ME338A
CONTINUUM MECHANICS

lecture notes 08

thursday, january 28th, 2010

Volumetric–deviatoric decomposition

in analogy to the strain tensor $\boldsymbol{\epsilon}$, the stress tensor $\boldsymbol{\sigma}$ can be additively decomposed into a volumetric part $\boldsymbol{\sigma}^{\text{vol}}$ and a traceless deviatoric part $\boldsymbol{\sigma}^{\text{dev}}$

volumetric – deviatoric decomposition of stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{vol}} + \boldsymbol{\sigma}^{\text{dev}} \tag{3.1.21}$$

with volumetric and deviatoric stress tensor $\boldsymbol{\sigma}^{\text{vol}}$ and $\boldsymbol{\sigma}^{\text{dev}}$

$$\text{tr}(\boldsymbol{\sigma}^{\text{vol}}) = \text{tr}(\boldsymbol{\sigma}) \qquad \text{tr}(\boldsymbol{\sigma}^{\text{dev}}) = 0 \tag{3.1.22}$$

• volumetric second order tensor $\boldsymbol{\sigma}^{\text{vol}}$

$$\boldsymbol{\sigma}^{\text{vol}} = \frac{1}{3}[\boldsymbol{\sigma} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\text{vol}} : \boldsymbol{\sigma} \tag{3.1.23}$$

upon double contraction volumetric fourth order unit tensor $\mathbb{I}^{\text{vol}}$ extracts volumetric part $\boldsymbol{\sigma}^{\text{vol}}$ of stress tensor

$$\begin{aligned} \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\,\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\,\delta_{ij}\delta_{kl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{3.1.24}$$

• deviatoric second order tensor $\boldsymbol{\sigma}^{\text{dev}}$

$$\boldsymbol{\sigma}^{\text{dev}} = \boldsymbol{\sigma} - \frac{1}{3}[\boldsymbol{\sigma} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{\sigma} \tag{3.1.25}$$

upon double contraction deviatoric fourth order unit tensor $\mathbb{I}^{\text{dev}}$ extracts deviatoric part of stress tensor

$$\begin{aligned} \mathbb{I}^{\text{dev}} &= \mathbb{I}^{\text{sym}} - \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{sym}} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{dev}} &= \left[\tfrac{1}{2}\,\delta_{ik}\delta_{jl} + \tfrac{1}{2}\,\delta_{il}\delta_{jk} - \tfrac{1}{3}\,\delta_{ij}\delta_{kl}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{3.1.26}$$

Volumetric stress

volumetric part $\boldsymbol{\sigma}^{\text{vol}}$ of stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}^{\text{vol}} = \tfrac{1}{3}\,[\boldsymbol{\sigma} : \boldsymbol{I}]\,\boldsymbol{I} = \tfrac{1}{3}\,[\boldsymbol{I} \otimes \boldsymbol{I}] : \boldsymbol{\sigma} = \mathbb{I}^{\text{vol}} : \boldsymbol{\sigma} \tag{3.1.27}$$

interpretation of trace as hydrostatic pressure

$$p = \tfrac{1}{3}\text{tr}(\boldsymbol{\sigma}) = \tfrac{1}{3}\,\boldsymbol{\sigma} : \boldsymbol{I} = \tfrac{1}{3}\,(\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{3.1.28}$$

index representation

$$\boldsymbol{\sigma}^{\text{vol}} = \sigma^{\text{vol}}_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.1.29}$$

matrix representation of coordinates $[\sigma^{\text{vol}}_{ij}]$

$$[\sigma^{\text{vol}}_{ij}] = p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad p = \frac{1}{3}\,\text{tr}(\boldsymbol{\sigma}) \tag{3.1.30}$$

volumetric stress tensor $\boldsymbol{\sigma}^{\text{vol}}$ is a spherical second order tensor as $\boldsymbol{\sigma}^{\text{vol}} = p\,\boldsymbol{I}$

volumetric stress tensor $\boldsymbol{\sigma}^{\text{vol}}$ contains the hydrostatic pressure part of the total stress tensor $\boldsymbol{\sigma}$

Deviatoric stress

deviatoric stress tensor $\boldsymbol{\sigma}^{\text{dev}}$ preserves the volume and containes the remaining part of the total stress tensor $\boldsymbol{\sigma}$

deviatoric part $\boldsymbol{\sigma}^{\text{dev}}$ of the stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}^{\text{dev}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{vol}} = \boldsymbol{\sigma} - \frac{1}{3}\left[\boldsymbol{\sigma} : \boldsymbol{I}\right]\boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{\sigma} \tag{3.1.31}$$

index representation

$$\boldsymbol{\sigma}^{\text{dev}} = \sigma_{ij}^{\text{dev}}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.1.32}$$

matrix representation of coordinates $[\sigma_{ij}^{\text{dev}}]$

$$[\sigma_{ij}^{\text{dev}}] = \frac{1}{3}\begin{bmatrix} 2\sigma_{11} - \sigma_{22} - \sigma_{33} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & 2\sigma_{22} - \sigma_{11} - \sigma_{33} & \sigma_{13} \\ \sigma_{31} & \sigma_{32} & 2\sigma_{33} - \sigma_{11} - \sigma_{22} \end{bmatrix} \tag{3.1.33}$$

trace of deviatoric stresss $\text{tr}\left(\boldsymbol{\sigma}^{\text{dev}}\right)$

