ME338A
CONTINUUM MECHANICS

lecture notes 05

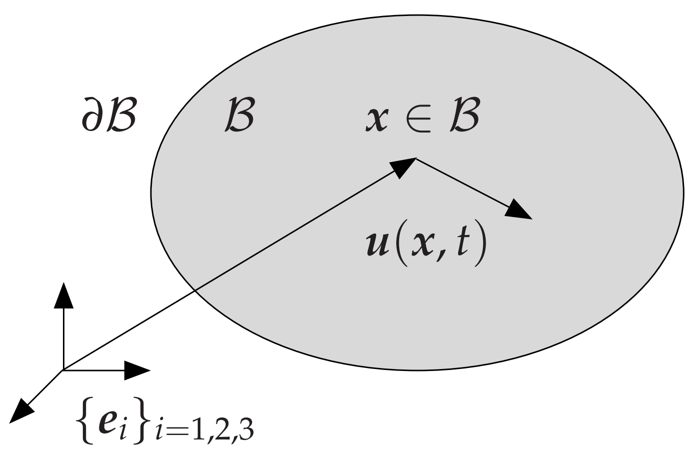
tuesday, january 19th, 2010

2 Kinematics

restriction to geometrically linear theory, valid if local strains remain small

2.1 Motion

consider a material body $\mathcal{B}$ as a simply connected subset of the Euclidian space $\mathcal{R}^3$ as $\mathcal{B} \subset \mathcal{R}^3$, with the boundary being denoted as $\partial\mathcal{B}$, a material point is defined as a point of the body $\boldsymbol{x} \in \mathcal{B}$



motion of a body $\mathcal{B} \subset \mathcal{R}^3$ characterized through time dependent vector field of displacements $\boldsymbol{u} \in \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{u} : \mathcal{B} \times \mathcal{R} \rightarrow \mathcal{R}^3 \qquad \boldsymbol{u}(\boldsymbol{x}, t) = u_i(\boldsymbol{x}, t)\, \boldsymbol{e}_i \tag{2.1.1}$$

2.2 Rates of kinematic quantities

2.2.1 Velocity

vector field of velocities $\boldsymbol{v} \in \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{v} : \mathcal{B} \times \mathcal{R} \rightarrow \mathcal{R}^3 \qquad \boldsymbol{v}(\boldsymbol{x}, t) = v_i(\boldsymbol{x}, t)\, \boldsymbol{e}_i \tag{2.2.1}$$

velocity field $\boldsymbol{v}$ defined through rate of change of displacement field $\boldsymbol{u}$

$$\boldsymbol{v}(\boldsymbol{x}, t) = \mathrm{D}_t \boldsymbol{u}(\boldsymbol{x}, t) = \left.\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x}\mathrm{fixed}} \tag{2.2.2}$$

velocity vector

$$\boldsymbol{v} = [v_1, v_2, v_3]^{\mathrm{t}} = \mathrm{D}_t [u_1, u_2, u_3]^{\mathrm{t}} \tag{2.2.3}$$

common notation in the literature $\boldsymbol{v} = \dot{\boldsymbol{u}}$

2.2.2 Acceleration

vector field of accelerations $\boldsymbol{a} \in \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{a} : \mathcal{B} \times \mathcal{R} \rightarrow \mathcal{R}^3 \qquad \boldsymbol{a}(\boldsymbol{x}, t) = a_i(\boldsymbol{x}, t)\, \boldsymbol{e}_i \tag{2.2.4}$$

acceleration field $\boldsymbol{a}$ defined through rate of change of velocity field $\boldsymbol{v}$

$$\boldsymbol{a}(\boldsymbol{x}, t) = \mathrm{D}_t \boldsymbol{v}(\boldsymbol{x}, t) = \left.\frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x}\mathrm{fixed}} = \left.\frac{\partial^2 \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t^2}\right|_{\boldsymbol{x}\mathrm{fixed}} \tag{2.2.5}$$

acceleration vector

$$\boldsymbol{a} = [a_1, a_2, a_3]^{\mathrm{t}} = \mathrm{D}_t [v_1, v_2, v_3]^{\mathrm{t}} = \mathrm{D}_t^2 [u_1, u_2, u_3]^{\mathrm{t}} \tag{2.2.6}$$

common notation in the literature $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{u}}$

