ME338A
CONTINUUM MECHANICS

lecture notes 07

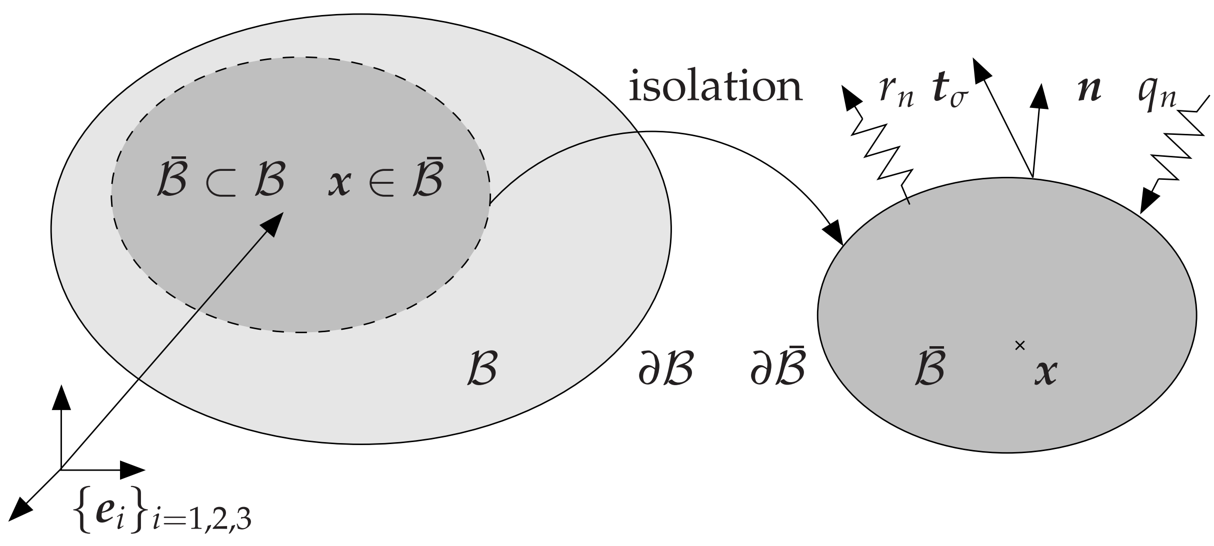
tuesday, january 26th, 2010

3 Balance equations

3.1 Basic ideas

• until now:
kinematics, i.e. characterization of deformation of a material body $\mathcal{B}$ without studying its physical cause

• now:
balance equations, i.e. general statements that characterize the cause of cause of the motion of any body $\mathcal{B}$



basic strategy

• isolation of an arbitrary subset $\bar{\mathcal{B}}$ of the body $\mathcal{B}$

• characterization of the influence of the remaining body $\mathcal{B}\backslash\bar{\mathcal{B}}$ on $\bar{\mathcal{B}}$ through phenomenological quantities, i.e. the contact mass flux r, the contact stress $\boldsymbol{t}_\sigma$, the contact heat flux q

• definition of basic physical quantities, i.e. the mass m, the linear momentum I, the moment of momentum D and the energy E of subset $\bar{\mathcal{B}}$

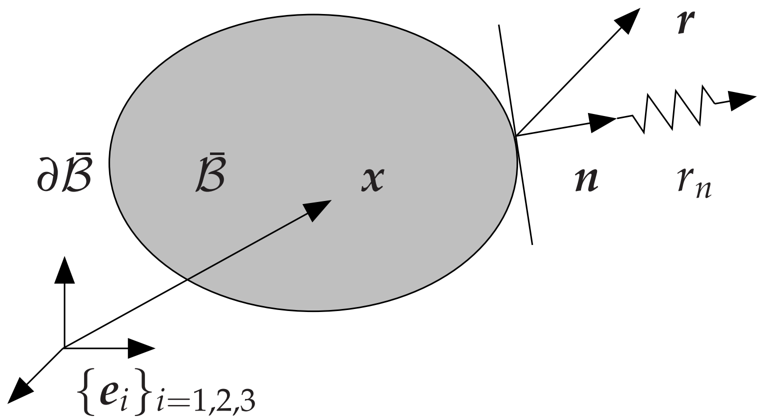
• postulate of balance of these quantities renders global balance equations for subset $\bar{\mathcal{B}}$

