ME338A
CONTINUUM MECHANICS

lecture notes 03

tuesday, january 12th, 2010

Invariants of second order tensors

the following property of the scalar triple product

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) \tag{1.1.67}$$

introduces three scalar–valued quantities I_A, II_A, III_A associated with the second order tensor $\boldsymbol{A}$

$$\begin{aligned} [\boldsymbol{A}\cdot\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] + [\boldsymbol{u}, \boldsymbol{A}\cdot\boldsymbol{v}, \boldsymbol{w}] + [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}\cdot\boldsymbol{w}] &= I_A[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \\ [\boldsymbol{u}, \boldsymbol{A}\cdot\boldsymbol{v}, \boldsymbol{A}\cdot\boldsymbol{w}] + [\boldsymbol{A}\cdot\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}\cdot\boldsymbol{w}] + [\boldsymbol{A}\cdot\boldsymbol{u}, \boldsymbol{A}\cdot\boldsymbol{v}, \boldsymbol{w}] &= II_A[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \\ [\boldsymbol{A}\cdot\boldsymbol{u}, \boldsymbol{A}\cdot\boldsymbol{v}, \boldsymbol{A}\cdot\boldsymbol{w}] &= III_A[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \end{aligned} \tag{1.1.68}$$

the proof of II_A, III_A being invariant for different base systems $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is similar to the one for I_A

I_A, II_A, III_A are called the three principal invariants of $\boldsymbol{A}$ which can be expressed as

$$\begin{aligned} I_A &= \operatorname{tr}(\boldsymbol{A}) & \partial_{\boldsymbol{A}}\, I_A &= \boldsymbol{I} \\ II_A &= \tfrac{1}{2}\left[\operatorname{tr}^2(\boldsymbol{A}) - \operatorname{tr}(\boldsymbol{A}^2)\right] & \partial_{\boldsymbol{A}}\, II_A &= I_A\, \boldsymbol{I} - \boldsymbol{A} \\ III_A &= \det(\boldsymbol{A}) & \partial_{\boldsymbol{A}}\, III_A &= III_A\, \boldsymbol{A}^{-\mathrm{t}} \end{aligned} \tag{1.1.69}$$

alternatively, we could work with the three basic invariants $\bar{I}_A$, $\bar{II}_A$, $\bar{III}_A$ of $\boldsymbol{A}$ which are more common in the context of anisotropy

$$\begin{aligned} \bar{I}_A &= \boldsymbol{A}^1 : \boldsymbol{I} \\ \bar{II}_A &= \boldsymbol{A}^2 : \boldsymbol{I} \\ \bar{III}_A &= \boldsymbol{A}^3 : \boldsymbol{I} \end{aligned} \tag{1.1.70}$$

Trace of second order tensors

trace $\operatorname{tr}(\boldsymbol{A})$ of a second order tensor $\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v}$ introduces a scalar $\operatorname{tr}(\boldsymbol{A}) \in \mathcal{R}$

$$\operatorname{tr}(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v} \tag{1.1.71}$$

such that $\operatorname{tr}(\boldsymbol{A})$ is the sum of the diagonal entries A_{ii} of $\boldsymbol{A}$

$$\begin{aligned} \operatorname{tr}(\boldsymbol{A}) &= \operatorname{tr}(A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \\ &= A_{ij} \operatorname{tr}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = A_{ij}\, \boldsymbol{e}_i \cdot \boldsymbol{e}_j \\ &= A_{ij}\, \delta_{ij} = A_{ii} = A_{11} + A_{22} + A_{33} \end{aligned} \tag{1.1.72}$$

with

$$I_A = \bar{I}_A = \operatorname{tr}(\boldsymbol{A}) \tag{1.1.73}$$

properties of the trace of second order tensors

$$\begin{aligned} &\operatorname{tr}(\boldsymbol{I}) = 3 \\ &\operatorname{tr}(\boldsymbol{A}^{\mathrm{t}}) = \operatorname{tr}(\boldsymbol{A}) \\ &\operatorname{tr}(\boldsymbol{A} \cdot \boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B} \cdot \boldsymbol{A}) \\ &\operatorname{tr}(\alpha\, \boldsymbol{A} + \beta\, \boldsymbol{B}) = \alpha \operatorname{tr}(\boldsymbol{A}) + \beta \operatorname{tr}(\boldsymbol{B}) \\ &\operatorname{tr}(\boldsymbol{A} \cdot \boldsymbol{B}^{\mathrm{t}}) = \boldsymbol{A} : \boldsymbol{B} \\ &\operatorname{tr}(\boldsymbol{A}) = \operatorname{tr}(\boldsymbol{A} \cdot \boldsymbol{I}) = \boldsymbol{A} : \boldsymbol{I} \end{aligned} \tag{1.1.74}$$

