ME338A
CONTINUUM MECHANICS

lecture notes 04

thursday, january 14th, 2010

1.2 Tensor analysis

1.2.1 Derivatives

consider smooth, differentiable scalar field Φ with

- scalar argument $\Phi : \quad \mathcal{R} \to \mathcal{R}; \quad \Phi(x) = \alpha$
- vectorial argument $\Phi : \quad \mathcal{R}^3 \to \mathcal{R}; \quad \Phi(\boldsymbol{x}) = \alpha$
- tensorial argument $\Phi : \quad \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}; \quad \Phi(\boldsymbol{X}) = \alpha$

Frechet derivative

$$
\begin{aligned}
&\bullet \text{ scalar} && \mathrm{D}\,\Phi(x) = \frac{\partial \Phi(x)}{\partial x} = \partial_x \Phi(x)\\
&\bullet \text{ vectorial} && \mathrm{D}\,\Phi(\boldsymbol{x}) = \frac{\partial \Phi(\boldsymbol{x})}{\partial \boldsymbol{x}} = \partial_{\boldsymbol{x}} \Phi(\boldsymbol{x})\\
&\bullet \text{ tensorial} && \mathrm{D}\,\Phi(\boldsymbol{X}) = \frac{\partial \Phi(\boldsymbol{X})}{\partial \boldsymbol{X}} = \partial_{\boldsymbol{X}} \Phi(\boldsymbol{X})
\end{aligned}
\tag{1.2.1}
$$

Gateaux derivative

Gateaux derivative as particular Frechet derivative with respect to directions u, $\boldsymbol{u}$ and $\boldsymbol{U}$

$$
\begin{aligned}
&\bullet \text{ scalar} && \mathrm{D}\,\Phi(x)\;\; u = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(x + \epsilon\, u)\,|_{\epsilon=0} \quad \forall\, u \in \mathcal{R}\\
&\bullet \text{ vectorial} && \mathrm{D}\,\Phi(\boldsymbol{x}) \cdot \boldsymbol{u} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(\boldsymbol{x} + \epsilon\, \boldsymbol{u})\,|_{\epsilon=0} \quad \forall\, \boldsymbol{u} \in \mathcal{R}^3\\
&\bullet \text{ tensorial} && \mathrm{D}\,\Phi(\boldsymbol{X}) : \boldsymbol{U} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(\boldsymbol{X} + \epsilon\, \boldsymbol{U})|_{\epsilon=0} \quad \forall\, \boldsymbol{U} \in \mathcal{R}^3 \otimes \mathcal{R}^3
\end{aligned}
\tag{1.2.2}
$$

in what follows in particular vectorial arguments, e.g., point position $\boldsymbol{x}$ or displacement $\boldsymbol{u}$

the Gateaux derivative is a simple mechanism to determine derivatives with respect to vectors or tensor, by rewriting them as derivatives with respect to a scalar ϵ

example: derivatives of invariants

use the Gateaux derivative

$$\mathrm{D}\Phi(\boldsymbol{A}) : \Delta\boldsymbol{A} = \frac{\partial\Phi(\boldsymbol{A})}{\partial\boldsymbol{A}} : \Delta\boldsymbol{A} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0}$$

to determine the derivatives of the three invariants I_A, II_A, III_A of the second order tensor $\boldsymbol{A}$ with respect to $\boldsymbol{A}$ itself!

in words: the Gateaux derivative D of a scalar field Φ (in this case the invariant I_A, II_A, III_A) along a given direction $\boldsymbol{U}$ (in this case $\Delta\boldsymbol{A}$) is the derivative of the field Φ at position $\boldsymbol{X}$ (in this case $\boldsymbol{A}$) "perturbed" by the direction $\epsilon\boldsymbol{U}$ (in this case $\epsilon\,\Delta\boldsymbol{A}$) with respect to ϵ evaluated at $\epsilon = 0$

setting $\epsilon = 0$ filters out the linear terms, i.e., all higher order terms are set to zero, the Gateaux derivative is therefore also referred to as linearization, it corresponds to the first term of a Taylor series

derivative of first invariant $I_A = \mathrm{tr}\,(\boldsymbol{A}) = \boldsymbol{A} : \boldsymbol{I}$

$$
\begin{aligned}
\mathrm{D}I_A(\boldsymbol{A}) : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\mathrm{tr}(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}|_{\epsilon=0} \\
&= \quad \Delta\boldsymbol{A} : \boldsymbol{I}|_{\epsilon=0} = \boldsymbol{I} : \Delta\boldsymbol{A}
\end{aligned}
$$