• localization of global balance equations renders local balance equations at point $\boldsymbol{x} \in \bar{\mathcal{B}}$

3.1.1 Concept of mass flux

the contact mass flux r_n at a point $\boldsymbol{x}$ is a scalar of the unit [mass/time/surface area]

the contact mass flux r_n characterizes the transport of matter normal to the tangent plane to an imaginary surface passing through this point with normal vector $\boldsymbol{n}$



definition of contact heat flux q_n in analogy to Cauchy's postulate, lemma and theorem originally introduced for the momentum flux in §3.1.2

Cauchy's postulate

$$r_n = r_n\left(\boldsymbol{x}, \boldsymbol{n}\right) \tag{3.1.1}$$

Cauchy's lemma

$$r_n(\boldsymbol{x}, \boldsymbol{n}) = -r_n(\boldsymbol{x}, -\boldsymbol{n}) \tag{3.1.2}$$

Cauchy's theorem

the contact mass flux r_n can be expressed as linear function of the surface normal $\boldsymbol{n}$ and the mass flux vector $\boldsymbol{r}$

$$r_n = \boldsymbol{r} \cdot \boldsymbol{n} \tag{3.1.3}$$

Mass flux vector

the vector field $\boldsymbol{r}$ is called mass flux vector

$$\boldsymbol{r} = r_i \, \boldsymbol{e}_i \tag{3.1.4}$$

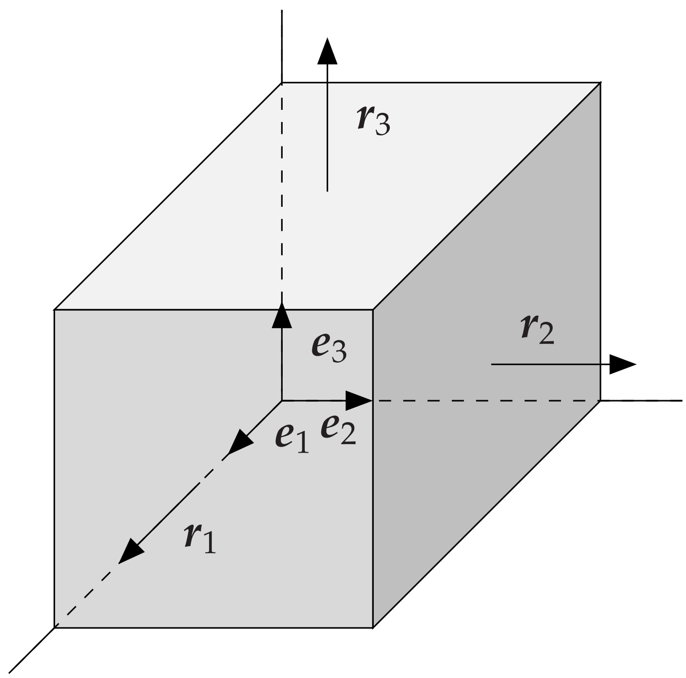
Cauchy's theorem

$$r_n = \boldsymbol{r} \cdot \boldsymbol{n} \tag{3.1.5}$$

index representation

$$r_n = (r_i \, \boldsymbol{e}_i) \cdot (n_j \, \boldsymbol{e}_j) = r_i \, n_j \, \delta_{ij} = r_i \, n_i \tag{3.1.6}$$

geometric interpretation



the coordinates r_i characterize the transport of matter through the planes parallel to the coordinate planes

in classical closed system continuum mechanics (here) the mass flux vector vanishes identically

examples of mass flux: transport of chemical reactants in chemomechanics or cell migration in biomechanics

