol{\sigma}, \theta, ...) \quad \rightarrow \quad \boldsymbol{\epsilon} = \mathrm{D}_{\boldsymbol{\sigma}} g \quad \text{and} \quad h = \mathrm{D}_\theta g \tag{3.8.5}$$

Legendre-Fenchel transform $\theta \rightarrow h$

$$\eta\,(\boldsymbol{\sigma}, h) = \inf_{\boldsymbol{\theta}} \left(g\,(\boldsymbol{\sigma}, \theta) - h\,\theta \right) = g\,(\boldsymbol{\sigma}, \theta(h)) - h\,\theta(h) \tag{3.8.6}$$

enthalpy

$$\eta = \eta\,(\boldsymbol{\epsilon}, h, ...) \quad \rightarrow \quad \boldsymbol{\epsilon} = \mathrm{D}_{\sigma} \eta \quad \text{and} \quad \theta = -\mathrm{D}_h \eta \tag{3.8.7}$$

Legendre-Fenchel transform $\boldsymbol{\sigma} \rightarrow \boldsymbol{\epsilon}$

$$i\,(\boldsymbol{\epsilon}, h) = \sup_{\sigma} \left(\boldsymbol{\epsilon} : \boldsymbol{\sigma} - \eta\,(\boldsymbol{\sigma}, h) \right) = \boldsymbol{\epsilon} : \boldsymbol{\sigma}(\boldsymbol{\epsilon}) - \eta\,(\boldsymbol{\sigma}(\boldsymbol{\epsilon}), h) \tag{3.8.8}$$