Determinant of second order tensors

determinant $\det(\boldsymbol{A})$ of second order tensor $\boldsymbol{A}$ introduces a scalar $\det(\boldsymbol{A}) \in \mathcal{R}$

$$\begin{aligned} \det(\boldsymbol{A}) &= \det(A_{ij}) = \tfrac{1}{6}\, e_{ijk}\, e_{abc}\, A_{ia}\, A_{jb}\, A_{kc} \\ &= A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} \\ &- A_{11}A_{23}A_{32} - A_{22}A_{31}A_{13} - A_{33}A_{12}A_{21} \end{aligned} \quad (1.1.75)$$

with

$$III_A = \det(\boldsymbol{A}) \quad (1.1.76)$$

determinant defining vector product $\boldsymbol{u} \times \boldsymbol{v}$

$$\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} u_1 & v_1 & \boldsymbol{e}_1 \\ u_2 & v_2 & \boldsymbol{e}_2 \\ u_3 & v_3 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 & - & u_3 v_2 \\ u_3 v_1 & - & u_1 v_3 \\ u_1 v_2 & - & u_2 v_1 \end{bmatrix} \quad (1.1.77)$$

determinant defining scalar triple vector product $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \quad (1.1.78)$$

properties of determinant of a second order tensors

$$\begin{aligned} &\det(\boldsymbol{I}) = 1 \\ &\det(\boldsymbol{A}^{\mathrm{t}}) = \det(\boldsymbol{A}) \\ &\det(\alpha\, \boldsymbol{A}) = \alpha^3 \det(\boldsymbol{A}) \\ &\det(\boldsymbol{A} \cdot \boldsymbol{B}) = \det(\boldsymbol{A}) \det(\boldsymbol{B}) \\ &\det(\boldsymbol{u} \otimes \boldsymbol{v}) = 0 \end{aligned} \quad (1.1.79)$$

Inverse of second order tensors

if $\det(\boldsymbol{A}) \neq 0$

existence of inverse $\boldsymbol{A}^{-1}$ of second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I} \tag{1.1.80}$$

in particular

$$\boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{u} \qquad \boldsymbol{A}^{-1} \cdot \boldsymbol{v} = \boldsymbol{u} \tag{1.1.81}$$

properties of inverse of two second order tensors

$$\begin{aligned} (\boldsymbol{A}^{-1})^{-1} &= \boldsymbol{A} \\ (\alpha\, \boldsymbol{A}^{-1})^{-1} &= \alpha^{-1}\, \boldsymbol{A} \\ (\boldsymbol{A} \cdot \boldsymbol{B})^{-1} &= \boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1} \end{aligned} \tag{1.1.82}$$

determinant $\det(\boldsymbol{A}^{-1})$ of inverse of $\boldsymbol{A}$

$$\det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A}) \tag{1.1.83}$$

adjoint $\boldsymbol{A}^{\mathrm{adj}}$ of a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A}^{\mathrm{adj}} = \det(\boldsymbol{A})\, \boldsymbol{A}^{-1} \tag{1.1.84}$$

cofactor $\boldsymbol{A}^{\mathrm{cof}}$ of a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A}^{\mathrm{cof}} = \det(\boldsymbol{A})\, \boldsymbol{A}^{-\mathrm{t}} = (\boldsymbol{A}^{\mathrm{adj}})^{\mathrm{t}} \tag{1.1.85}$$

with

$$\partial_{\boldsymbol{A}} \det(\boldsymbol{A}) = \det(\boldsymbol{A})\, \boldsymbol{A}^{-\mathrm{t}} = III_{\boldsymbol{A}}\, \boldsymbol{A}^{-\mathrm{t}} = \boldsymbol{A}^{\mathrm{cof}} \tag{1.1.86}$$

