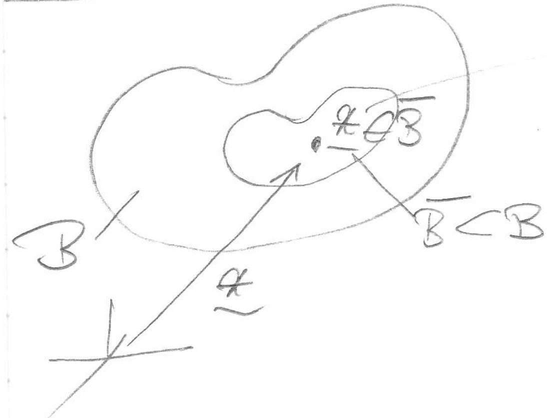
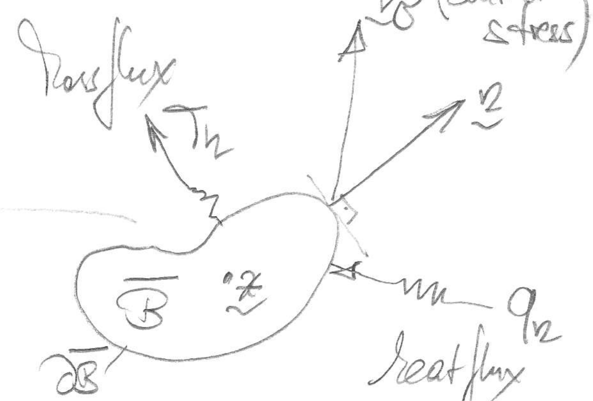
Jan 28, 2010

3. BALANCE EQUATIONS

Recall: the basic steps –

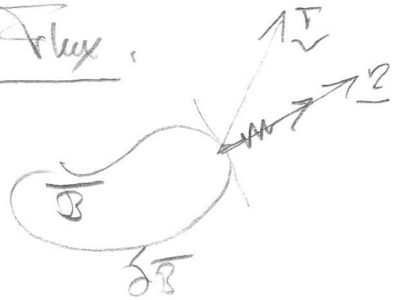




Action at a vicinity –

- Cut out $\bar{B} \subset B$
- Characterize the action at vicinity such as the mass flux (out) T_n, the stress traction vector $\underset{\sim}{t}_\sigma$, and the heat flux q_n
- Postulate the balance of basic physical quantities.
  - Mass m
  - Linear Momentum $\underset{\sim}{I}$
  - Moment of Momentum $\underset{\sim}{D}$
  - Energy $\mathcal{E}$
- Localize the global balance equations to obtain local balance equations at point $\underset{\sim}{x} \in \bar{B}$.

Mass Flux.



- $T_n = T_n(\underset{\sim}{x}, \underset{\sim}{n})$ (Cauchy's postulate)
- $T_n(\underset{\sim}{x}, \underset{\sim}{n}) = -T_n(\underset{\sim}{x}, -\underset{\sim}{n})$ (Cauchy's lemma)
- $T_n = \underset{\sim}{T} \cdot \underset{\sim}{n}$ (Cauchy's theorem)

Today:

Concept of Stress:

