ME338A
CONTINUUM MECHANICS

lecture notes 06

thursday, january 21st, 2010

2.3.4 Volumetric–deviatoric decomposition

a material volume element can deform volumetrically and deviatorically, volumetric deformation conserves the shape (i.e. no changes in angles, no sliding) while deviatoric (isochoric) deformation conserves the volume

volumetric – deviatoric decomposition of strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\text{vol}} + \boldsymbol{\epsilon}^{\text{dev}} \tag{2.3.24}$$

with volumetric and deviatoric strain tensor $\boldsymbol{\epsilon}^{\text{vol}}$ and $\boldsymbol{\epsilon}^{\text{dev}}$

$$\text{tr}(\boldsymbol{\epsilon}^{\text{vol}}) = \text{tr}(\boldsymbol{\epsilon}) \qquad \text{tr}(\boldsymbol{\epsilon}^{\text{dev}}) = 0 \tag{2.3.25}$$

- volumetric second order tensor $\boldsymbol{\epsilon}^{\text{vol}}$

$$\boldsymbol{\epsilon}^{\text{vol}} = \frac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\, \boldsymbol{I} = \mathbb{I}^{\text{vol}} : \boldsymbol{\epsilon} \tag{2.3.26}$$

upon double contraction volumetric fourth order unit tensor $\mathbb{I}^{\text{vol}}$ extracts volumetric part $\boldsymbol{\epsilon}^{\text{vol}}$ of strain tensor

$$\begin{aligned} \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\, \boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\, \delta_{ij}\delta_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{2.3.27}$$

- deviatoric second order tensor $\boldsymbol{\epsilon}^{\text{dev}}$

$$\boldsymbol{\epsilon}^{\text{dev}} = \boldsymbol{\epsilon} - \frac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\, \boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{\epsilon} \tag{2.3.28}$$

upon double contraction deviatoric fourth order unit tensor $\mathbb{I}^{\text{dev}}$ extracts deviatoric part of strain tensor

$$\begin{aligned} \mathbb{I}^{\text{dev}} &= \mathbb{I}^{\text{sym}} - \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{sym}} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{dev}} &= \left[\tfrac{1}{2}\, \delta_{ik}\delta_{jl} + \tfrac{1}{2}\, \delta_{il}\delta_{jk} - \tfrac{1}{3}\, \delta_{ij}\delta_{kl}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{2.3.29}$$

Volumetric strain

volumetric deformation is characterized through the volume dilatation $e \in \mathcal{R}$, i.e. difference of deformed volume and original volume $\mathrm{d}v - \mathrm{d}V$ scaled by original volume $\mathrm{d}V$

$$e = \frac{\mathrm{d}v - \mathrm{d}V}{\mathrm{d}V} \begin{aligned} &= (1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33}) - 1 \\ &= \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + \mathcal{O}(\epsilon_{ij}^2) \end{aligned} \quad (2.3.30)$$

neglection of higher order terms: trace of strain tensor $\mathrm{tr}\,(\boldsymbol{\epsilon}) = \boldsymbol{\epsilon} : \boldsymbol{I} \in \mathcal{R}$ as characteristic measure for volume changes

$$e = \mathrm{div}\,\boldsymbol{u} = \nabla \boldsymbol{u} : \boldsymbol{I} = \boldsymbol{\epsilon} : \boldsymbol{I} = \mathrm{tr}\,(\boldsymbol{\epsilon}) \quad (2.3.31)$$

volumetric part $\boldsymbol{\epsilon}^{\mathrm{vol}}$ of strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon}^{\mathrm{vol}} = \tfrac{1}{3}\, e\, \boldsymbol{I} = \tfrac{1}{3}\, [\boldsymbol{\epsilon} : \boldsymbol{I}]\, \boldsymbol{I} = \tfrac{1}{3}\, [\boldsymbol{I} \otimes \boldsymbol{I}] : \boldsymbol{\epsilon} = \mathbb{I}^{\mathrm{vol}} : \boldsymbol{\epsilon} \quad (2.3.32)$$

index representation

$$\boldsymbol{\epsilon}^{\mathrm{vol}} = \epsilon_{ij}^{\mathrm{vol}}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad (2.3.33)$$

matrix representation of coordinates $[\epsilon_{ij}^{\mathrm{vol}}]$

$$[\epsilon_{ij}^{\mathrm{vol}}] = \frac{1}{3}\, e \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad e = \mathrm{tr}(\boldsymbol{\epsilon}) \quad (2.3.34)$$