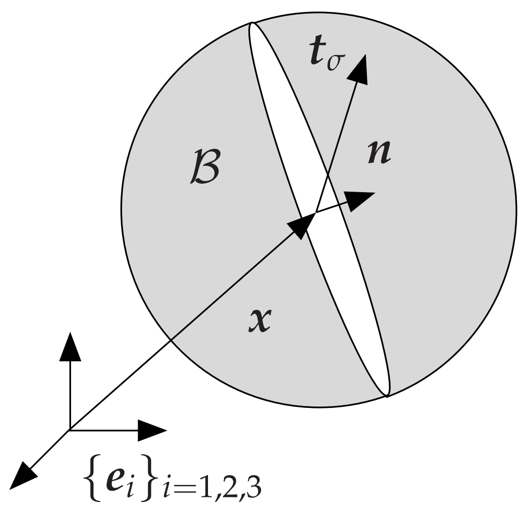
3.1.2 Concept of stress

traction vector

$$\boldsymbol{t}_\sigma = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{f}}{\Delta a} = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}a} \qquad (3.1.7)$$

interpretation as surface force per unit surface area

Cauchy's postulate



the traction vector $\boldsymbol{t}_\sigma$ at a point $\boldsymbol{x}$ can be expressed exclusively in terms of the point $\boldsymbol{x}$ and the normal $\boldsymbol{n}$ to the tangent plane to an imaginary surface passing through this point

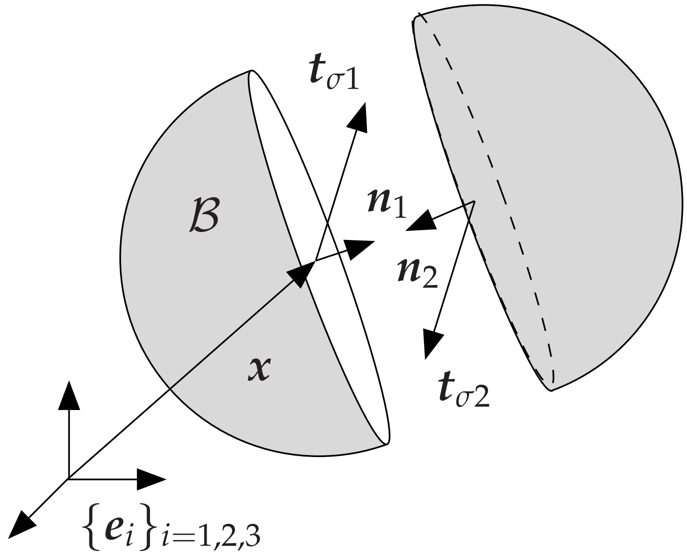
traction vector

$$\boldsymbol{t}_\sigma = \boldsymbol{t}_\sigma(\boldsymbol{x}, \boldsymbol{n}) \qquad (3.1.8)$$

Cauchy's lemma

the traction vectors acting on opposite sides of a surface are equal in magnitude and opposite in sign

$$\boldsymbol{t}_{\sigma 1}(\boldsymbol{x}, \boldsymbol{n}_1) = -\boldsymbol{t}_{\sigma 2}(\boldsymbol{x}, \boldsymbol{n}_2) \qquad (3.1.9)$$



generalization with $\boldsymbol{n} = \boldsymbol{n}_1 = -\boldsymbol{n}_2$ and $\boldsymbol{t}_\sigma = \boldsymbol{t}_{\sigma 1}$

$$\boldsymbol{t}_\sigma(\boldsymbol{x}, \boldsymbol{n}) = -\boldsymbol{t}_\sigma(\boldsymbol{x}, -\boldsymbol{n}) \qquad (3.1.10)$$