$$
\mathrm{D}I_A(\boldsymbol{A}) = \frac{\partial I_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \underline{\underline{\boldsymbol{I}}}
$$

remark: I_A is linear in $\boldsymbol{A}$, its Gateaux derivative is constant, there are no terms in ϵ once we take the derivative $\mathrm{d}/\mathrm{d}\epsilon$

derivative of second invariant $II_A = \frac{1}{2}\left[[\boldsymbol{A} : \boldsymbol{I}]^2 + \boldsymbol{A} : \boldsymbol{A}^{\mathrm{t}}\right]$

$$
\begin{aligned}
\mathrm{D}II_A(\boldsymbol{A}) : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\tfrac{1}{2}\mathrm{tr}^2(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}) - \tfrac{1}{2}\mathrm{tr}(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})^2|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\tfrac{1}{2}[[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}]^2 \\
&- \quad \tfrac{1}{2}[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : [\boldsymbol{A}^{\mathrm{t}} + \epsilon\Delta\boldsymbol{A}^{\mathrm{t}}]|_{\epsilon=0} \\
&= \quad [\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}]\Delta\boldsymbol{A} : \boldsymbol{I} \\
&- \quad \tfrac{1}{2}\Delta\boldsymbol{A} : \boldsymbol{A}^{\mathrm{t}} - \tfrac{1}{2}\boldsymbol{A} : \Delta\boldsymbol{A}^{\mathrm{t}} - \epsilon\Delta\boldsymbol{A} : \Delta\boldsymbol{A}^{\mathrm{t}}|_{\epsilon=0} \\
&= \quad [\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} - \boldsymbol{A}^{\mathrm{t}}] : \Delta\boldsymbol{A}
\end{aligned}
$$

$$
\mathrm{D}II_A(\boldsymbol{A}) = \frac{\partial II_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \underline{\underline{\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} - \boldsymbol{A}^{\mathrm{t}}}}
$$

here, we have used the following identity

$$
\mathrm{tr}(\boldsymbol{A}^2) = (\boldsymbol{A} \cdot \boldsymbol{A}) : \boldsymbol{I} = \boldsymbol{A} : \boldsymbol{A}^{\mathrm{t}}
$$

remark: II_A is quadratic in $\boldsymbol{A}$, its Gateaux derivative is linear, once we took the derivative $\mathrm{d}/\mathrm{d}\epsilon$, the higher order term in ϵ is filtered out by setting $\epsilon = 0$

derivative of third invariant $III_A = \det(\boldsymbol{A})$

$$\begin{aligned}
\mathrm{D}III_A(\boldsymbol{A}) : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A}\cdot[\boldsymbol{I} + \boldsymbol{A}^{-1}\cdot\epsilon\Delta\boldsymbol{A}])|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A})\cdot\det(\epsilon\boldsymbol{A}^{-1}\cdot\Delta\boldsymbol{A} + \boldsymbol{I})|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A})\cdot[(\epsilon\lambda_{A^{-1}\cdot\Delta A1} + 1) \\
&\qquad (\epsilon\lambda_{A^{-1}\cdot\Delta A2} + 1)(\epsilon\lambda_{A^{-1}\cdot\Delta A3} + 1)]\,|_{\epsilon=0} \\
&= \det(\boldsymbol{A})\cdot[\lambda_{A^{-1}\cdot\Delta A1} + \lambda_{A^{-1}\cdot\Delta A2} + \lambda_{A^{-1}\cdot\Delta A3}] \\
&= \det(\boldsymbol{A})\cdot\mathrm{tr}(\boldsymbol{A}^{-1}\cdot\Delta\boldsymbol{A}) \\
&= \det(\boldsymbol{A})\cdot\boldsymbol{A}^{-\mathrm{t}} : \Delta\boldsymbol{A}
\end{aligned}$$

$$\mathrm{D}III_A(\boldsymbol{A}) = \frac{\partial III_A(\boldsymbol{A})}{\partial\boldsymbol{A}} = \underline{\underline{\det(\boldsymbol{A})\cdot\boldsymbol{A}^{-\mathrm{t}}}}$$

here, we have used the following expression for the determinant of $(\epsilon\boldsymbol{A}^{-1}\cdot\Delta\boldsymbol{A})$ expressed through the characteristic polynom for the eigenvalue $\lambda = -1$

