ME338A
CONTINUUM MECHANICS

lecture notes 09

tuesday, february 02nd, 2010

3.2 Balance of mass

total mass m of a body $\bar{\mathcal{B}}$

$$m := \int_{\bar{\mathcal{B}}} \mathrm{d}\, m \tag{3.2.1}$$

mass density

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{\mathrm{d} m}{\mathrm{d} v} \qquad \mathrm{d}\, m = \rho \, \mathrm{d} V \tag{3.2.2}$$

$$m = \int_{\bar{\mathcal{B}}} \rho \, \mathrm{d} V \tag{3.2.3}$$

mass exchange of body $\bar{\mathcal{B}}$ with environment m^{sur} and m^{vol}

$$m^{\mathrm{sur}} := \int_{\partial \bar{\mathcal{B}}} r_n \, \mathrm{d}\, A \qquad m^{\mathrm{vol}} := \int_{\bar{\mathcal{B}}} \mathcal{R} \, \mathrm{d}\, V \tag{3.2.4}$$

with contact mass flux $r_n = \boldsymbol{r} \cdot \boldsymbol{n}$ and mass source $\mathcal{R}$

3.2.1 Global form of balance of mass

″The time rate of change of the total mass m of a body $\bar{\mathcal{B}}$ is balanced with the mass exchange due to contact mass flux m^{sur} and the at-a-distance mass exchange m^{vol}.″

$$\mathrm{D}_t m = m^{\mathrm{sur}} + m^{\mathrm{vol}} \tag{3.2.5}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho \, \mathrm{d} V = \int_{\partial \bar{\mathcal{B}}} r_n \, \mathrm{d}\, A + \int_{\bar{\mathcal{B}}} \mathcal{R} \, \mathrm{d}\, V \tag{3.2.6}$$

3.2.2 Local form of balance of mass

modification of rate term $\mathrm{D}_t\, m$

$$\mathrm{D}_t\, m = \mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho \, \mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t \rho \, \mathrm{d}V \tag{3.2.7}$$

modification of surface term m^{sur}

$$m^{\text{sur}} = \int_{\partial\bar{\mathcal{B}}} r_n \, \mathrm{d}\, A \overset{\text{Cauchy}}{=} \int_{\partial\bar{\mathcal{B}}} \boldsymbol{r} \cdot \boldsymbol{n} \, \mathrm{d}\, A \overset{\text{Gauss}}{=} \int_{\bar{\mathcal{B}}} \operatorname{div}(\boldsymbol{r}) \, \mathrm{d}\, V \tag{3.2.8}$$

and thus

$$\int_{\bar{\mathcal{B}}} \mathrm{D}_t \rho \, \mathrm{d}V = \int_{\bar{\mathcal{B}}} \operatorname{div}(\boldsymbol{r}) \, \mathrm{d}\, V + \int_{\bar{\mathcal{B}}} \mathcal{R} \, \mathrm{d}\, V \tag{3.2.9}$$

for arbitrary bodies $\bar{\mathcal{B}} \rightarrow$ local form of mass balance

$$\mathrm{D}_t \rho = \operatorname{div}(\boldsymbol{r}) + \mathcal{R} \tag{3.2.10}$$

Classical continuum mechanics of closed systems

in classical closed system continuum mechanics (here), $\boldsymbol{r} = \mathbf{0}$ and $\mathcal{R} = 0$, such that the mass density ρ is constant in time

$$\mathrm{D}_t\, \rho(\boldsymbol{x}, t) = 0 \qquad \rho = \rho(\boldsymbol{x}) = \text{const} \tag{3.2.11}$$

typically $\boldsymbol{r} \neq \mathbf{0}$ and $\mathcal{R} \neq 0$ only in bio– or chemomechanics