2.3 Gradients of kinematic quantities

2.3.1 Displacement gradient

second order tensor field of displacement gradient $\boldsymbol{H} \in \mathcal{R}^3 \otimes \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{H} : \mathcal{B} \times \mathcal{R} \rightarrow \mathcal{R}^3 \otimes \mathcal{R}^3 \quad \boldsymbol{H}(\boldsymbol{x}, t) = H_{ij}(\boldsymbol{x}, t)\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.1}$$

displacement gradient $\boldsymbol{H}$ defined through gradient of displacement field $\boldsymbol{u}$

$$\boldsymbol{H}(\boldsymbol{x}, t) = \nabla \boldsymbol{u}(\boldsymbol{x}, t) = \left. \frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \right|_{t \text{fixed}} \tag{2.3.2}$$

index representation

$$H_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{\partial u_i}{\partial X_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = u_{i,j}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.3}$$

matrix representation of coordinates H_{ij}

$$[H_{ij}] = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \tag{2.3.4}$$

- displacement gradient $\boldsymbol{H} = \nabla \boldsymbol{u}$ is non–symmetric, $\boldsymbol{H} \neq \boldsymbol{H}^{\mathrm{t}}$
- displacement gradient $\boldsymbol{H} = \nabla \boldsymbol{u}$ does not vanish for pointwise rigid body motion

Symmetric-skew-symmetric decomposition

symmetric–skew-symmetric decomposition of displacement gradient $\boldsymbol{H} = \nabla \boldsymbol{u}$

$$\boldsymbol{H} = \frac{1}{2}[\boldsymbol{H} + \boldsymbol{H}^{\mathrm{t}}] + \frac{1}{2}[\boldsymbol{H} - \boldsymbol{H}^{\mathrm{t}}] = \boldsymbol{\epsilon} + \boldsymbol{\omega} \tag{2.3.5}$$

with symmetric and skew-symmetric second order tensor $\boldsymbol{\epsilon} = \boldsymbol{H}^{\mathrm{sym}}$ and $\boldsymbol{\omega} = \boldsymbol{H}^{\mathrm{skw}}$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{t}} \qquad \boldsymbol{\omega} = -\boldsymbol{\omega}^{\mathrm{t}} \tag{2.3.6}$$

• geometrically linear strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \frac{1}{2}[\boldsymbol{H} + \boldsymbol{H}^{\mathrm{t}}] = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla^{\mathrm{t}} \boldsymbol{u}] = \nabla^{\mathrm{sym}} \boldsymbol{u} = \mathbb{I}^{\mathrm{sym}} : \nabla \boldsymbol{u} \tag{2.3.7}$$

upon double contraction symmetric fourth order unit tensor $\mathbb{I}^{\mathrm{sym}}$ extracts symmetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\mathrm{sym}} &= \tfrac{1}{2}\,[\mathbb{I} + \mathbb{I}^{\mathrm{t}}] \\ \mathbb{I}^{\mathrm{sym}} &= \tfrac{1}{2}\,[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{2.3.8}$$

• geometrically linear rotation tensor $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \frac{1}{2}[\boldsymbol{H} - \boldsymbol{H}^{\mathrm{t}}] = \frac{1}{2}[\nabla \boldsymbol{u} - \nabla^{\mathrm{t}} \boldsymbol{u}] = \nabla^{\mathrm{skw}} \boldsymbol{u} = \mathbb{I}^{\mathrm{skw}} : \nabla \boldsymbol{u} \tag{2.3.9}$$

upon double contraction skew-symmetric fourth order unit tensor $\mathbb{I}^{\mathrm{skw}}$ extracts skew-symmetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\mathrm{skw}} &= \tfrac{1}{2}\,[\mathbb{I} - \mathbb{I}^{\mathrm{t}}] \\ \mathbb{I}^{\mathrm{skw}} &= \tfrac{1}{2}\,[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{2.3.10}$$