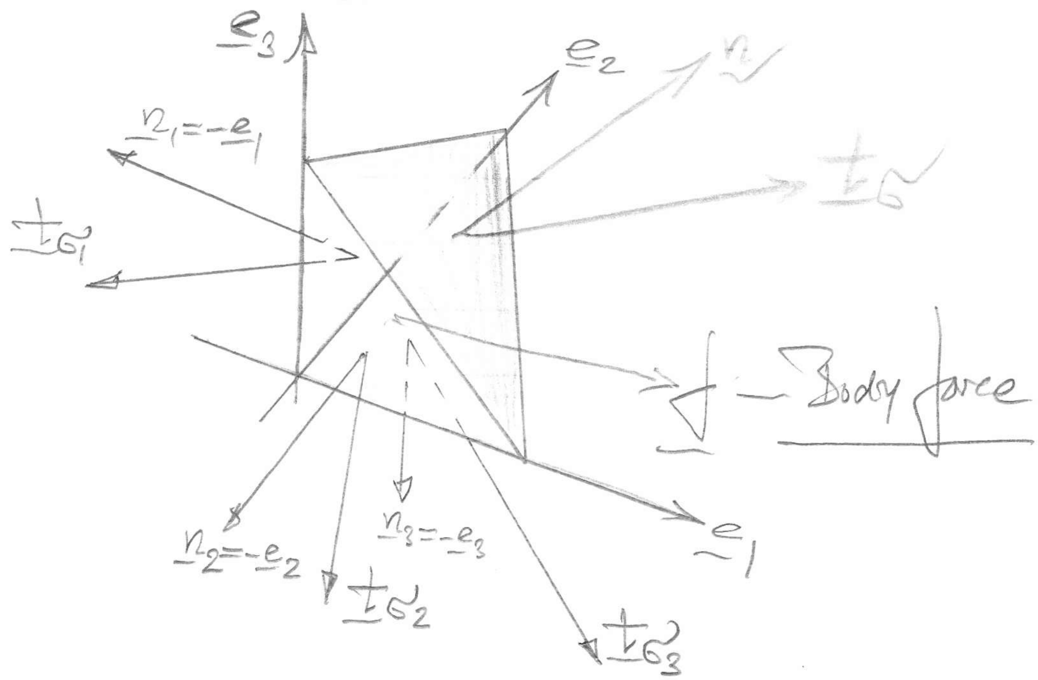
Stress traction vector: $\underline{t}_\sigma := \lim_{\Delta a \to 0} \frac{\Delta \underline{f}}{\Delta a} = \frac{d\underline{f}}{da}$

Cauchy's postulate $\underline{t}_\sigma = \underline{t}_\sigma(\underline{x}, \underline{n})$

Cauchy's Lemma $\underline{t}_\sigma(\underline{x}, \underline{n}) = -\underset{\sim}{t}_\sigma(\underline{x}, -\underline{n})$

Cauchy's Theorem: $\underline{t}_\sigma(\underline{x}, \underline{n}) = \underline{\sigma}^t(\underline{x}) \cdot \underset{\sim}{n}$

Consider a tetrahedron



Directional cosines $da_j = da\, \underline{n} \cdot \underline{e}_j \quad \forall j = 1,2,3$

Force Balance: $\underline{t}_\sigma(\underline{n})\, da + \sum_{j=1}^{3} \underline{t}_{\sigma_j}(\underline{n}_j)\, da_j + \underline{f}\, dv = \underset{\sim}{0}$

$$\lim_{\frac{dv}{da} \to 0} \; \underset{\sim}{t}_\sigma(\underset{\sim}{n})\,da - \sum \underset{\sim}{t}_{\sigma_j}(\underset{\sim}{e}_j)\cdot \underset{\sim}{e}_j\,da + \underset{\sim}{f}\,dv = 0$$

$$= \underset{\sim}{t}_\sigma - \left(\underset{\sim}{t}_{\sigma_1}\, \underset{\sim}{n}\cdot\underset{\sim}{e}_1 + \underset{\sim}{t}_{\sigma_2}\, \underset{\sim}{n}\cdot\underset{\sim}{e}_2 + \underset{\sim}{t}_{\sigma_3}\, \underset{\sim}{n}\cdot\underset{\sim}{e}_3\right) = \underset{\sim}{0}$$

$$\underset{\sim}{t}_\sigma = \left[\underset{\sim}{t}_{\sigma_1}\underset{\sim}{e}_1 + \underset{\sim}{t}_{\sigma_2}\underset{\sim}{e}_2 + \underset{\sim}{t}_{\sigma_3}\underset{\sim}{e}_3\right]\cdot \underset{\sim}{n}$$