$$\begin{aligned}
\det(\epsilon\boldsymbol{A}^{-1}\cdot\Delta\boldsymbol{A} - \lambda\boldsymbol{I}) &= (\epsilon\lambda_{A^{-1}\cdot\Delta A1} - \lambda) \\
&\qquad (\epsilon\lambda_{A^{-1}\cdot\Delta A2} - \lambda)(\epsilon\lambda_{A^{-1}\cdot\Delta A3} - \lambda)
\end{aligned}$$

reformulation with the help of index notation

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{A}^{-1}\cdot\Delta\boldsymbol{A}) &= (\boldsymbol{A}^{-1}\cdot\Delta\boldsymbol{A}) : \boldsymbol{I} \\
&= (A^{-1}_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot(\Delta A_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l) : (\delta_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n) \\
&= (A^{-1}_{ij}\Delta A_{jl}\boldsymbol{e}_i\otimes\boldsymbol{e}_l) : (\delta_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n) \\
&= A^{-1}_{ij}\Delta A_{jl}\delta_{im}\delta_{ln}\delta_{mn} = \underline{\underline{\boldsymbol{A}^{-\mathrm{t}} : \Delta\boldsymbol{A}}}
\end{aligned}$$

remark: III_A is cubic in $\boldsymbol{A}$

derivative of function with tensorial argument show that the derivative of the function $\Phi(\boldsymbol{A})$ with respect to its tensorial arguement $\boldsymbol{A}$ is given through the component-wise derivative with respect to the individual tensorial entries A_{ij}

index representation of second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$\begin{aligned} \boldsymbol{A} &= A_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + A_{13}\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\ &+ A_{21}\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + A_{22}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ &+ A_{31}\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \end{aligned}$$

extraction of individual components A_{ij} through "projection" onto the base vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$

$$A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A} \cdot \boldsymbol{e}_j = \boldsymbol{e}_i \cdot (A_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l) \cdot \boldsymbol{e}_j = \delta_{ik}\, A_{kl}\, \delta_{lj} = A_{ij}$$

and thus

$$\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}} = \frac{\partial \Phi}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial \boldsymbol{A}} = \frac{\partial \Phi}{\partial A_{ij}} \frac{\partial \boldsymbol{e}_i \cdot \boldsymbol{A} \cdot \boldsymbol{e}_j}{\partial \boldsymbol{A}} = \frac{\partial \Phi}{\partial A_{ij}}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$\begin{aligned} \frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}} &= \frac{\partial \Phi}{\partial A_{11}}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \frac{\partial \Phi}{\partial A_{12}}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \frac{\partial \Phi}{\partial A_{13}}\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\ &+ \frac{\partial \Phi}{\partial A_{21}}\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \frac{\partial \Phi}{\partial A_{22}}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \frac{\partial \Phi}{\partial A_{23}}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ &+ \frac{\partial \Phi}{\partial A_{31}}\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + \frac{\partial \Phi}{\partial A_{32}}\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + \frac{\partial \Phi}{\partial A_{33}}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \end{aligned}$$

remark: derivatives of functions with respect to tensors can be derived individually for each component

derivative of first invariant I_A wrt A use the component-wise derivation to determine the derivative of the first invariant I_A with respect to its tensor and validate the result derived previously

$$I_A = \mathrm{tr}\boldsymbol{A} = A_{11} + A_{22} + A_{33}$$

$$\frac{\partial I_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \left[\frac{\partial I_A}{\partial A_{ij}}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$\left[\frac{\partial I_A}{\partial A_{ij}}\right] = \begin{bmatrix} \frac{\partial I_A}{\partial A_{11}} & \frac{\partial I_A}{\partial A_{12}} & \frac{\partial I_A}{\partial A_{13}} \\ \frac{\partial I_A}{\partial A_{21}} & \frac{\partial I_A}{\partial A_{22}} & \frac{\partial I_A}{\partial A_{23}} \\ \frac{\partial I_A}{\partial A_{31}} & \frac{\partial I_A}{\partial A_{32}} & \frac{\partial I_A}{\partial A_{33}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{\frac{\partial I_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \boldsymbol{I}}}$$

remark: derivatives of functions with respect to tensors can be derived individually for each component

derivative of A wrt itself

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} = \left[\frac{\partial A_{ij}}{\partial A_{kl}}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \left[\frac{\partial A_{ij}}{\partial A_{kl}}\right] = \delta_{ik}\delta_{jl}$$