1.1.3 Spectral decomposition

eigenvalue problem of arbitrary second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} \cdot \boldsymbol{n}_A = \lambda_A \, \boldsymbol{n}_A \qquad [\, \boldsymbol{A} - \lambda_A \, \boldsymbol{I} \,] \cdot \boldsymbol{n}_A = \boldsymbol{0} \tag{1.1.87}$$

solution introduces eigenvector(s) $\boldsymbol{n}_{Ai}$ and eigenvalue(s) λ_{Ai}

$$\det (\boldsymbol{A} - \lambda_A \, \boldsymbol{I}) = 0 \tag{1.1.88}$$

alternative representation in terms of scalar triple product

$$[\boldsymbol{A} \cdot \boldsymbol{u} - \lambda_A \boldsymbol{u}, \boldsymbol{A} \cdot \boldsymbol{v} - \lambda_A \boldsymbol{v}, \boldsymbol{A} \cdot \boldsymbol{w} - \lambda_A \boldsymbol{w}] = 0 \tag{1.1.89}$$

removal of arbitrary factor $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$ yields characteristic equation

$$\lambda_A^3 - I_A \, \lambda_A^2 + II_A \, \lambda_A - III_A = 0 \tag{1.1.90}$$

roots of characteristic equations are principal invariants of $\boldsymbol{A}$

$$\begin{aligned} I_A &= \operatorname{tr}(\boldsymbol{A}) \\ II_A &= \tfrac{1}{2} \left[\operatorname{tr}^2(\boldsymbol{A}) - \operatorname{tr}(\boldsymbol{A}^2) \right] \\ III_A &= \det(\boldsymbol{A}) \end{aligned} \tag{1.1.91}$$

spectral decomposition of $\boldsymbol{A}$

$$\boldsymbol{A} = \sum_{i=1}^{3} \lambda_{Ai} \, \boldsymbol{n}_{Ai} \otimes \boldsymbol{n}_{Ai} \tag{1.1.92}$$

Cayleigh–Hamilton theorem:
a tensor $\boldsymbol{A}$ satisfies its own characteristic equation

$$\boldsymbol{A}^3 - I_A \, \boldsymbol{A}^2 + II_A \, \boldsymbol{A} - III_A \, \boldsymbol{I} = \boldsymbol{0} \tag{1.1.93}$$

1.1.4 Symmetric – skew-symmetric decomposition

symmetric – skew-symmetric decomposition of second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = \frac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\mathrm{t}}] + \frac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\mathrm{t}}] = \boldsymbol{A}^{\mathrm{sym}} + \boldsymbol{A}^{\mathrm{skw}} \tag{1.1.94}$$

with symmetric and skew-symmetric second order tensor $\boldsymbol{A}^{\mathrm{sym}}$ and $\boldsymbol{A}^{\mathrm{skw}}$

$$\boldsymbol{A}^{\mathrm{sym}} = \left(\boldsymbol{A}^{\mathrm{sym}}\right)^{\mathrm{t}} \qquad \boldsymbol{A}^{\mathrm{skw}} = -\left(\boldsymbol{A}^{\mathrm{skw}}\right)^{\mathrm{t}} \tag{1.1.95}$$

• symmetric second order tensor $\boldsymbol{A}^{\mathrm{sym}}$

$$\boldsymbol{A}^{\mathrm{sym}} = \frac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\mathrm{t}}] = \mathbb{I}^{\mathrm{sym}} : \boldsymbol{A} \tag{1.1.96}$$

upon double contraction symmetric fourth order unit tensor $\mathbb{I}^{\mathrm{sym}}$ extracts symmetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\mathrm{sym}} &= \tfrac{1}{2}\left[\mathbb{I} + \mathbb{I}^{\mathrm{t}}\right] \\ \mathbb{I}^{\mathrm{sym}} &= \tfrac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{1.1.97}$$

• skew–symmetric second order tensor $\boldsymbol{A}^{\mathrm{skw}}$

$$\boldsymbol{A}^{\mathrm{skw}} = \frac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\mathrm{t}}] = \mathbb{I}^{\mathrm{skw}} : \boldsymbol{A} \tag{1.1.98}$$

upon double contraction skew-symmetric fourth order unit tensor $\mathbb{I}^{\mathrm{skw}}$ extracts skew-symmetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\mathrm{skw}} &= \tfrac{1}{2}\left[\mathbb{I} - \mathbb{I}^{\mathrm{t}}\right] \\ \mathbb{I}^{\mathrm{skw}} &= \tfrac{1}{2}\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{1.1.99}$$