• incompressibility is characterized through $\mathrm{div}\,\boldsymbol{u} = 0$

• volumetric strain tensor $\boldsymbol{\epsilon}^{\mathrm{vol}}$ is a spherical second order tensor as $\boldsymbol{\epsilon}^{\mathrm{vol}} = \frac{1}{3} e \boldsymbol{I}$

• volumetric strain tensor $\boldsymbol{\epsilon}^{\mathrm{vol}}$ contains the volume changing, shape preserving part of the total strain tensor $\boldsymbol{\epsilon}$

Deviatoric strain

deviatoric strain tensor $\boldsymbol{\epsilon}^{\text{dev}}$ preserves the volume and contains the remaining part of the total strain tensor $\boldsymbol{\epsilon}$

deviatoric part $\boldsymbol{\epsilon}^{\text{dev}}$ of the strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon}^{\text{dev}} = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{\text{vol}} = \boldsymbol{\epsilon} - \frac{1}{3}\left[\boldsymbol{\epsilon} : \boldsymbol{I}\right]\boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{\epsilon} \tag{2.3.35}$$

index representation

$$\boldsymbol{\epsilon}^{\text{dev}} = \epsilon^{\text{dev}}_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.36}$$

matrix representation of coordinates $[\epsilon^{\text{dev}}_{ij}]$

$$[\epsilon^{\text{dev}}_{ij}] = \begin{bmatrix} \frac{1}{3}[2\epsilon_{11} - \epsilon_{22} - \epsilon_{33}] & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \frac{1}{3}[2\epsilon_{22} - \epsilon_{11} - \epsilon_{33}] & \epsilon_{13} \\ \epsilon_{31} & \epsilon_{32} & \frac{1}{3}[2\epsilon_{33} - \epsilon_{11} - \epsilon_{22}] \end{bmatrix} \tag{2.3.37}$$

trace of deviatoric strains $\text{tr}\left(\boldsymbol{\epsilon}^{\text{dev}}\right)$

$$\begin{aligned} \text{tr}\left(\boldsymbol{\epsilon}^{\text{dev}}\right) &= \frac{1}{3}\left[2\epsilon_{11} - \epsilon_{22} - \epsilon_{33}\right] \\ &+ \frac{1}{3}\left[2\epsilon_{22} - \epsilon_{11} - \epsilon_{33}\right] \\ &+ \frac{1}{3}\left[2\epsilon_{33} - \epsilon_{11} - \epsilon_{22}\right] = 0 \end{aligned} \tag{2.3.38}$$

• deviatoric strain tensor $\boldsymbol{\epsilon}^{\text{dev}}$ is a traceless second order tensor as $\text{tr}\left(\boldsymbol{\epsilon}^{\text{dev}}\right) = 0$

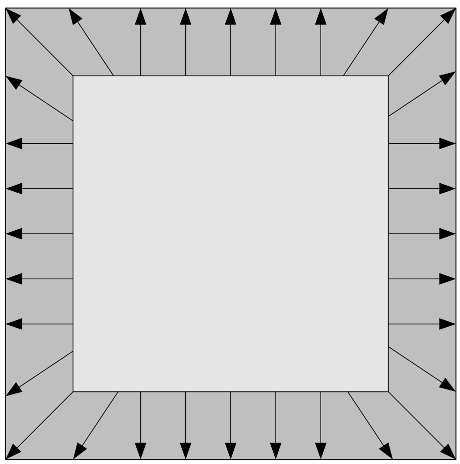
• deviatoric strain tensor $\boldsymbol{\epsilon}^{\text{dev}}$ contains the shape changing, volume preserving part of the total strain tensor $\boldsymbol{\epsilon}$