$$\underline{\underline{\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} = \boldsymbol{I}\overline{\otimes}\boldsymbol{I} = \mathbb{I}}}$$

remark: always keep in mind that the derivative of $\boldsymbol{A}$ with respect to itself is not(!) 1 but the fourth order unity tensor $\mathbb{I}$

derivative of inverse A^{-1} wrt A determine the derivative of the inverse of $\boldsymbol{A}$ wrt $\boldsymbol{A}$ by using the definition of the inverse $\boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$

$$\frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} = \left[\frac{\partial A_{ij}^{-1}}{\partial A_{kl}} \right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

derivative of identity with respect to $\boldsymbol{A}$

$$\frac{\partial \delta_{im}}{\partial A_{kl}} = \frac{\partial A_{ij}^{-1} A_{jm}}{\partial A_{kl}} = \frac{\partial A_{ij}^{-1}}{\partial A_{kl}} A_{jm} + A_{ij}^{-1} \frac{\partial A_{jm}}{\partial A_{kl}} = \mathbb{O}$$

and thus

$$\left[\frac{\partial A_{ij}^{-1}}{\partial A_{kl}} \right] = -A_{ij}^{-1} \frac{\partial A_{jm}}{\partial A_{kl}} A_{mj}^{-1} = -A_{ij}^{-1} \delta_{jk} \delta_{ml} A_{jm}^{-\mathrm{t}} = -A_{ik}^{-1} \, A_{jl}^{-\mathrm{t}}$$

$$\underline{\underline{\frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} = -\boldsymbol{A}^{-1} \overline{\otimes} \boldsymbol{A}^{-\mathrm{t}}}}$$

remark: the derivative of the inverse of a tensor wrt the tensor itself is negative. it can be determined with a trick by using the definition of the inverse.

1.2.2 Gradient

consider vector valued scalar and vector field $f(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$f : \mathcal{B} \to \mathcal{R} \qquad f : \boldsymbol{x} \to f\;(\boldsymbol{x})$$
$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f}\;(\boldsymbol{x})$$

Gradient of a scalar field

gradient $\nabla f(\boldsymbol{x})$ of vector valued scalar field $f(\boldsymbol{x})$

$$\nabla f(\boldsymbol{x}) = \frac{\partial f(\boldsymbol{x})}{\partial x_i} = f_{,i}(\boldsymbol{x})\;\boldsymbol{e}_i \tag{1.2.3}$$

and thus

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} f_{,1} \\ f_{,2} \\ f_{,3} \end{bmatrix} \tag{1.2.4}$$

gradient of scalar field renders a vector field

Gradient of a vector field

gradient $\nabla \boldsymbol{f}(\boldsymbol{x})$ of vector valued vector field $\boldsymbol{f}(\boldsymbol{x})$

$$\nabla \boldsymbol{f}(\boldsymbol{x}) = \frac{\partial f_i(\boldsymbol{x})}{\partial x_j} = f_{i,j}(\boldsymbol{x})\;\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.2.5}$$

and thus

$$\nabla \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix} \tag{1.2.6}$$

gradient of vector field renders a (second order) tensor field

1.2.3 Divergence

consider vector valued vector and tensor field $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{F}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$\begin{aligned} \boldsymbol{f} &: \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{f} &: \boldsymbol{x} \to \boldsymbol{f}\ (\boldsymbol{x}) \\ \boldsymbol{F} &: \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{F} &: \boldsymbol{x} \to \boldsymbol{F}\ (\boldsymbol{x}) \end{aligned}$$

Divergence of a vector field

divergence $\nabla \boldsymbol{f}\,(\boldsymbol{x})$ of vector valued vector field $\boldsymbol{f}\,(\boldsymbol{x})$

$$\operatorname{div}(\boldsymbol{f}\,(\boldsymbol{x})) = \operatorname{tr}(\nabla \boldsymbol{f}\,(\boldsymbol{x})) = \nabla \boldsymbol{f}\,(\boldsymbol{x}) : \boldsymbol{I} \tag{1.2.7}$$

with $\nabla \boldsymbol{f}\,(\boldsymbol{x}) = f_{i,j}(\boldsymbol{x})\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j$

$$\operatorname{div}(\boldsymbol{f}\,(\boldsymbol{x})) = f_{i,i}(\boldsymbol{x}) = f_{1,1} + f_{2,2} + f_{3,3} \tag{1.2.8}$$

