ME338A
CONTINUUM MECHANICS

lecture notes 10

thursday, february 4th, 2010

Classical continuum mechanics of closed systems

in classical closed system continuum mechanics (here), $\boldsymbol{r} = \boldsymbol{0}$ and $\mathcal{R} = 0$, such that the mass density ρ is constant in time, i.e. $\mathrm{D}_t\,(\rho\,\boldsymbol{v}) = \rho\,\mathrm{D}_t\boldsymbol{v}$ and thus

$$\rho\,\mathrm{D}_t\boldsymbol{v} = \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{b} \tag{3.3.12}$$

the above equation is referred to as 'Cauchy's first equation of motion', Cauchy [1827]

3.4 Balance of angular momentum

total angular momentum $\boldsymbol{l}$ of a body $\bar{\mathcal{B}}$

$$\boldsymbol{l} := \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \mathrm{D}_t\boldsymbol{u}\,\mathrm{d}\,m = \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{v}\,\mathrm{d}\,m = \int_{\bar{\mathcal{B}}} \rho\,\boldsymbol{x} \times \boldsymbol{v}\,\mathrm{d}V \tag{3.4.1}$$

angular momentum exchange of body $\bar{\mathcal{B}}$ with environment through momentum from contact forces $\boldsymbol{l}^{\mathrm{sur}}$ and momentum from at-a-distance forces $\boldsymbol{l}^{\mathrm{vol}}$

$$\boldsymbol{l}^{\mathrm{sur}} := \int_{\partial\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{t}_\sigma\,\mathrm{d}\,A \qquad \boldsymbol{l}^{\mathrm{vol}} := \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{b}\,\mathrm{d}\,V \tag{3.4.2}$$

with momentum due to contact/surface forces $\boldsymbol{x} \times \boldsymbol{t}_\sigma = \boldsymbol{x} \times \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}$ and volume forces $\boldsymbol{x} \times \boldsymbol{b}$, assumption: no additional external torques

3.4.1 Global form of balance of angular momentum

"The time rate of change of the total angular momentum $\boldsymbol{l}$ of a body $\bar{\mathcal{B}}$ is balanced with the angular momentum exchange due to contact momentum flux / surface force $\boldsymbol{l}^{\mathrm{sur}}$ and due to the at-a-distance momentum exchange / volume force $\boldsymbol{l}^{\mathrm{vol}}$."

$$\mathrm{D}_t\boldsymbol{l} = \boldsymbol{l}^{\mathrm{sur}} + \boldsymbol{l}^{\mathrm{vol}} \tag{3.4.3}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho\, \boldsymbol{x} \times \boldsymbol{v}\, \mathrm{d}V = \int_{\partial\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{t}_\sigma\, \mathrm{d}\,A + \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{b}\, \mathrm{d}\,V \tag{3.4.4}$$

3.4.2 Local form of balance of angular momentum

modification of rate term $\mathrm{D}_t\, \boldsymbol{p}$

$$\mathrm{D}_t\, \boldsymbol{l} = \mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho\, \boldsymbol{x} \times \boldsymbol{v} \mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t(\rho\, \boldsymbol{x} \times \boldsymbol{v}) \mathrm{d}V \tag{3.4.5}$$

modification of surface term $\boldsymbol{l}^{\mathrm{sur}}$

$$\begin{aligned} \boldsymbol{l}^{\mathrm{sur}} &= \int_{\partial\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{t}_\sigma\, \mathrm{d}\,A \overset{\text{Cauchy}}{=} \int_{\partial\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}\, \mathrm{d}\,A \\ &\overset{\text{Gauss}}{=} \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{x} \times \boldsymbol{\sigma}^{\mathrm{t}})\, \mathrm{d}\,V \\ &= \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}})\, \mathrm{d}\,V + \int_{\bar{\mathcal{B}}} \nabla\boldsymbol{x} \times \boldsymbol{\sigma}^{\mathrm{t}}\, \mathrm{d}\,V \end{aligned} \tag{3.4.6}$$

and thus

$$\begin{aligned} \int_{\bar{\mathcal{B}}} \mathrm{D}_t(\rho\, \boldsymbol{x} \times \boldsymbol{v})\, \mathrm{d}V &= \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}})\, \mathrm{d}\,V \\ &+ \int_{\bar{\mathcal{B}}} \boldsymbol{I} \times \boldsymbol{\sigma}^{\mathrm{t}}\, \mathrm{d}\,V + \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{b}\, \mathrm{d}\,V \end{aligned} \tag{3.4.7}$$

for arbitrary bodies $\bar{\mathcal{B}} \rightarrow$ local form of ang. mom. balance

$$\mathrm{D}_t(\rho\, \boldsymbol{x} \times \boldsymbol{v}) = \boldsymbol{x} \times \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{I} \times \boldsymbol{\sigma}^{\mathrm{t}} + \boldsymbol{x} \times \boldsymbol{b} \tag{3.4.8}$$