Symmetric tensors

symmetric part $\boldsymbol{A}^{\rm sym}$ of a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A}^{\rm sym} = \frac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\rm t}] \qquad \boldsymbol{A}^{\rm sym} = (\boldsymbol{A}^{\rm sym})^{\rm t} \tag{1.1.100}$$

alternative representation

$$\boldsymbol{A}^{\rm sym} = \mathbb{I}^{\rm sym} : \boldsymbol{A} \tag{1.1.101}$$

whereby symmetric fourth order tensor $\mathbb{I}^{\rm sym}$ extracts symmetric part $\boldsymbol{A}^{\rm sym}$ of second order tensor $\boldsymbol{A}$

a symmetric second order tensor $\boldsymbol{S} = \boldsymbol{A}^{\rm sym}$ processes three real eigenvalues $\{\lambda_{Si}\}_{i=1,2,3}$ and three corresponding orthogonal eigenvectors $\{\boldsymbol{n}_{Si}\}_{i=1,2,3}$, such that the spectral representation of $\boldsymbol{S}$ takes the following form

$$\boldsymbol{S} = \sum_{i=1}^{3} \lambda_{Si}\,(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \tag{1.1.102}$$

three invariants I_S, II_S, III_S of symmetric tensor $\boldsymbol{S} = \boldsymbol{A}^{\rm sym}$

$$\begin{aligned} I_S &= \lambda_{S1} + \lambda_{S2} + \lambda_{S3} \\ II_S &= \lambda_{S2}\,\lambda_{S3} + \lambda_{S3}\,\lambda_{S1} + \lambda_{S1}\,\lambda_{S2} \\ III_S &= \lambda_{S1}\,\lambda_{S2}\,\lambda_{S3} \end{aligned} \tag{1.1.103}$$

square root $\sqrt{\boldsymbol{S}}$, inverse $\boldsymbol{S}^{-1}$, exponent $\exp(\boldsymbol{S})$ and logarithm $\ln(\boldsymbol{S})$, of positive semi-definite symmetric tensor $\boldsymbol{S}$ for which $\lambda_{Si} \geq 0$

$$\begin{aligned} \sqrt{\boldsymbol{S}} &= \textstyle\sum_{i=1}^{3} \sqrt{\lambda_{Si}}\;(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \boldsymbol{S}^{-1} &= \textstyle\sum_{i=1}^{3} \lambda_{Si}^{-1}\;(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \exp(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3} \exp(\lambda_{Si})\;(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \ln(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3} \ln(\lambda_{Si})\;(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \end{aligned} \tag{1.1.104}$$

Skew-symmetric tensors

skew-symmetric part $\boldsymbol{A}^{\text{skw}}$ of a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A}^{\text{skw}} = \frac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\text{t}}] \qquad \boldsymbol{A}^{\text{skw}} = -(\boldsymbol{A}^{\text{skw}})^{\text{t}} \tag{1.1.105}$$

alternative representation

$$\boldsymbol{A}^{\text{skw}} = \mathbb{I}^{\text{skw}} : \boldsymbol{A} \tag{1.1.106}$$

whereby skew-symmetric fourth order tensor $\mathbb{I}^{\text{skw}}$ extracts skew-symmetric part $\boldsymbol{A}^{\text{skw}}$ of second order tensor $\boldsymbol{A}$

a skew-symmetric second order tensor $\boldsymbol{W} = \boldsymbol{A}^{\text{skw}}$ posses three independent entries, three entries vanish identically, three are equal to the negative of the independent entries, these define the axial vector $\boldsymbol{w}$

$$\boldsymbol{w} = -\frac{1}{2} \overset{3}{\boldsymbol{e}} : \boldsymbol{W} \qquad \boldsymbol{w} = - \overset{3}{\boldsymbol{e}} \cdot \boldsymbol{w} \tag{1.1.107}$$

associated with each skew-symmetric tensor $\boldsymbol{W} = \boldsymbol{A}^{\text{skw}}$

$$\boldsymbol{W} \cdot \boldsymbol{p} = \boldsymbol{w} \times \boldsymbol{p} \tag{1.1.108}$$

three invariants I_W, II_W, III_W of skew-symmetric tensor $\boldsymbol{W}$

$$\begin{aligned} I_W &= \text{tr}(\boldsymbol{W}) &&= 0 \\ II_W &= \boldsymbol{w} \cdot \boldsymbol{w} && \\ III_W &= \det(\boldsymbol{W}) &&= 0 \end{aligned} \tag{1.1.109}$$