3.3 Balance of linear momentum

total momentum $\boldsymbol{p}$ of a body $\bar{\mathcal{B}}$

$$\boldsymbol{p} := \int_{\bar{\mathcal{B}}} \mathrm{D}_t \boldsymbol{u} \, \mathrm{d}\, m = \int_{\bar{\mathcal{B}}} \boldsymbol{v} \, \mathrm{d}\, m = \int_{\bar{\mathcal{B}}} \rho \, \boldsymbol{v} \, \mathrm{d}V \tag{3.3.1}$$

momentum exchange of body $\bar{\mathcal{B}}$ with environment through contact forces $\boldsymbol{f}^{\text{sur}}$ and at-a-distance forces $\boldsymbol{f}^{\text{vol}}$

$$\boldsymbol{f}^{\text{sur}} := \int_{\partial\bar{\mathcal{B}}} \boldsymbol{t}_\sigma \, \mathrm{d}\, A \qquad \boldsymbol{f}^{\text{vol}} := \int_{\bar{\mathcal{B}}} \boldsymbol{b} \, \mathrm{d}\, V \tag{3.3.2}$$

with contact/surface force $\boldsymbol{t}_\sigma = \boldsymbol{\sigma}^{\text{t}} \cdot \boldsymbol{n}$ and volume force $\boldsymbol{b}$

3.3.1 Global form of balance of momentum

"The time rate of change of the total momentum $\boldsymbol{p}$ of a body $\bar{\mathcal{B}}$ is balanced with the momentum exchange due to contact momentum flux / surface force $\boldsymbol{f}^{\text{sur}}$ and the at-a-distance momentum exchange / volume force $\boldsymbol{f}^{\text{vol}}$."

$$\mathrm{D}_t \boldsymbol{p} = \boldsymbol{f}^{\text{sur}} + \boldsymbol{f}^{\text{vol}} \tag{3.3.3}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho \, \boldsymbol{v} \, \mathrm{d}V = \int_{\partial\bar{\mathcal{B}}} \boldsymbol{t}_\sigma \, \mathrm{d}\, A + \int_{\bar{\mathcal{B}}} \boldsymbol{b} \, \mathrm{d}\, V \tag{3.3.4}$$

3.3.2 Local form of balance of momentum

modification of rate term $\mathrm{D}_t \, \boldsymbol{p}$

$$\mathrm{D}_t \, \boldsymbol{p} = \mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho \, \boldsymbol{v} \mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t(\rho \, \boldsymbol{v}) \mathrm{d}V \tag{3.3.5}$$

modification of surface term $\boldsymbol{f}^{\mathrm{sur}}$

$$\boldsymbol{f}^{\mathrm{sur}} = \int_{\partial\bar{\mathcal{B}}} \boldsymbol{t}_\sigma \,\mathrm{d}\,A \overset{\text{Cauchy}}{=} \int_{\partial\bar{\mathcal{B}}} \boldsymbol{\sigma}_{\mathrm{t}} \cdot \boldsymbol{n} \,\mathrm{d}\,A \overset{\text{Gauss}}{=} \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}})\,\mathrm{d}\,V \tag{3.3.6}$$

and thus

$$\int_{\bar{\mathcal{B}}} \mathrm{D}_t(\rho\,\boldsymbol{v})\,\mathrm{d}V = \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}})\,\mathrm{d}\,V + \int_{\bar{\mathcal{B}}} \boldsymbol{b}\,\mathrm{d}\,V \tag{3.3.7}$$

for arbitrary bodies $\bar{\mathcal{B}}$ $\rightarrow$ local form of momentum balance

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$$\mathrm{D}_t(\rho\,\boldsymbol{v}) = \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{b} \tag{3.3.8}$$

3.3.3 Reduction with lower order balance equations

balance equations are typically modified with the help of lower order balance equations

$$\mathrm{D}_t(\rho\,\boldsymbol{v}) = \boldsymbol{v}\,\mathrm{D}_t\,\rho + \rho\,\mathrm{D}_t\,\boldsymbol{v} = \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{b} \tag{3.3.9}$$

with balance of mass multiplied by velocity $\boldsymbol{v}$

$$\boldsymbol{v}\,\mathrm{D}_t\,\rho = \boldsymbol{v}\,\mathrm{div}\,(\boldsymbol{r}) + \boldsymbol{v}\,\mathcal{R}_0 = \mathrm{div}\,(\boldsymbol{v}\otimes\boldsymbol{r}) - \nabla\boldsymbol{v}\cdot\boldsymbol{r} + \boldsymbol{v}\,\mathcal{R}_0 \tag{3.3.10}$$

local momentum balance in reduced format

$$\rho\,\mathrm{D}_t\boldsymbol{v} = \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}} - \nabla\boldsymbol{v}\otimes\boldsymbol{r}) + \boldsymbol{b} + \nabla\boldsymbol{v}\cdot\boldsymbol{r} - \boldsymbol{v}\,\mathcal{R}_0 \tag{3.3.11}$$