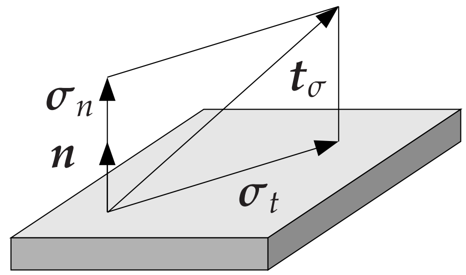
$$\begin{aligned} \text{tr}\left(\boldsymbol{\sigma}^{\text{dev}}\right) &= \frac{1}{3}\left[2\sigma_{11} - \sigma_{22} - \sigma_{33}\right] \\ &+ \frac{1}{3}\left[2\sigma_{22} - \sigma_{11} - \sigma_{33}\right] \\ &+ \frac{1}{3}\left[2\sigma_{33} - \sigma_{11} - \sigma_{22}\right] = 0 \end{aligned} \tag{3.1.34}$$

deviatoric stress tensor $\boldsymbol{\sigma}^{\text{dev}}$ is a traceless second order tensor as $\text{tr}\left(\boldsymbol{\sigma}^{\text{dev}}\right) = 0$

deviatoric stress tensor $\boldsymbol{\sigma}^{\text{dev}}$ contains the hydrostatic pressure free part of the total stress tensor $\boldsymbol{\sigma}$

Normal–shear decomposition

assume we are interested in the stress σ_n normal to a particular plane characterized through its normal $\boldsymbol{n}$, i.e. the normal projection of the stress vector $\boldsymbol{t}_\sigma$



$$\sigma_n = \boldsymbol{t}_\sigma \cdot \boldsymbol{n} = [\boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}] \cdot \boldsymbol{n} = \boldsymbol{\sigma}^{\mathrm{t}} : [\boldsymbol{n} \otimes \boldsymbol{n}] = \boldsymbol{\sigma}^{\mathrm{t}} : \boldsymbol{N} \qquad (3.1.35)$$

normal–shear (tangential) decomposition of stress vector $\boldsymbol{t}_\sigma$

$$\boldsymbol{t}_\sigma = \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_t \qquad (3.1.36)$$

normal stress vector – stress in direction of $\boldsymbol{n}$

$$\boldsymbol{\sigma}_n = [\boldsymbol{\sigma}^{\mathrm{t}} : [\boldsymbol{n} \otimes \boldsymbol{n}]]\, \boldsymbol{n} = \boldsymbol{\sigma}^{\mathrm{t}} : [\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] \qquad (3.1.37)$$

shear (tangential) stress vector – stress in the plane

$$\begin{aligned} \boldsymbol{\sigma}_t &= \boldsymbol{t}_\sigma - \boldsymbol{\sigma}_n = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} - \boldsymbol{\sigma}^{\mathrm{t}} : [\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] \\ &= \boldsymbol{\sigma}^{\mathrm{t}} : [\mathbb{I}^{\mathrm{sym}} \cdot \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] = \boldsymbol{\sigma}^{\mathrm{t}} : \boldsymbol{T} \end{aligned} \qquad (3.1.38)$$

amount of shear stress τ_n

$$||\tau_n||^2 = (\boldsymbol{t}_\sigma - \boldsymbol{\sigma}_n) \cdot (\boldsymbol{t}_\sigma - \boldsymbol{\sigma}_n) = \boldsymbol{t}_\sigma \cdot \boldsymbol{t}_\sigma - 2\boldsymbol{t}_\sigma \cdot \boldsymbol{\sigma}_n + \sigma_n^2 \boldsymbol{n} \cdot \boldsymbol{n} \qquad (3.1.39)$$

and thus

$$\tau_n = ||\boldsymbol{\sigma}_t|| = \sqrt{\boldsymbol{\sigma}_t \cdot \boldsymbol{\sigma}_t} = \sqrt{\boldsymbol{t}_\sigma \cdot \boldsymbol{t}_\sigma - \sigma_n^2} \qquad (3.1.40)$$

in general, i.e. for an arbitrary direction $\boldsymbol{n}$, we have normal and shear contributions to the stress vector, however, three particular directions $\{\boldsymbol{n}_{\sigma\, i}\}_{i=1,2,3}$ can be identified, for which $\boldsymbol{t}_\sigma = \boldsymbol{\sigma}_n$ and thus $\boldsymbol{\sigma}_t = \boldsymbol{0}$, the corresponding $\{\boldsymbol{n}_{\sigma\, i}\}_{i=1,2,3}$ are called prinicpal stress directions and $\{\sigma_{n\, i}\}_{i=1,2,3} = \{\lambda_{\sigma i}\}_{i=1,2,3}$ are the principal stresses