Cauchy's theorem

the traction vector $\boldsymbol{t}_\sigma$ can be expressed as a linear map of the surface normal $\boldsymbol{n}$ mapped via the transposed stress tensor $\boldsymbol{\sigma}^{\mathrm{t}}$

$$\boldsymbol{t}_\sigma = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} \qquad (3.1.11)$$

accordingly with $\boldsymbol{n} = \boldsymbol{n}_1 = -\boldsymbol{n}_2$ and $\boldsymbol{t}_\sigma = \boldsymbol{t}_{\sigma 1}$

$$\begin{aligned} \boldsymbol{t}_{\sigma 1} &= \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}_1 = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} = \boldsymbol{t}_\sigma \\ \boldsymbol{t}_{\sigma 2} &= \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}_2 = -\boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} = -\boldsymbol{t}_\sigma \end{aligned} \qquad (3.1.12)$$

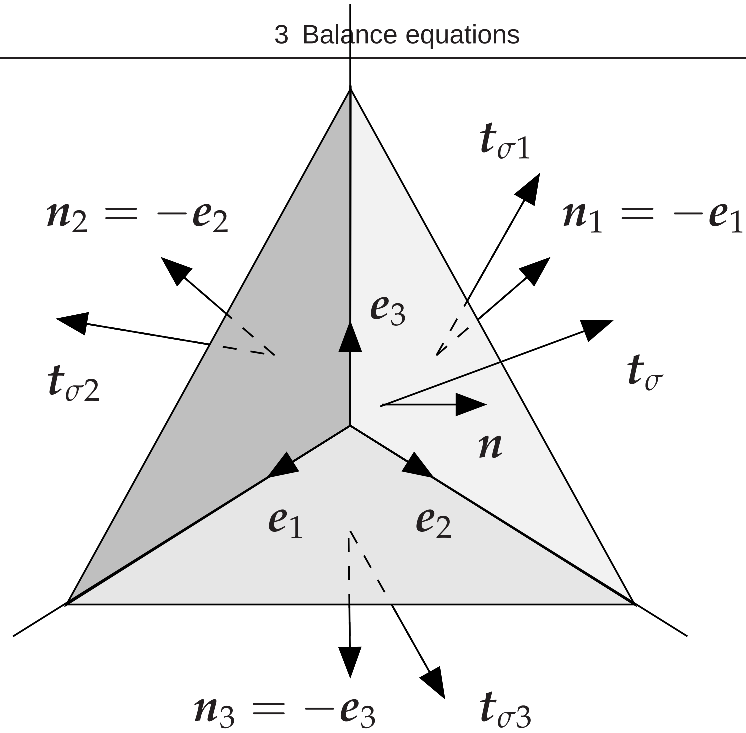
Cauchy tetraeder

balance of momentum (pointwise)

$$\boldsymbol{t}_\sigma(\boldsymbol{n})\, da = -\boldsymbol{t}_\sigma(\boldsymbol{n}_i)\, da_i = \boldsymbol{t}_\sigma(\boldsymbol{e}_i)\, da_i = \boldsymbol{t}_{\sigma i}\, da_i \qquad (3.1.13)$$

surface theorem, area fractions from Gauss theorem

$$\boldsymbol{n} da = -\boldsymbol{n}_i da_i = \boldsymbol{e}_i da_i \qquad \frac{da_i}{da} = \boldsymbol{e}_i \cdot \boldsymbol{n} = \cos\angle(\boldsymbol{e}_i, \boldsymbol{n}) \quad (3.1.14)$$



traction vector as linear map of surface normal

$$\boldsymbol{t}_\sigma(\boldsymbol{n}) = \boldsymbol{t}_{\sigma i}\frac{da_i}{da} = \boldsymbol{t}_{\sigma i}\cos\angle(\boldsymbol{e_i},\boldsymbol{n}) = \boldsymbol{t}_{\sigma i}\,[\boldsymbol{e_i}\cdot\boldsymbol{n}] = [\boldsymbol{t}_{\sigma i}\otimes\boldsymbol{e_i}]\cdot\boldsymbol{n} \tag{3.1.15}$$

compare $\boldsymbol{t}_\sigma(\boldsymbol{n}) = \boldsymbol{\sigma}^{\mathrm{t}}\cdot\boldsymbol{n}$
interpretation of second order stress tensor as $\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{t}_{\sigma i}\otimes\boldsymbol{e}_i$