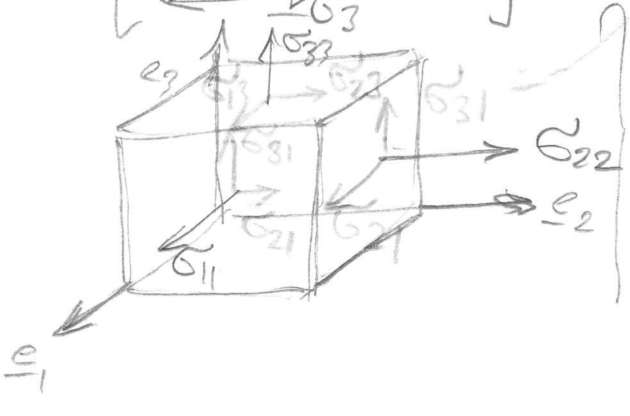
$$\underset{\sim}{t}_\sigma = \underline{\left[\underset{\sim}{t}_{\sigma_j} \otimes \underset{\sim}{e}_j\right]}\cdot \underset{\sim}{n} =: \underline{\underset{\sim}{\sigma}^t \cdot \underset{\sim}{n}}$$

$$\therefore \quad \underset{\sim}{\sigma}^t = \underset{\sim}{t}_{\sigma_j} \otimes \underset{\sim}{e}_j = t_{\sigma_{ji}}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$= \sigma_{ji}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$\boxed{\underset{\sim}{t}_{\sigma_j} = \sigma_{ji}\, \underset{\sim}{e}_i}$$

$$[\sigma_{ij}] = \begin{bmatrix} \longleftarrow \underset{\sim}{t}_{\sigma_1} \longrightarrow \\ \longleftarrow \underset{\sim}{t}_{\sigma_2} \longrightarrow \\ \longleftarrow \underset{\sim}{t}_{\sigma_3} \longrightarrow \end{bmatrix} \qquad \boxed{t_i = \sigma_{ji}\, n_j}$$



Physical interpretation

Volumetric – Deviatoric Decomposition

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{vol} + \underset{\sim}{\sigma}^{dev}$$

where

$$\underset{\sim}{\sigma}^{vol} := \frac{1}{3}\mathrm{tr}(\underset{\sim}{\sigma})\,\underset{\sim}{1} = \frac{1}{3}(\underset{\sim}{\sigma} : \underset{\sim}{1})\,\underset{\sim}{1}$$

$$= \underset{\sim}{\sigma} : \frac{1}{3}\underset{\sim}{1} \otimes \underset{\sim}{1} = \underset{\sim}{\sigma} : \underset{\approx}{\mathbb{I}}^{vol}$$

$$\therefore \quad \underset{\approx}{\mathbb{I}}^{vol} := \frac{1}{3}\underset{\sim}{I} \otimes \underset{\sim}{I} \quad *$$

$$\underset{\sim}{\sigma}^{dev} = \underset{\sim}{\sigma} - \underset{\sim}{\sigma}^{vol}$$

$$= \underset{\sim}{\sigma} : \underset{\approx}{\mathbb{I}}^{sym} - \underset{\sim}{\sigma} : \underset{\approx}{\mathbb{I}}^{vol}$$

$$= \underset{\sim}{\sigma} : \left[\underset{\approx}{\mathbb{I}}^{sym} - \underset{\approx}{\mathbb{I}}^{vol}\right]$$

$$= \underset{\sim}{\sigma} : \underset{\approx}{\mathbb{I}}^{dev}$$

$$\therefore \quad \underset{\approx}{\mathbb{I}}^{dev} = \underset{\approx}{\mathbb{I}}^{sym} - \underset{\approx}{\mathbb{I}}^{vol} \left(= \underset{\approx}{\mathbb{P}} \right) \quad *$$