3.4.3 Reduction with lower order balance equations

modification with the help of lower order balance equations

$$\begin{aligned} \mathrm{D}_t(\rho\, \boldsymbol{x} \times \boldsymbol{v}) &= \mathrm{D}_t\, \boldsymbol{x} \times (\rho\, \boldsymbol{v}) + \boldsymbol{x} \times \mathrm{D}_t\, (\rho\, \boldsymbol{v}) \\ &= \boldsymbol{x} \times \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{I} \times \boldsymbol{\sigma}^{\mathrm{t}} + \boldsymbol{x} \times \boldsymbol{b} \end{aligned} \qquad (3.4.9)$$

position vector $\boldsymbol{x}$ fixed, thus $\mathrm{D}_t\, \boldsymbol{x} = \boldsymbol{0}$, with balance of linear momentum multiplied by position $\boldsymbol{x}$

$$\boldsymbol{x} \times \mathrm{D}_t(\rho\, \boldsymbol{v}) = \boldsymbol{x} \times \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{x} \times \boldsymbol{b} \qquad (3.4.10)$$

local angular momentum balance in reduced format

$$\boldsymbol{I} \times \boldsymbol{\sigma}^{\mathrm{t}} = \overset{3}{\boldsymbol{e}} : \boldsymbol{\sigma} = -2\,\mathrm{axl}\,(\boldsymbol{\sigma}) = \boldsymbol{0} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}} \qquad (3.4.11)$$

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the above equation is referred to as 'Cauchy's second equation of motion', Cauchy [1827]

in the absense of couple stresses, surface and body couples, the stress tensor is symmetric, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$, else: micropolar / Cosserat continua

vector product of second order tensors $\boldsymbol{A} \times \boldsymbol{B} = \overset{3}{\boldsymbol{e}} : [\boldsymbol{A} \cdot \boldsymbol{B}^{\mathrm{t}}]$

3.5 Balance of energy

total energy E of a body $\bar{\mathcal{B}}$ as the sum of the kinetic energy K and the internal energy I

$$
\begin{aligned}
E &:= \int_{\bar{\mathcal{B}}} e \;\mathrm{d}V = \int_{\bar{\mathcal{B}}} k + i \,\mathrm{d}V = K + I \\
K &:= \int_{\bar{\mathcal{B}}} k \;\mathrm{d}V = \int_{\bar{\mathcal{B}}} \frac{1}{2} \rho \, \boldsymbol{v} \cdot \boldsymbol{v} \,\mathrm{d}V \\
I &:= \int_{\bar{\mathcal{B}}} i \;\mathrm{d}V
\end{aligned}
\tag{3.5.1}
$$

energy exchange of body $\bar{\mathcal{B}}$ with environment e^{sur} and e^{vol}

$$
\begin{aligned}
E^{\text{sur}} &:= \int_{\partial\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{t}_\sigma \,\mathrm{d}\,A - \int_{\partial\bar{\mathcal{B}}} q_n \,\mathrm{d}\,A \\
E^{\text{vol}} &:= \int_{\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{b} \,\mathrm{d}\,V + \int_{\partial\bar{\mathcal{B}}} \mathcal{Q} \,\mathrm{d}\,V
\end{aligned}
\tag{3.5.2}
$$

with contact heat flux $q_n = \boldsymbol{r} \cdot \boldsymbol{n}$ and heat source $\mathcal{Q}$

3.5.1 Global form of balance of energy

"The time rate of change of the total energy E of a body $\bar{\mathcal{B}}$ is balanced with the energy exchange due to contact energy flux E^{sur} and the at-a-distance energy exchange E^{vol}."

$$
\mathrm{D}_t E = E^{\text{sur}} + E^{\text{vol}} \tag{3.5.3}
$$

and thus

$$
\mathrm{D}_t \int_{\bar{\mathcal{B}}} e \,\mathrm{d}V = \int_{\partial\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{t}_\sigma - q_n \,\mathrm{d}A + \int_{\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{b} + \mathcal{Q} \mathrm{d}V \tag{3.5.4}
$$

3.5.2 Local form of balance of energy

modification of rate term $\mathrm{D}_t\, E$

$$\mathrm{D}_t\, E = \mathrm{D}_t \int_{\bar{\mathcal{B}}} e \,\mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t e \,\mathrm{d}V \tag{3.5.5}$$

modification of surface term E^{sur}

$$\begin{aligned} E^{\text{sur}} &= \int_{\partial\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{t}_\sigma - q_n \,\mathrm{d}\, A \\ &\overset{\text{Cauchy}}{=} \int_{\partial\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} - \boldsymbol{q} \cdot \boldsymbol{n} \,\mathrm{d}\, A \\ &\overset{\text{Gauss}}{=} \int_{\bar{\mathcal{B}}} \operatorname{div}(\boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}}) + \operatorname{div}(-\boldsymbol{q}) \,\mathrm{d}\, V \end{aligned} \tag{3.5.6}$$

and thus

$$\int_{\bar{\mathcal{B}}} \mathrm{D}_t e \,\mathrm{d}V = \int_{\bar{\mathcal{B}}} \operatorname{div}(\boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}} - \boldsymbol{q}) \,\mathrm{d}\, V + \int_{\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{b} + \mathcal{Q} \,\mathrm{d}\, V \tag{3.5.7}$$

for arbitrary bodies $\bar{\mathcal{B}} \rightarrow$ local form of energy balance

$$\mathrm{D}_t e = \operatorname{div}(\boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}} - \boldsymbol{q}) + \boldsymbol{v} \cdot \boldsymbol{b} + \mathcal{Q} \tag{3.5.8}$$