Volumetric–deviatoric decomposition

- examples of purely volumetric deformation

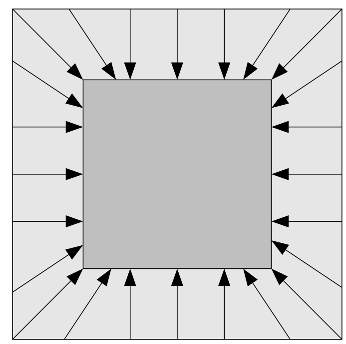
$$\boldsymbol{\epsilon}^{\mathrm{vol}} = \frac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\mathrm{vol}} : \boldsymbol{\epsilon} \qquad \mathrm{tr}(\boldsymbol{\epsilon}^{\mathrm{vol}}) = \mathrm{tr}(\boldsymbol{\epsilon}) \qquad (2.3.39)$$

expansion



$e > 0$ and $\boldsymbol{\epsilon}^{\mathrm{dev}} = \mathbf{0}$

compression

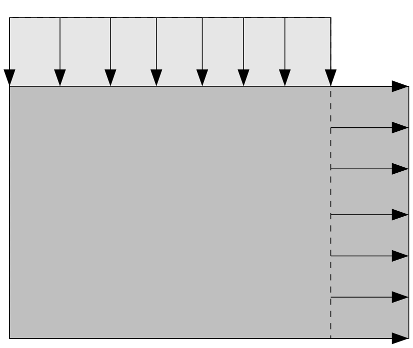


$e < 0$ and $\boldsymbol{\epsilon}^{\mathrm{dev}} = \mathbf{0}$

- examples of purely deviatoric deformation

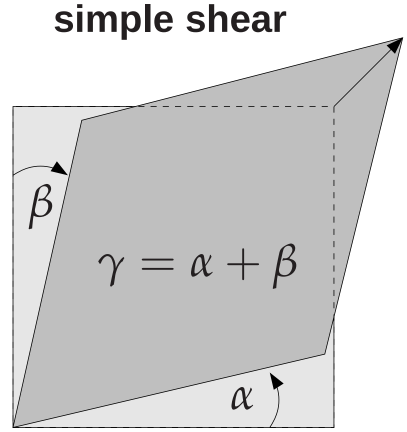
$$\boldsymbol{\epsilon}^{\mathrm{dev}} = \boldsymbol{\epsilon} - \frac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\mathrm{dev}} : \boldsymbol{\epsilon} \qquad \mathrm{tr}(\boldsymbol{\epsilon}^{\mathrm{dev}}) = 0 \qquad (2.3.40)$$

pure shear



$e = 0$ and $\boldsymbol{\epsilon}^{\mathrm{dev}} \neq \mathbf{0}$

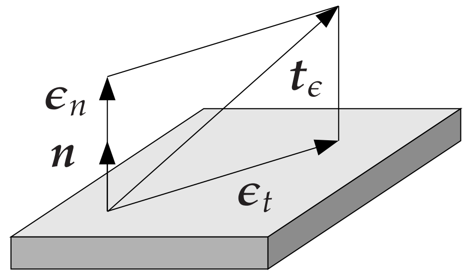
simple shear



$e = 0$ and $\boldsymbol{\epsilon}^{\mathrm{dev}} \neq \mathbf{0}$

2.3.5 Strain vector

assume we are interested in strain on a plane characterized through its normal $\boldsymbol{n}$, strain vector $\boldsymbol{t}_\epsilon$ acting on plane given through normal projection of strain tensor $\boldsymbol{\epsilon}$



$$\boldsymbol{t}_\epsilon = \boldsymbol{\epsilon} \cdot \boldsymbol{n} \tag{2.3.41}$$

index representation

$$\begin{aligned} \boldsymbol{t}_\epsilon &= (\epsilon_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (n_k\, \boldsymbol{e}_k) \\ &= \epsilon_{ij}\, n_k \delta_{jk}\, \boldsymbol{e}_i = \epsilon_{ij}\, n_j\, \boldsymbol{e}_i = t_{\epsilon i}\, \boldsymbol{e}_i \end{aligned} \tag{2.3.42}$$

representation of coordinates $[t_{\epsilon i}]$

$$\begin{bmatrix} t_{\epsilon 1} \\ t_{\epsilon 2} \\ t_{\epsilon 3} \end{bmatrix} = \begin{bmatrix} \epsilon_{11}\, n_1 & + & \epsilon_{12}\, n_2 & + & \epsilon_{13}\, n_3 \\ \epsilon_{21}\, n_1 & + & \epsilon_{22}\, n_2 & + & \epsilon_{23}\, n_3 \\ \epsilon_{31}\, n_1 & + & \epsilon_{32}\, n_2 & + & \epsilon_{33}\, n_3 \end{bmatrix} \tag{2.3.43}$$