Principal stresses

assume stress tensor $\boldsymbol{\sigma}^{\mathrm{t}}$ to be known at $\boldsymbol{x} \in \mathcal{B}$, principal stresses $\{\lambda_{\sigma\, i}\}_{i=1,2,3}$ and principal stress directions $\{\boldsymbol{n}_{\sigma\, i}\}_{i=1,2,3}$ can be derived from solution of special eigenvalue problem according to §1.1.3

$$\boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}_{\sigma\, i} = \lambda_{\sigma\, i}\, \boldsymbol{n}_{\sigma\, i} \qquad \left[\, \boldsymbol{\sigma}^{\mathrm{t}} - \lambda_{\sigma\, i}\,\right] \cdot \boldsymbol{n}_{\sigma\, i} = \boldsymbol{0} \tag{3.1.41}$$

solution

$$\det\left(\boldsymbol{\sigma}^{\mathrm{t}} - \lambda_{\sigma}\, \boldsymbol{I}\right) = 0 \tag{3.1.42}$$

or in terms of roots of characteristic equation

$$\lambda_{\sigma}^{3} - I_{\sigma}\, \lambda_{\sigma}^{2} + II_{\sigma}\, \lambda_{\sigma} - III_{\sigma} = 0 \tag{3.1.43}$$

roots of characteristic equation in terms of principal invariants of $\boldsymbol{\sigma}^{\mathrm{t}}$

$$\begin{aligned} I_{\sigma} &= \mathrm{tr}\,(\boldsymbol{\sigma}^{\mathrm{t}}) &&= \lambda_{\sigma 1} + \lambda_{\sigma 2} + \lambda_{\sigma 3} \\ II_{\sigma} &= \tfrac{1}{2}[\mathrm{tr}^{2}(\boldsymbol{\sigma}^{\mathrm{t}}) - \mathrm{tr}\,(\boldsymbol{\sigma}^{\mathrm{t}\,2})] &&= \lambda_{\sigma 2}\lambda_{\sigma 3} + \lambda_{\sigma 3}\lambda_{\sigma 1} + \lambda_{\sigma 1}\lambda_{\sigma 2} \\ III_{\sigma} &= \det\,(\boldsymbol{\sigma}^{\mathrm{t}}) &&= \lambda_{\sigma 1}\, \lambda_{\sigma 2}\, \lambda_{\sigma 3} \end{aligned} \tag{3.1.44}$$

spectral representation of $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}^{\mathrm{t}} = \sum_{i=1}^{3} \lambda_{\sigma i}\, \boldsymbol{n}_{\sigma i} \otimes \boldsymbol{n}_{\sigma i} \tag{3.1.45}$$

principal stresses $\lambda_{\sigma i}$ are purely normal, no shear stress τ_n in principal directions, i.e. $\boldsymbol{t}_{\sigma i} = \boldsymbol{\sigma}_n = \lambda_{\sigma i}\, \boldsymbol{n}_{\sigma i}$ and $\boldsymbol{\sigma}_t = \boldsymbol{0}$ thus $\tau_n = 0$

due to symmetry of stresses $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$, stress tensor posseses three real eigenvalues $\{\lambda_{\sigma\, i}\}_{i=1,2,3}$, corresponding eigendirections $\{\boldsymbol{n}_{\sigma\, i}\}_{i=1,2,3}$ are thus orthogonal $\boldsymbol{n}_{\sigma i} \cdot \boldsymbol{n}_{\sigma j} = \delta_{ij}$

Special case of plane stress

dimensional reduction in case of plane stress with vanishing stresses $\sigma_{13} = \sigma_{23} = \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$ in out of plane direction, e.g. for flat sheets

$$\boldsymbol{\sigma} = \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.1.46}$$

matrix representation of coordinates $[\sigma_{ij}]$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.1.47}$$

Voigt representation of stress

three dimensional second order stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.1.48}$$

matrix representation of coordinates $[\sigma_{ij}]$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \tag{3.1.49}$$

due to symmetry $[\sigma_{ij}] = [\sigma_{ji}]$ and thus $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{31} = \sigma_{13}$, stress tensor $\boldsymbol{\sigma}$ contains only six independent components $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}$,it proves convenient to represent second order tensor $\boldsymbol{\sigma}$ through a vector $\underline{\sigma}$

$$\underline{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^{\mathrm{t}} \tag{3.1.50}$$