1.1.5 Volumetric – deviatoric decomposition

volumetric – deviatoric decomposition of second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{A}^{\text{vol}} + \boldsymbol{A}^{\text{dev}} \tag{1.1.110}$$

with volumetric and deviatoric second order tensor $\boldsymbol{A}^{\text{vol}}$ and $\boldsymbol{A}^{\text{dev}}$

$$\text{tr}(\boldsymbol{A}^{\text{vol}}) = \text{tr}(\boldsymbol{A}) \qquad \text{tr}(\boldsymbol{A}^{\text{dev}}) = 0 \tag{1.1.111}$$

• volumetric second order tensor $\boldsymbol{A}^{\text{vol}}$

$$\boldsymbol{A}^{\text{vol}} = \frac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\text{vol}} : \boldsymbol{A} \tag{1.1.112}$$

upon double contraction volumetric fourth order unit tensor $\mathbb{I}^{\text{vol}}$ extracts volumetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\,\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\,\delta_{ij}\delta_{kl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{1.1.113}$$

• deviatoric second order tensor $\boldsymbol{A}^{\text{dev}}$

$$\boldsymbol{A}^{\text{dev}} = \boldsymbol{A} - \frac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{A} = 0 \tag{1.1.114}$$

upon double contraction deviatoric fourth order unit tensor $\mathbb{I}^{\text{dev}}$ extracts deviatoric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\text{dev}} &= \mathbb{I}^{\text{sym}} - \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{sym}} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{dev}} &= \left[\tfrac{1}{2}\,\delta_{ik}\delta_{jl} + \tfrac{1}{2}\,\delta_{il}\delta_{jk} - \tfrac{1}{3}\,\delta_{ij}\delta_{kl}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{1.1.115}$$

1.1.6 Orthogonal tensors

a second order tensor $\boldsymbol{Q}$ is called orthogonal if its inverse $\boldsymbol{Q}^{-1}$ is identical to its transpose $\boldsymbol{Q}^{\mathrm{t}}$

$$\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathrm{t}} \qquad \Leftrightarrow \qquad \boldsymbol{Q}^{\mathrm{t}} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathrm{t}} = \boldsymbol{I} \tag{1.1.116}$$

a second order tensor $\boldsymbol{A}$ can be decomposed multiplicatively into a positive definite symmetric tensor $\boldsymbol{U}^{\mathrm{t}} = \boldsymbol{U}$ or $\boldsymbol{V}^{\mathrm{t}} = \boldsymbol{V}$ with $\boldsymbol{a} \cdot \boldsymbol{U} \cdot \boldsymbol{a} \geq 0$ and $\boldsymbol{a} \cdot \boldsymbol{V} \cdot \boldsymbol{a} \geq 0$ and an orthogonal tensor $\boldsymbol{Q}^{\mathrm{t}} = \boldsymbol{Q}^{-1}$ as

$$\boldsymbol{A} = \boldsymbol{Q} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{Q} \tag{1.1.117}$$

with S0(3) being the special orthogonal group, $\boldsymbol{Q} \in$ S0(3) if $\det(\boldsymbol{Q}) = +1$, then $\boldsymbol{Q}$ is called proper orthogonal

a proper orthogonal tensor $\boldsymbol{Q} \in$ S0(3) has an eigenvalue equal to one $\lambda_Q = 1$ introducing an eigenvector $\boldsymbol{n}_Q$ such that

$$\boldsymbol{Q} \cdot \boldsymbol{n}_Q = \boldsymbol{n}_Q \tag{1.1.118}$$

let $\{\boldsymbol{n}_{Qi}\}_{i=1,2,3}$ be a Cartesian basis containing the vector $\boldsymbol{n}_Q$, then matrix representation of coordinates Q_{ij}

$$[Q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\cos\varphi & +\sin\varphi \\ 0 & -\sin\varphi & +\cos\varphi \end{bmatrix} \tag{1.1.119}$$

geometric interpretation: $\boldsymbol{Q}$ characterizes a finite rotation around the axis $\boldsymbol{n}_Q$ with $\boldsymbol{Q} \cdot \boldsymbol{n}_Q = \boldsymbol{n}_Q$, i.e. associated with $\lambda_Q = 1$