alternative interpretation: assume we are interested in strains along a particular material direction, i.e. the stretch of a fiber at $\boldsymbol{x} \in \mathcal{B}$ characterized through its normal $\boldsymbol{n}$ with $||\boldsymbol{n}|| = 1$

stretch as change of displacement vector $\boldsymbol{u}$ in the direction of $\boldsymbol{n}$ given through the Gateaux derivative §**??**

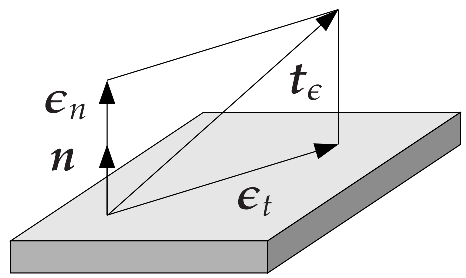
$$\begin{aligned} \mathrm{D}\,\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \boldsymbol{u}(\boldsymbol{x} + \epsilon\, \boldsymbol{n})|_{\epsilon=0} \\ &= \underbrace{\nabla \boldsymbol{u}(\boldsymbol{x} + \epsilon\, \boldsymbol{n})}_{\text{outer derviative}} \cdot \underbrace{\boldsymbol{n}}_{\text{inner derivative}} |_{\epsilon=0} = \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n} \end{aligned} \tag{2.3.44}$$

recall that $\nabla \boldsymbol{u} = \nabla^{\text{sym}} \boldsymbol{u} + \nabla^{\text{skw}} \boldsymbol{u} = \boldsymbol{\epsilon} + \boldsymbol{\omega}$ whereby rotation $\boldsymbol{\omega} = \nabla^{\text{skw}} \boldsymbol{u}$ does not induce strain, thus

$$\boldsymbol{t}_\epsilon = \nabla^{\text{sym}} \boldsymbol{u} \cdot \boldsymbol{n} = \epsilon \cdot \boldsymbol{n} \tag{2.3.45}$$

2.3.6 Normal–shear decomposition

assume we are interested in strain along a particular fiber characterized through its normal $\boldsymbol{n}$, stretch of fiber ϵ_n given through normal projection of strain vector $\boldsymbol{t}_\epsilon$



$$\epsilon_n = \boldsymbol{t}_\epsilon \cdot \boldsymbol{n} \tag{2.3.46}$$

alternative interpretation: stretch of a line element can be understood as the projection of change of displacement in the direction of $\boldsymbol{n}$ as $\mathrm{D}\boldsymbol{u} \cdot \boldsymbol{n} = \nabla \boldsymbol{u} \cdot \boldsymbol{n}$ onto the direction $\boldsymbol{n}$

$$\epsilon_n = \boldsymbol{n} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{n} = \boldsymbol{\epsilon} : [\boldsymbol{n} \otimes \boldsymbol{n}] \tag{2.3.47}$$

normal-shear (tangential) decomposition of strain vector $\boldsymbol{t}_\epsilon$

$$\boldsymbol{t}_\epsilon = \boldsymbol{\epsilon}_n + \boldsymbol{\epsilon}_t \tag{2.3.48}$$

normal strain vector – stretch of fibers in direction of n

$$\boldsymbol{\epsilon}_n = \boldsymbol{\epsilon} : [\boldsymbol{n} \otimes \boldsymbol{n}]\, \boldsymbol{n} \tag{2.3.49}$$

shear (tangential) strain vector – sliding of fibers parallel to $\boldsymbol{n}$

$$\boldsymbol{\epsilon}_t = \boldsymbol{t}_\epsilon - \boldsymbol{\epsilon}_n = \boldsymbol{\epsilon} : [\mathbb{I}^{\mathrm{sym}} \cdot \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] \tag{2.3.50}$$