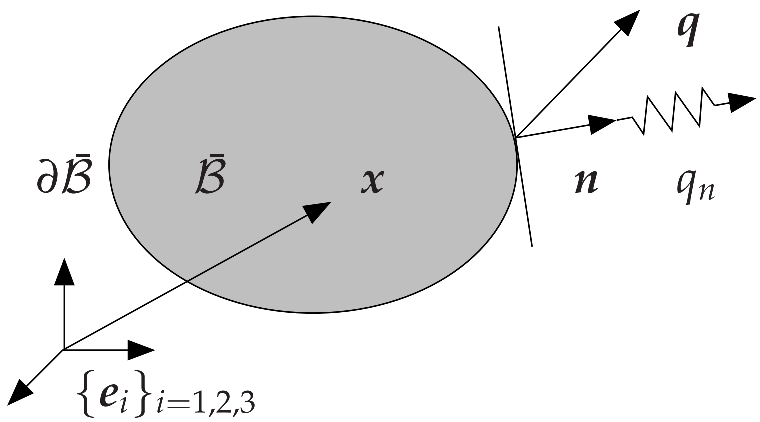
vector representation $\underline{\sigma}$ of stress $\boldsymbol{\sigma}$ in case of plane stress

$$\underline{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}]^{\mathrm{t}} \tag{3.1.51}$$

3.1.3 Concept of heat flux

the contact heat flux q_n at a point $\boldsymbol{x}$ is a scalar of the unit [energy/time/surface area]

the contact heat flux q_n characterizes the energy transport normal to the tangent plane to an imaginary surface passing through this point with normal vector $\boldsymbol{n}$



definition of contact heat flux q_n in analogy to Cauchy's postulate, lemma and theorem originally introduced for the momentum flux in §3.1.2

Cauchy's postulate

$$q_n = q_n(\boldsymbol{x}, \boldsymbol{n}) \tag{3.1.52}$$

Cauchy's lemma

$$q_n(\boldsymbol{x}, \boldsymbol{n}) = -q_n(\boldsymbol{x}, -\boldsymbol{n}) \tag{3.1.53}$$

Cauchy's theorem

the contact heat flux q_n can be expressed as linear function of the surface normal $\boldsymbol{n}$ and the heat flux vector $\boldsymbol{q}$

$$q_n = \boldsymbol{q} \cdot \boldsymbol{n} \tag{3.1.54}$$

Heat flux vector

the vector field $\boldsymbol{q}$ is called heat flux vector

$$\boldsymbol{q} = q_i \, \boldsymbol{e}_i \tag{3.1.55}$$

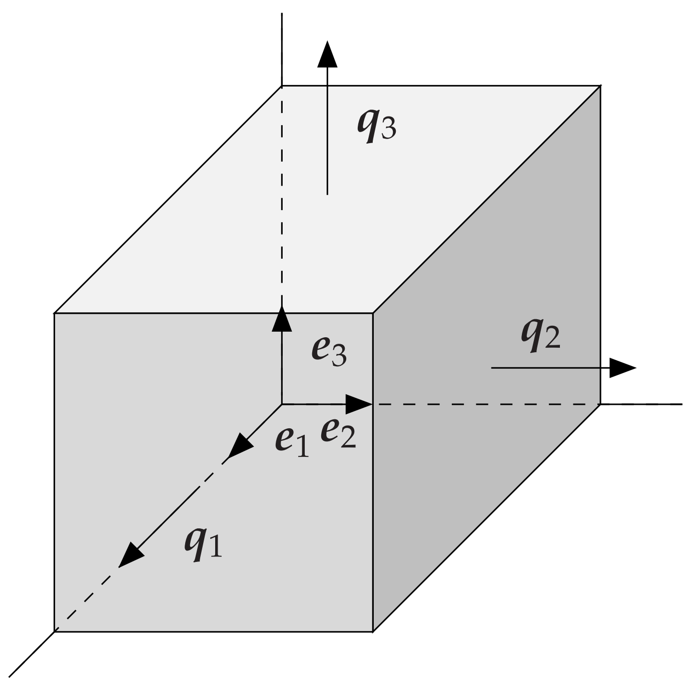
Cauchy's theorem

$$q_n = \boldsymbol{q} \cdot \boldsymbol{n} \tag{3.1.56}$$

index representation

$$q_n = (q_i \, \boldsymbol{e}_i) \cdot (n_j \, \boldsymbol{e}_j) = q_i \, n_j \, \delta_{ij} = q_i \, n_i \tag{3.1.57}$$

geometric interpretation



the coordinates q_i characterize the heat energy transport through the planes parallel to the coordinate planes

in continuum mechanics of adiabatic systems the heat flux vector vanishes identically