Owing to the symmetry of $\underset{\approx}{\mathbb{I}}^{vol}$ and $\underset{\approx}{\mathbb{I}}^{dev}$, we can equivalently have

$$\underset{\sim}{\sigma}^{vol} = \underset{\approx}{\mathbb{I}}^{vol} : \underset{\sim}{\sigma} \; ; \; \underset{\sim}{\sigma}^{dev} := \underset{\approx}{\mathbb{I}}^{dev} : \underset{\sim}{\sigma}$$

* See Notes for the indicial representation

Observe that $tr(\underset{\sim}{\sigma}^{dev}) = 0$, and $\underset{\sim}{\sigma}^{vol} = p\,\underset{\sim}{1}$

where $p \triangleq \frac{1}{3} tr\,\underset{\sim}{\sigma}$ → negative pressure.

* See the Notes for the matrix representation

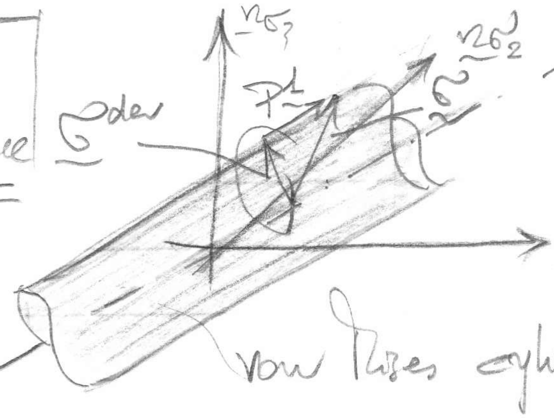
* EXAMPLES: Metal plasticity, Incompressible Elasticity.

Explain where does the factor $\sqrt{\frac{2}{3}}\,\sigma_Y$ come from in the von Mises yield criterion:

$$\phi(\underset{\sim}{\sigma}) = \|\underset{\sim}{\sigma}^{dev}\| - \sqrt{\frac{2}{3}}\,\sigma_Y \le 0$$

Based on the uniaxial-tension test

Principal-Stress-Space

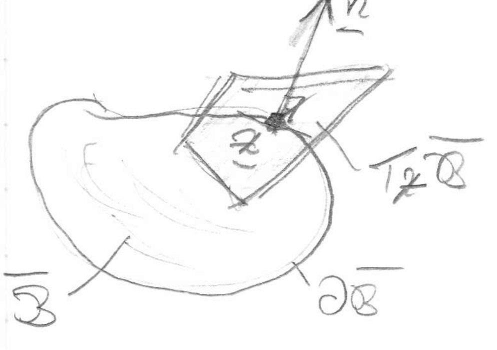


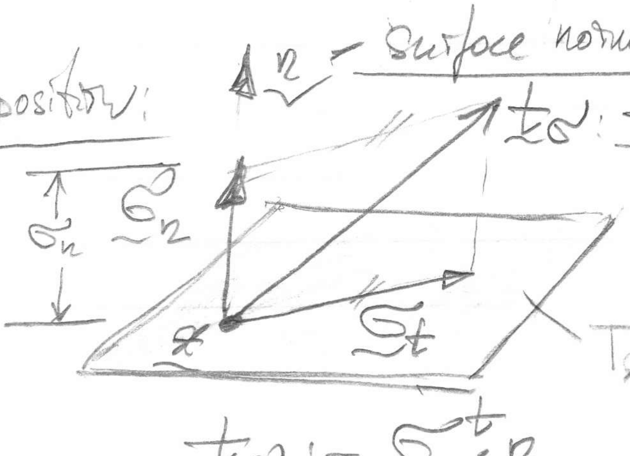
$\underset{\sim}{1}p \;(\underset{\sim}{n}_{\sigma_1} + \underset{\sim}{n}_{\sigma_2} + \underset{\sim}{n}_{\sigma_3})$

Hydrostatic axis

von Mises cylinder (yield surface)

