3.2 Balance of mass

total mass $m$ of a body $\bar{B}$

$$m := \int_{\bar{B}} d\,m$$  \hspace{1cm} (3.2.1)

mass density

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{d\,m}{d\,V}$$  \hspace{1cm} (3.2.2)

$$m = \int_{\bar{B}} \rho \,d\,V$$  \hspace{1cm} (3.2.3)

mass exchange of body $\bar{B}$ with environment $m^{\text{sur}}$ and $m^{\text{vol}}$

$$m^{\text{sur}} := \int_{\partial \bar{B}} r_n \,d\,A \quad m^{\text{vol}} := \int_{\bar{B}} R \,d\,V$$  \hspace{1cm} (3.2.4)

with contact mass flux $r_n = r \cdot n$ and mass source $R$

3.2.1 Global form of balance of mass

"The time rate of change of the total mass $m$ of a body $\bar{B}$ is balanced with the mass exchange due to contact mass flux $m^{\text{sur}}$ and the at-a-distance mass exchange $m^{\text{vol}}$."

$$D_t m = m^{\text{sur}} + m^{\text{vol}}$$  \hspace{1cm} (3.2.5)

and thus

$$D_t \int_{\bar{B}} \rho \,d\,V = \int_{\partial \bar{B}} r_n \,d\,A + \int_{\bar{B}} R \,d\,V$$  \hspace{1cm} (3.2.6)
3.2.2 Local form of balance of mass

modification of rate term $D_t m$

$$D_t m = D_t \int_{\bar{B}} \rho \, dV \quad \equiv \quad \int_{\bar{B}} D_t \rho \, dV$$  \hfill (3.2.7)

modification of surface term $m_{\text{sur}}$

$$m_{\text{sur}} = \int_{\partial \bar{B}} r \cdot n \, dA \quad \equiv \quad \int_{\partial \bar{B}} \mathbf{r} \cdot \mathbf{n} \, dA \quad \equiv \quad \int_{\bar{B}} \text{div} (\mathbf{r}) \, dV$$  \hfill (3.2.8)

and thus

$$\int_{\bar{B}} D_t \rho \, dV = \int_{\bar{B}} \text{div} (\mathbf{r}) \, dV + \int_{\bar{B}} \mathcal{R} \, dV$$  \hfill (3.2.9)

for arbitrary bodies $\bar{B} \rightarrow$ local form of mass balance

$$D_t \rho = \text{div} (\mathbf{r}) + \mathcal{R}$$  \hfill (3.2.10)

Classical continuum mechanics of closed systems

in classical closed system continuum mechanics (here), $\mathbf{r} = 0$ and $\mathcal{R} = 0$, such that the mass density $\rho$ is constant in time

$$D_t \rho (x,t) = 0 \quad \rho = \rho (x) = \text{const}$$  \hfill (3.2.11)

typically $\mathbf{r} \neq 0$ and $\mathcal{R} \neq 0$ only in bio– or chemomechanics
3.3 Balance of linear momentum

Total momentum $p$ of a body $\bar{B}$

$$p := \int_{\bar{B}} D_t u \, dm = \int_{\bar{B}} v \, dm = \int_{\bar{B}} \rho \, v \, dV$$  \hspace{1cm} (3.3.1)

Momentum exchange of body $\bar{B}$ with environment through contact forces $f^{\text{sur}}$ and at-a-distance forces $f^{\text{vol}}$

$$f^{\text{sur}} := \int_{\partial \bar{B}} t_\sigma \, dA \quad f^{\text{vol}} := \int_{\bar{B}} b \, dV$$ \hspace{1cm} (3.3.2)

with contact/surface force $t_\sigma = \sigma^t \cdot n$ and volume force $b$

3.3.1 Global form of balance of momentum

"The time rate of change of the total momentum $p$ of a body $\bar{B}$ is balanced with the momentum exchange due to contact momentum flux / surface force $f^{\text{sur}}$ and the at-a-distance momentum exchange / volume force $f^{\text{vol}}."$

$$D_t p = f^{\text{sur}} + f^{\text{vol}}$$ \hspace{1cm} (3.3.3)

and thus

$$D_t \int_{\bar{B}} \rho \, v \, dV = \int_{\partial \bar{B}} t_\sigma \, dA + \int_{\bar{B}} b \, dV$$ \hspace{1cm} (3.3.4)

3.3.2 Local form of balance of momentum

Modification of rate term $D_t p$

$$D_t p = D_t \int_{\bar{B}} \rho \, v \, dV_{\text{fixed}} \equiv \int_{\bar{B}} D_t (\rho \, v) \, dV$$ \hspace{1cm} (3.3.5)
3 Balance equations

modification of surface term $f^{\text{sur}}$

$$f^{\text{sur}} = \int_{\partial \overline{B}} t^\sigma \, dA \overset{\text{Cauchy}}{=} \int_{\partial \overline{B}} \sigma^t \cdot n \, dA \overset{\text{Gauss}}{=} \int_{\overline{B}} \text{div} \left( \sigma^t \right) \, dV$$

(3.3.6)

and thus

$$\int_{\overline{B}} D_t(\rho \, v) \, dV = \int_{\overline{B}} \text{div} \left( \sigma^t \right) \, dV + \int_{\overline{B}} b \, dV$$

(3.3.7)

for arbitrary bodies $\overline{B} \rightarrow$ local form of momentum balance

$$D_t(\rho \, v) = \text{div} \left( \sigma^t \right) + b$$

(3.3.8)

3.3.3 Reduction with lower order balance equations

balance equations are typically modified with the help of lower order balance equations

$$D_t(\rho \, v) = v \, D_t \rho + \rho \, D_t v = \text{div} \left( \sigma^t \right) + b$$

(3.3.9)

with balance of mass multiplied by velocity $v$

$$v \, D_t \rho = v \, \text{div} \, (r) + v \, R_0 = \text{div} \left( v \otimes r \right) - \nabla v \cdot r + v \, R_0$$

(3.3.10)

local momentum balance in reduced format

$$\rho \, D_t v = \text{div} \left( \sigma^t - \nabla v \otimes r \right) + b + \nabla v \cdot r - v \, R_0$$

(3.3.11)