divergence of a vector field renders a scalar field

Divergence of a tensor field

divergence $\nabla \boldsymbol{F}\,(\boldsymbol{x})$ of vector valued tensor field $\boldsymbol{F}\,(\boldsymbol{x})$

$$\operatorname{div}(\boldsymbol{F}\,(\boldsymbol{x})) = \operatorname{tr}(\nabla \boldsymbol{F}\,(\boldsymbol{x})) = \nabla \boldsymbol{F}\,(\boldsymbol{x}) : \boldsymbol{I} \tag{1.2.9}$$

with $\nabla \boldsymbol{F}\,(\boldsymbol{x}) = F_{ij,k}(\boldsymbol{x})\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$

$$\operatorname{div}(\boldsymbol{F}\,(\boldsymbol{x})) = F_{ij,j}(\boldsymbol{x}) = \begin{bmatrix} F_{11,1} + F_{12,2} + F_{13,3} \\ F_{21,1} + F_{22,2} + F_{23,3} \\ F_{31,1} + F_{32,2} + F_{33,3} \end{bmatrix} \tag{1.2.10}$$

divergence of a second order tensor field renders a vector field

1.2.4 Laplace operator

consider vector valued scalar and vector field $f(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$f : \mathcal{B} \to \mathcal{R} \qquad f : \boldsymbol{x} \to f\ (\boldsymbol{x})$$
$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f}\ (\boldsymbol{x})$$

Laplace operator acting on scalar field

Laplace operator $\Delta f(\boldsymbol{x})$ acting on vector valued scalar field $f(\boldsymbol{x})$

$$\Delta f\ (\boldsymbol{x}) = \mathrm{div}(\nabla(f\ (\boldsymbol{x}))) \tag{1.2.11}$$

and thus

$$\Delta f\ (\boldsymbol{x}) = f_{,ii} = f_{,11} + f_{,22} + f_{,33} \tag{1.2.12}$$

Laplace operator acting on scalar field renders a scalar field

Laplace operator acting on vector field

Laplace operator $\Delta \boldsymbol{f}\ (\boldsymbol{x})$ acting on vector valued vector field $\boldsymbol{f}\ (\boldsymbol{x})$

$$\Delta \boldsymbol{f}\ (\boldsymbol{x}) = \mathrm{div}(\nabla(\boldsymbol{f}\ (\boldsymbol{x}))) \tag{1.2.13}$$

and thus

$$\Delta \boldsymbol{f}\ (\boldsymbol{x}) = f_{i,jj} = \begin{bmatrix} f_{1,11} + f_{1,22} + f_{1,33} \\ f_{2,11} + f_{2,22} + f_{2,33} \\ f_{3,11} + f_{3,22} + f_{3,33} \end{bmatrix} \tag{1.2.14}$$

Laplace operator acting on vector field renders a vector field

Useful transformations

consider scalar, vector and second order tensor field $\alpha(\boldsymbol{x})$, $\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{x})$ and $\boldsymbol{A}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$
\begin{aligned}
\alpha &: \mathcal{B} \to \mathcal{R} & \alpha &: \boldsymbol{x} \to \alpha\ (\boldsymbol{x}) \\
\boldsymbol{u} &: \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{u} &: \boldsymbol{x} \to \boldsymbol{u}\ (\boldsymbol{x}) \\
\boldsymbol{v} &: \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{v} &: \boldsymbol{x} \to \boldsymbol{v}\ (\boldsymbol{x}) \\
\boldsymbol{A} &: \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{A} &: \boldsymbol{x} \to \boldsymbol{A}\ (\boldsymbol{x})
\end{aligned}
$$