2.3.2 Strain

second order tensor field of (geometrically linear) strains $\boldsymbol{\epsilon} \in \mathcal{R}^3 \otimes \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{\epsilon} : \mathcal{B} \times \mathcal{R} \to \mathcal{R}^3 \otimes \mathcal{R}^3 \quad \boldsymbol{\epsilon}(\boldsymbol{x}, t) = \epsilon_{ij}(\boldsymbol{x}, t)\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.11}$$

strain tensor $\boldsymbol{\epsilon}$ defined through symmetric part of gradient of displacement field $\boldsymbol{u}$

$$\boldsymbol{\epsilon}(\boldsymbol{x}, t) = \nabla^{\text{sym}} \boldsymbol{u}(\boldsymbol{x}, t) = \left[\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \right]^{\text{sym}} \tag{2.3.12}$$

index representation

$$\epsilon_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{1}{2} [u_{i,j} + u_{j,i}] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.13}$$

matrix representation of coordinates ϵ_{ij}

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2u_{1,1} & u_{1,2} + u_{2,1} & u_{1,3} + u_{3,1} \\ u_{2,1} + u_{1,2} & 2u_{2,2} & u_{2,3} + u_{3,2} \\ u_{3,1} + u_{1,3} & u_{3,2} + u_{2,3} & 2u_{3,3} \end{bmatrix} \tag{2.3.14}$$

strain tensor $\boldsymbol{\epsilon} = \nabla^{\text{sym}} \boldsymbol{u}$ is symmetric, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\text{t}}$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{21} & \epsilon_{31} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{32} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} = \epsilon_{ji} \tag{2.3.15}$$

ϵ_{ij} for $i = j$ ... diagonal entries: normal strain
ϵ_{ij} for $i \neq j$ ... off–diagonal entries: shear strain

2.3.3 Rotation

second order tensor field of (geometrically linear) rotation $\boldsymbol{\omega} \in \mathcal{R}^3 \otimes \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{\omega} : \mathcal{B} \times \mathcal{R} \rightarrow \mathcal{R}^3 \otimes \mathcal{R}^3 \quad \boldsymbol{\omega}(\boldsymbol{x}, t) = \omega_{ij}(\boldsymbol{x}, t)\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.16}$$

rotation tensor $\boldsymbol{\omega}$ defined through skew–symmetric part of gradient of displacement field $\boldsymbol{u}$

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \nabla^{\text{skw}} \boldsymbol{u}(\boldsymbol{x}, t) = \left[\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \right]^{\text{skw}} \tag{2.3.17}$$

index representation

$$\omega_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{1}{2} [u_{i,j} - u_{j,i}] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.18}$$

matrix representation of coordinates ω_{ij}

$$[\omega_{ij}] = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & u_{1,2} - u_{2,1} & u_{1,3} - u_{3,1} \\ u_{2,1} - u_{1,2} & 0 & u_{2,3} - u_{3,2} \\ u_{3,1} - u_{1,3} & u_{3,2} - u_{2,3} & 0 \end{bmatrix} \tag{2.3.19}$$

rotation tensor $\boldsymbol{\omega} = \nabla^{\text{skw}} \boldsymbol{u}$ is skew-symmetric, $\boldsymbol{\omega} = -\boldsymbol{\omega}^{\text{t}}$

$$[\omega_{ij}] = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix} = - \begin{bmatrix} 0 & \omega_{21} & \omega_{31} \\ \omega_{12} & 0 & \omega_{32} \\ \omega_{13} & \omega_{23} & 0 \end{bmatrix} = -\omega_{ji} \tag{2.3.20}$$

corresponding axial vector $\boldsymbol{w} = -1/2 \stackrel{3}{\boldsymbol{e}} : \boldsymbol{\omega}$

$$\boldsymbol{w} = [w_1, w_2, w_3]^{\text{t}} = -[\omega_{23}, \omega_{31}, \omega_{12}]^{\text{t}} \tag{2.3.21}$$