amount of sliding γ_n

$$\gamma_n = 2\,||\boldsymbol{\epsilon}_t|| = 2\,\sqrt{\boldsymbol{\epsilon}_t \cdot \boldsymbol{\epsilon}_t} = 2\,\sqrt{\boldsymbol{t}_\epsilon \cdot \boldsymbol{t}_\epsilon - \epsilon_n^2} \tag{2.3.51}$$

in general, i.e. for an arbitrary direction $\boldsymbol{n}$, we have normal and shear contributions to the strain vector, however, three particular directions $\{\boldsymbol{n}_{\epsilon\, i}\}_{i=1,2,3}$ can be identified, for which $\boldsymbol{t}_\epsilon = \boldsymbol{\epsilon}_n$ and thus $\boldsymbol{\epsilon}_t = \boldsymbol{0}$, the corresponding $\{\boldsymbol{n}_{\epsilon\, i}\}_{i=1,2,3}$ are called principal strain directions and $\{\epsilon_{n\, i}\}_{i=1,2,3} = \{\lambda_{\epsilon i}\}_{i=1,2,3}$ are the principal strains or stretches

2.3.7 Principal strains – stretches

assume strain tensor $\boldsymbol{\epsilon}$ to be known at $\boldsymbol{x} \in \mathcal{B}$, principal strains $\{\lambda_{\epsilon\,i}\}_{i=1,2,3}$ and principal strain directions $\{\boldsymbol{n}_{\epsilon\,i}\}_{i=1,2,3}$ can be derived from solution of special eigenvalue problem according to §1.1.3

$$\boldsymbol{\epsilon} \cdot \boldsymbol{n}_{\epsilon\,i} = \lambda_{\epsilon\,i}\, \boldsymbol{n}_{\epsilon\,i} \qquad [\,\boldsymbol{\epsilon} - \lambda_{\epsilon\,i}\,] \cdot \boldsymbol{n}_{\epsilon\,i} = \boldsymbol{0} \tag{2.3.52}$$

solution

$$\det(\boldsymbol{\epsilon} - \lambda_\epsilon\, \boldsymbol{I}) = 0 \tag{2.3.53}$$

or in terms of roots of characteristic equation

$$\lambda_\epsilon^3 - I_\epsilon\, \lambda_\epsilon^2 + II_\epsilon\, \lambda_\epsilon - III_\epsilon = 0 \tag{2.3.54}$$

roots of characteristic equations in terms of principal invariants of $\boldsymbol{\epsilon}$

$$\begin{aligned} I_\epsilon &= \operatorname{tr}(\boldsymbol{\epsilon}) &&= \lambda_{\epsilon 1} + \lambda_{\epsilon 2} + \lambda_{\epsilon 3} \\ II_\epsilon &= \tfrac{1}{2}[\operatorname{tr}^2(\boldsymbol{\epsilon}) - \operatorname{tr}(\boldsymbol{\epsilon}^2)] &&= \lambda_{\epsilon 2}\lambda_{\epsilon 3} + \lambda_{\epsilon 3}\lambda_{\epsilon 1} + \lambda_{\epsilon 1}\lambda_{\epsilon 2} \\ III_\epsilon &= \det(\boldsymbol{\epsilon}) &&= \lambda_{\epsilon 1}\, \lambda_{\epsilon 2}\, \lambda_{\epsilon 3} \end{aligned} \tag{2.3.55}$$

spectral representation of $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \sum_{i=1}^{3} \lambda_{\epsilon i}\, \boldsymbol{n}_{\epsilon i} \otimes \boldsymbol{n}_{\epsilon i} \tag{2.3.56}$$

principal strains (stretches) $\lambda_{\epsilon i}$ are purely normal, no shear deformation (sliding) γ_n in principal directions, i.e. $\boldsymbol{t}_{\epsilon i} = \boldsymbol{\epsilon}_n = \lambda_{\epsilon i}\, \boldsymbol{n}_{\epsilon i}$ and $\boldsymbol{\epsilon}_t = \boldsymbol{0}$ thus $\gamma_n = 0$

due to symmetry of strains $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{t}}$, strain tensor possesses three real eigenvalues $\{\lambda_{\epsilon\,i}\}_{i=1,2,3}$, corresponding eigendirections $\{\boldsymbol{n}_{\epsilon\,i}\}_{i=1,2,3}$ are thus orthogonal $\boldsymbol{n}_{\epsilon i} \cdot \boldsymbol{n}_{\epsilon j} = \delta_{ij}$