Stress tensor

Cauchy stress (true stress)

$$\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{t}_{\sigma i}\otimes\boldsymbol{e}_i = \sigma_{ji}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \qquad \boldsymbol{\sigma} = \boldsymbol{e}_i\otimes\boldsymbol{t}_{\sigma i} = \sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{3.1.16}$$

Cauchy theorem

$$\boldsymbol{t}_\sigma = \boldsymbol{\sigma}^{\mathrm{t}}\cdot\boldsymbol{n} \tag{3.1.17}$$

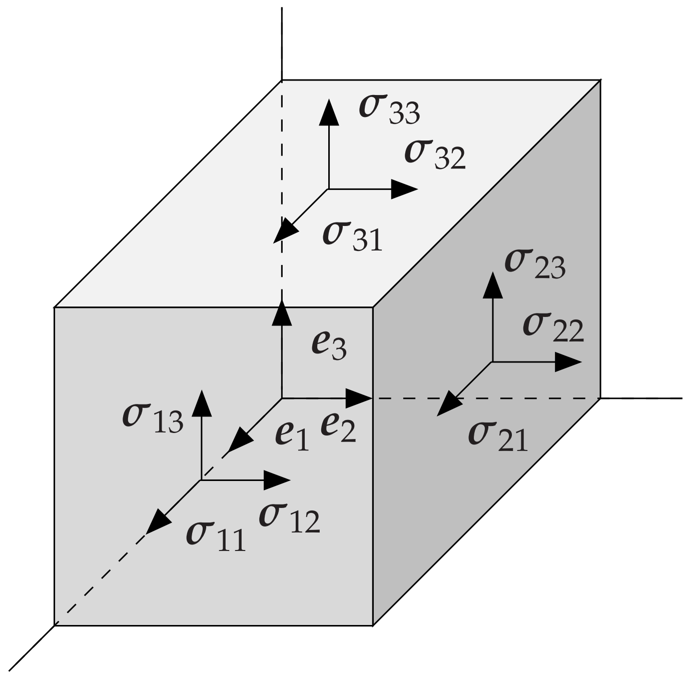
index representation

$$\boldsymbol{t}_\sigma = \sigma_{ji}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\cdot n_k\boldsymbol{e}_k = \sigma_{ji}n_k\delta_{jk}\boldsymbol{e}_i = \sigma_{ji}n_j\boldsymbol{e}_i = t_i\boldsymbol{e}_i \tag{3.1.18}$$

matrix representation of tensor coordinates of σ_{ij}

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{t}^{\mathrm{t}}_{\sigma 1} \\ \boldsymbol{t}^{\mathrm{t}}_{\sigma 2} \\ \boldsymbol{t}^{\mathrm{t}}_{\sigma 3} \end{bmatrix} \tag{3.1.19}$$

geometric interpretation



with traction vectors on surfaces

$$\begin{aligned} \boldsymbol{t}_{\sigma 1} &= [\ \sigma_{11}\ \ \sigma_{12}\ \ \sigma_{13}\ ]^{\mathrm{t}} \\ \boldsymbol{t}_{\sigma 2} &= [\ \sigma_{21}\ \ \sigma_{22}\ \ \sigma_{23}\ ]^{\mathrm{t}} \\ \boldsymbol{t}_{\sigma 3} &= [\ \sigma_{31}\ \ \sigma_{32}\ \ \sigma_{33}\ ]^{\mathrm{t}} \end{aligned} \tag{3.1.20}$$

first index ... surface normal
second index ... direction (coordinate of traction vector)

diagonal entries ... normal stresses
non–diagonal entries .. shear stresses