important transformations

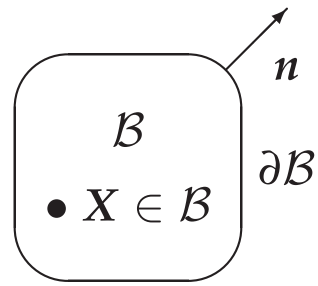
$$
\begin{aligned}
\nabla\ (\alpha\, \boldsymbol{u}) &= \boldsymbol{u} \otimes \nabla\alpha + \alpha\, \nabla \boldsymbol{u} \\
\nabla\ (\boldsymbol{u} \cdot \boldsymbol{v}) &= \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} \\
\operatorname{div}\ (\alpha\, \boldsymbol{u}) &= \alpha \operatorname{div}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla\alpha \\
\operatorname{div}\ (\alpha\, \boldsymbol{A}) &= \alpha \operatorname{div}(\boldsymbol{A}) + \boldsymbol{A} \cdot \nabla\alpha \\
\operatorname{div}\ (\boldsymbol{u} \cdot \boldsymbol{A}) &= \boldsymbol{u} \cdot \operatorname{div}(\boldsymbol{A}) + \boldsymbol{A} : \nabla \boldsymbol{u} \\
\operatorname{div}\ (\boldsymbol{u} \otimes \boldsymbol{v}) &= \boldsymbol{u} \operatorname{div}(\boldsymbol{v}) + \boldsymbol{v} \cdot \nabla \boldsymbol{u}^{\mathrm{t}}
\end{aligned}
\tag{1.2.15}
$$

index notation write eqns (1.2.15) in index notation!

$$
\begin{aligned}
(\ \alpha\, u_i\ )_{,j} &= u_i\, \alpha_{,j} + \alpha\, u_{i,j} \\
(\ u_i\, v_i\ )_{,j} &= u_i\, v_{i,j} + v_i\, u_{i,j} \\
(\ \alpha\, u_i\ )_{,i} &= \alpha\, u_{i,i} + u_i\, \alpha_{,i} \\
(\ \alpha\, A_{ij}\ )_{,j} &= \alpha\, A_{ij,j} + A_{ij}\, \alpha_{,j} \\
(u_i\, A_{ij})_{,j} &= u_i\, A_{ij,j} + A_{ij}\, u_{i,j} \\
(\ u_i\, v_j\ )_{,j} &= u_i\, v_{j,j} + v_j\, u_{i,j}
\end{aligned}
\tag{1.2.16}
$$

remark: sometimes index notation can be more illustrative!

1.2.5 Integral transformations



integral theorems define relations between surface integral $\int_{\partial\mathcal{B}} ...\mathrm{d}A$ and volume integral $\int_{\mathcal{B}} ...\mathrm{d}V$

consider scalar, vector and second order tensor field $\alpha(\boldsymbol{x})$, $\boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{A}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$
\begin{aligned}
&\alpha : \ \mathcal{B} \to \mathcal{R} && \alpha : \ \boldsymbol{x} \to \alpha \ \ (\boldsymbol{x}) \\
&\boldsymbol{u} : \ \mathcal{B} \to \mathcal{R}^3 && \boldsymbol{u} : \ \boldsymbol{x} \to \boldsymbol{u} \ \ (\boldsymbol{x}) \\
&\boldsymbol{A} : \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 && \boldsymbol{A} : \boldsymbol{x} \to \boldsymbol{A} \ \ (\boldsymbol{x})
\end{aligned}
$$

Integral theorem for scalar fields (Green theorem)

$$
\begin{aligned}
\int_{\partial\mathcal{B}} \alpha\, \boldsymbol{n}\ \mathrm{d}A &= \int_{\mathcal{B}} \nabla\alpha\ \ \mathrm{d}V \\
\int_{\partial\mathcal{B}} \alpha\, n_i\ \mathrm{d}A &= \int_{\mathcal{B}} \ \ \alpha_{,i}\ \mathrm{d}V
\end{aligned}
\tag{1.2.17}
$$

Integral theorem for vector fields (Gauss theorem)

$$
\begin{aligned}
\int_{\partial\mathcal{B}} \boldsymbol{u}\cdot\boldsymbol{n}\ \mathrm{d}A &= \int_{\mathcal{B}} \mathrm{div}(\boldsymbol{u})\ \mathrm{d}V \\
\int_{\partial\mathcal{B}} u_i\, n_i\ \mathrm{d}A &= \int_{\mathcal{B}} \ \ u_{i,i}\ \ \mathrm{d}V
\end{aligned}
\tag{1.2.18}
$$

Integral theorem for tensor fields (Gauss theorem)

$$
\begin{aligned}
\int_{\partial\mathcal{B}} \boldsymbol{A}\cdot\boldsymbol{n}\ \ \mathrm{d}A &= \int_{\mathcal{B}} \mathrm{div}(\boldsymbol{A})\ \mathrm{d}V \\
\int_{\partial\mathcal{B}} A_{ij}\, n_j\ \mathrm{d}A &= \int_{\mathcal{B}} \ \ A_{ij,j}\ \ \mathrm{d}V
\end{aligned}
\tag{1.2.19}
$$