2.3.8 Compatibility

until now, we have assumed the displacement field $\boldsymbol{u}(\boldsymbol{x}, t)$ to be given, such that the strain field $\boldsymbol{\epsilon} = \nabla^{\text{sym}} \boldsymbol{u}$ could have been derived uniquely as partial derivative of $\boldsymbol{u}$ with respect to the position $\boldsymbol{x}$ at fixed time t
assume now, that for a given strain field $\boldsymbol{\epsilon}(\boldsymbol{x}, t)$, we want to know whether these strains $\boldsymbol{\epsilon}$ are compatible with a continuous single–valued displacement field $\boldsymbol{u}$

symmetric second order incompatibility tensor

$$\boldsymbol{\eta} = \text{crl}(\text{crl}(\boldsymbol{\epsilon})) \tag{2.3.57}$$

index representation of incompatibility tensor

$$\boldsymbol{\eta} = \eta_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \overset{3}{e}_{ikm} \, \epsilon_{kn,ml} \, \overset{3}{e}_{jln} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{0} \tag{2.3.58}$$

coordinate representation of compatibility condition

$$\epsilon_{kl,mn} + \epsilon_{mn,kl} - \epsilon_{ml,kn} - \epsilon_{kn,ml} = 0 \tag{2.3.59}$$

valid $\forall \; k, l, m, n$, thus 81 equations which are partly redundant, six independent conditions

St. Venant compatibility conditions

$$\begin{array}{rcccccccl}
\eta_{11} & = & \epsilon_{22,33} & + & \epsilon_{33,22} & - & 2\,\epsilon_{23,32} & = & 0 \\
\eta_{22} & = & \epsilon_{33,11} & + & \epsilon_{11,33} & - & 2\,\epsilon_{31,13} & = & 0 \\
\eta_{33} & = & \epsilon_{11,22} & + & \epsilon_{22,11} & - & 2\,\epsilon_{12,21} & = & 0 \\
\eta_{12} & = & \epsilon_{13,32} & + & \epsilon_{23,31} & - & \epsilon_{33,12} - \epsilon_{12,33} & = & 0 \\
\eta_{23} & = & \epsilon_{21,13} & + & \epsilon_{31,12} & - & \epsilon_{11,23} - \epsilon_{23,11} & = & 0 \\
\eta_{31} & = & \epsilon_{32,21} & + & \epsilon_{12,23} & - & \epsilon_{22,31} - \epsilon_{31,22} & = & 0
\end{array} \tag{2.3.60}$$

incompatible displacement field, e.g. in dislocation theory

2.3.9 Special case of plane strain

dimensional reduction in case of plane strain with vanishing strains $\epsilon_{13} = \epsilon_{23} = \epsilon_{31} = \epsilon_{32} = \epsilon_{33} = 0$ in out of plane direction, e.g. in geomechanics

$$\boldsymbol{\epsilon} = \epsilon_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.61}$$

matrix representation of coordinates $[\epsilon_{ij}]$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.3.62}$$

2.3.10 Voigt representation of strain

three dimensional second order strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \epsilon_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.63}$$

matrix representation of coordinates $[\epsilon_{ij}]$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \tag{2.3.64}$$

due to symmetry $[\epsilon_{ij}] = [\epsilon_{ji}]$ and thus $\epsilon_{12} = \epsilon_{21}$, $\epsilon_{23} = \epsilon_{32}$, $\epsilon_{31} = \epsilon_{13}$, strain tensor $\boldsymbol{\epsilon}$ contains only six independent components $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{23}, \epsilon_{31}$,it proves convenient to represent second order tensor $\boldsymbol{\epsilon}$ through a vector $\underline{\epsilon}$

$$\underline{\epsilon} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\,\epsilon_{12}, 2\,\epsilon_{23}, 2\,\epsilon_{31}]^{\mathrm{t}} \tag{2.3.65}$$

vector representation $\underline{\epsilon}$ of strain $\boldsymbol{\epsilon}$ in case of plane strain

$$\underline{\epsilon} = [\epsilon_{11}, \epsilon_{22}, 2\,\epsilon_{12}]^{\mathrm{t}} \tag{2.3.66}$$