Symmetric-skew-symmetric decomposition

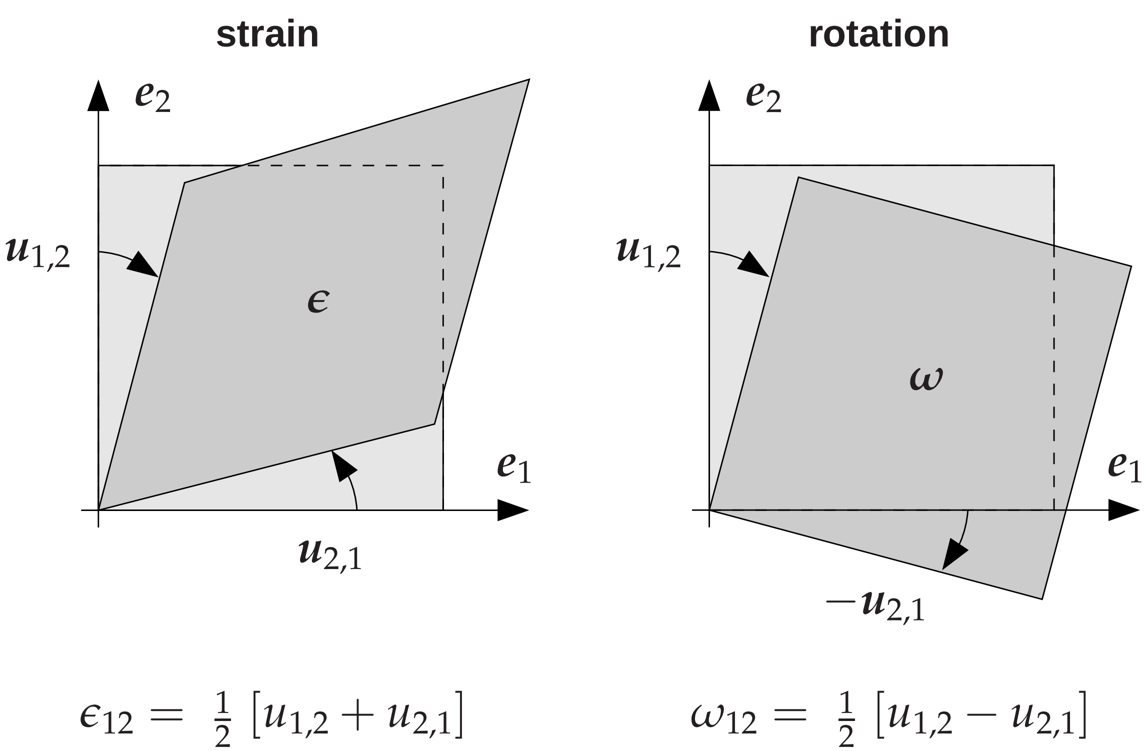
symmetric–skew-symmetric decomposition of displacement gradient $\boldsymbol{H} = \nabla \boldsymbol{u}$

$$\boldsymbol{H} = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla^{\mathrm{t}} \boldsymbol{u}] + \frac{1}{2}[\nabla \boldsymbol{u} - \nabla^{\mathrm{t}} \boldsymbol{u}] = \boldsymbol{\epsilon} + \boldsymbol{\omega} \qquad (2.3.22)$$

with symmetric and skew-symmetric second order tensor $\boldsymbol{\epsilon} = \nabla^{\mathrm{sym}} \boldsymbol{u}$ and $\boldsymbol{\omega} = \nabla^{\mathrm{skw}} \boldsymbol{u}$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{t}} \qquad \boldsymbol{\omega} = -\boldsymbol{\omega}^{\mathrm{t}} \qquad (2.3.23)$$

geometric interpretation:
representation of symmetric and skew–symmetric part of displacement gradient for two-dimensional case



symmetric part $\boldsymbol{\epsilon} = \nabla^{\mathrm{sym}} \boldsymbol{u}$ represents strain while skew-symmetric $\boldsymbol{\omega} = \nabla^{\mathrm{skw}} \boldsymbol{u}$ part represents rotation