3.5.3 Reduction with lower order balance equations

modification with the help of lower order balance equations with

$$\begin{aligned} \mathrm{D}_t\, e &= \mathrm{D}_t\,(\,k + i\,) = v\,\mathrm{D}_t\,(\rho\,\boldsymbol{v}) + \mathrm{D}_t\, i \\ &= \operatorname{div}(\boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}} - \boldsymbol{q}) + \boldsymbol{v} \cdot \boldsymbol{b} + \mathcal{Q} \end{aligned} \tag{3.5.9}$$

with

$$\mathrm{D}_t\, k = \mathrm{D}_t(\tfrac{1}{2}\rho\boldsymbol{v} \cdot \boldsymbol{v}) = \tfrac{1}{2}\mathrm{D}_t(\rho\boldsymbol{v}) \cdot \boldsymbol{v} + \tfrac{1}{2}\boldsymbol{v} \cdot \mathrm{D}_t(\rho\boldsymbol{v}) = \boldsymbol{v} \cdot \mathrm{D}_t(\rho\boldsymbol{v})$$

(3.5.10)

with balance of momentum multiplied by velocity $\boldsymbol{v}$

$$\boldsymbol{v}\,\mathrm{D}_t\,(\rho\,\boldsymbol{v}) = \boldsymbol{v}\,\mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{v}\,\boldsymbol{b} = \mathrm{div}\,(\boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}}) - \nabla\boldsymbol{v} : \boldsymbol{\sigma}^{\mathrm{t}} + \boldsymbol{v}\cdot\boldsymbol{b} \tag{3.5.11}$$

with stress power

$$\begin{aligned} \nabla\boldsymbol{v} : \boldsymbol{\sigma}^{\mathrm{t}} &= \boldsymbol{\sigma}^{\mathrm{t}} : \nabla\boldsymbol{v} = \boldsymbol{\sigma}^{\mathrm{t}} : \nabla\mathrm{D}_t\boldsymbol{\varphi} \\ &= \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\nabla\boldsymbol{\varphi} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\nabla^{\mathrm{sym}}\boldsymbol{\varphi} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} \end{aligned} \tag{3.5.12}$$

local energy balance in reduced format/balance of int. energy

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$$\mathrm{D}_t\, i = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} - \mathrm{div}\,(\boldsymbol{q}) + \mathcal{Q} \tag{3.5.13}$$

3.5.4 First law of thermodynamics

alternative definitions

$$\begin{aligned} \mathsf{p}^{\mathrm{ext}} &:= \mathrm{div}\,(\boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{v}\cdot\boldsymbol{b} && \text{... external mechanical power} \\ \mathsf{p}^{\mathrm{int}} &:= \boldsymbol{\sigma}^{\mathrm{t}} : \nabla\boldsymbol{v} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} && \text{... internal mechanical power} \\ \mathsf{q}^{\mathrm{ext}} &:= \mathrm{div}\,(-\boldsymbol{q}) + \mathcal{Q} && \text{... external thermal power} \end{aligned} \tag{3.5.14}$$

Balance of total energy / first law of thermodynamics

the rate of change of the total energy e, i.e. the sum of the kinetic energy k and the potential energy i is in equilibrium with the external mechanical power $\mathsf{p}^{\mathrm{ext}}$ and the external thermal power $\mathsf{q}^{\mathrm{ext}}$

$$D_t\, e = D_t\, k + D_t\, i = \mathsf{p}^{\text{ext}} + \mathsf{q}^{\text{ext}} \tag{3.5.15}$$

the above equation is typically referred to as ”principle of interconvertibility of heat and mechanical work” , Carnot [1832], Joule [1843], Duhem [1892]

first law of thermodynamics does not provide any information about the direction of a thermodynamic process

balance of kinetic energy

$$D_t\, k = \mathsf{p}^{\text{ext}} - \mathsf{p}^{\text{int}} \tag{3.5.16}$$

the balance of kinetic energy is an alternative statement of the balance of linear momentum

balance of internal energy

$$D_t\, i = \mathsf{q}^{\text{ext}} + \mathsf{p}^{\text{int}} \tag{3.5.17}$$

kinetic energy k and internal energy i are no conservation properties, they exchange the internal mechanical energy p^{int}