Normal-Shear Decomposition:





$$\underset{\sim}{t}_\sigma := \underset{\sim}{\sigma}^t \cdot \underset{\sim}{n}$$

Through the Cauchy theorem, we have

Normal Stress Vector

Tangential Stress Vector

$$\underset{\sim}{t}_\sigma := \underset{\sim}{\sigma}^t \cdot \underset{\sim}{n} = \underset{\sim}{\sigma}_n + \underset{\sim}{\sigma}_t$$

$$\sigma_n := \underset{\sim}{t}_\sigma \cdot \underset{\sim}{n} = \underset{\sim}{\sigma}^t : (\underset{\sim}{n} \otimes \underset{\sim}{n}) \quad \text{Normal Stress}$$

Thus,

$$\underset{\sim}{\sigma}_n = \sigma_n \underset{\sim}{n} = \left[\underset{\sim}{\sigma}^t : (\underset{\sim}{n} \otimes \underset{\sim}{n})\right] \underset{\sim}{n}$$

$$= \underset{\sim}{\sigma}^t : \left[\underset{\sim}{n} \otimes \underset{\sim}{n} \otimes \underset{\sim}{n}\right]$$

Normal stress Vector

and,

Shear

$$\underset{\sim}{\sigma}_t = \underset{\sim}{t}_\sigma - \underset{\sim}{\sigma}_n = \underset{\sim}{\sigma}^t \cdot \underset{\sim}{n} - \underset{\sim}{\sigma}^t : \left[\underset{\sim}{n} \otimes \underset{\sim}{n} \otimes \underset{\sim}{n}\right]$$

$$= \underset{\sim}{\sigma}^t : \left[\underset{\approx}{I}^{sym} \cdot \underset{\sim}{n} - \underset{\sim}{n} \otimes \underset{\sim}{n} \otimes \underset{\sim}{n}\right]$$

Tangential stress Vector

EXAMPLE: CRYSTAL PLASTICITY (SCHMID STRESS), SOIL PLASTICITY.

Then, the amount of shear stress τ_n is given by

$$\|\tau_n\|^2 = (\underset{\sim}{t}_\sigma - \underset{\sim}{\sigma}_n) \cdot (\underset{\sim}{t}_\sigma - \underset{\sim}{\sigma}_n) = \underset{\sim}{\sigma}_t \cdot \underset{\sim}{\sigma}_t$$

$$\|\tau_n\| = \sqrt{\underset{\sim}{t}_\sigma \cdot \underset{\sim}{t}_\sigma - \sigma_n^2}$$

PRINCIPAL STRESSES

Eigenvalue problem $\underset{\sim}{\sigma}^t \cdot \underline{n}_{\sigma_i} = \lambda_{\sigma_i} \underline{n}_{\sigma_i}$

$$\left[\underset{\sim}{\sigma}^t - \lambda_{\sigma_i} \underset{\sim}{1}\right] \cdot \underline{n}_{\sigma_i} = \underset{\sim}{0}$$

Characteristic Equation

$$\lambda_\sigma^3 - I_\sigma \lambda_\sigma^2 + II_\sigma \lambda_\sigma - III_\sigma = 0$$

where

$$I_\sigma := tr[\underset{\sim}{\sigma}^t] = \lambda_{\sigma_1} + \lambda_{\sigma_2} + \lambda_{\sigma_3}$$

$$II_\sigma := \frac{1}{2}\left[tr^2(\underset{\sim}{\sigma}^t) - tr(\underset{\sim}{\sigma}^{t2})\right] = \lambda_{\sigma_1}\lambda_{\sigma_2} + \lambda_{\sigma_2}\lambda_{\sigma_3} + \lambda_{\sigma_1}\lambda_{\sigma_3}$$

$$III_\sigma := det(\underset{\sim}{\sigma}^t) = \lambda_{\sigma_1}\lambda_{\sigma_2}\lambda_{\sigma_3}$$

This leads us to the spectral representation of the stress tensor

$$\boxed{\underset{\sim}{\sigma}^t = \sum_{i=1}^{3} \lambda_{\sigma_i} \underline{n}_{\sigma_i} \otimes \underline{n}_{\sigma_i}}$$

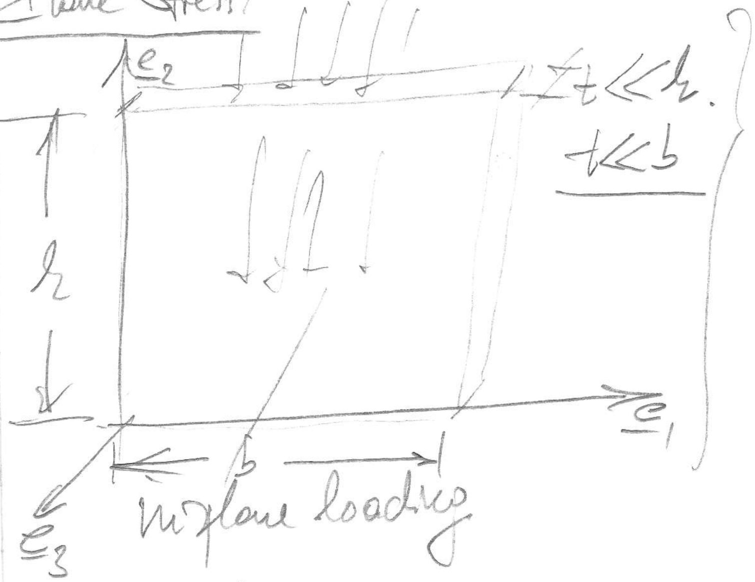
Observe that λ_{σ_i} are purely normal; that is, if we decompose $\underset{\sim}{\sigma}^t$ on a plane with a normal $\underline{n}_{\sigma_i}$; $\underset{\sim}{\sigma}_{n_i} = \lambda_{\sigma_i} \underline{n}_i$, $\underset{\sim}{\sigma}_t = \underset{\sim}{0}$

Derivation

$$\sum \left[\underset{\sim}{\sigma}^t \cdot \underline{n}_{\sigma_i} \otimes \underline{n}_{\sigma_i} = \lambda_{\sigma_i} \underline{n}_{\sigma_i}\right] \otimes \underline{n}_{\sigma_i}$$

$$\underset{\sim}{\sigma}^t \cdot \underbrace{\sum \underline{n}_{\sigma_i} \otimes \underline{n}_{\sigma_i}}_{\underset{\sim}{1}} = \sum \lambda_{\sigma_i} \underline{n}_{\sigma_i} \otimes \underline{n}_{\sigma_i} \quad q.e.d.$$

Plane Stress



$\sigma_{i3} = \sigma_{3i} = 0 \quad \forall i = 1,2,3$

Plane stress condition

Voigt Notation (Storage): As we will show $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^t$, we store the stress tensor

$$\begin{bmatrix} 1 & 4 & 6 \\ & 2 & 5 \\ & & 3 \end{bmatrix} \longrightarrow$$

In 3d. $\rightarrow \underset{\sim}{\sigma} = [\sigma_{11} \; \sigma_{22} \; \sigma_{33} \; \sigma_{12} \; \sigma_{23} \; \sigma_{13}]^t$

Concept of Heat Flux.

Analogous for:
- Mass flux
- Momentum flux
- Heat flux

Cauchy-Type descriptions

Cauchy's postulate $\quad q_n = \hat{q}_n(\underline{x}, \underline{n})$

Cauchy's Lemma $\quad q_n(\underline{x}, \underline{n}) = -q_n(\underline{x}, -\underline{n})$

Cauchy's Theorem $\quad q_n = \underset{\sim}{q} \cdot \underset{\sim}{n}$