ME338A
CONTINUUM MECHANICS

lecture notes 02

thursday, january 07th, 2010

1.1.2 Tensor algebra

Notation

Second order tensors

tensor (dyadic) product $\boldsymbol{u} \otimes \boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ introduces a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} \tag{1.1.29}$$

introducing $\boldsymbol{u} = u_i \, \boldsymbol{e}_i$ and $\boldsymbol{v} = v_j \, \boldsymbol{e}_j$ yields index representation of three-dimensional second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = A_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.1.30}$$

with $A_{ij} = u_i \, v_j$ coordinates (components) of $\boldsymbol{A}$ relative to the tensor basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, matrix representation of coordinates

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1.1.31}$$

transpose of second order tensor $\boldsymbol{A}^{\mathrm{t}}$

$$\boldsymbol{A}^{\mathrm{t}} = (\boldsymbol{u} \otimes \boldsymbol{v})^{\mathrm{t}} = \boldsymbol{v} \otimes \boldsymbol{u} \tag{1.1.32}$$

introducing $\boldsymbol{u} = u_i \, \boldsymbol{e}_i$ and $\boldsymbol{v} = v_j \, \boldsymbol{e}_j$ yields index representation of transpose of second order tensor $\boldsymbol{A}^{\mathrm{t}}$

$$\boldsymbol{A}^{\mathrm{t}} = A_{ji} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.1.33}$$

with $A_{ji} = v_j \, u_i$ coordinates (components) of $\boldsymbol{A}^{\mathrm{t}}$ relative to the tensor basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, matrix representation of coordinates

$$[A_{ji}] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \tag{1.1.34}$$

second order unit tensor $\boldsymbol{I}$ in terms of Kronecker symbol δ_{ij}

$$\boldsymbol{I} = \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.1.35}$$

matrix representation of coordinates δ_{ij}

$$[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1.36}$$

Third order tensors

tensor (dyadic) product $\boldsymbol{A} \otimes \boldsymbol{u}$ of second order tensor $\boldsymbol{A}$ and vectors $\boldsymbol{u}$ introduces a third order tensor $\overset{3}{\boldsymbol{a}}$

$$\overset{3}{\boldsymbol{a}} = \boldsymbol{A} \otimes \boldsymbol{u} \tag{1.1.37}$$

introducing $\boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\boldsymbol{u} = u_k\, \boldsymbol{e}_k$ yields index representation of three-dimensional third order tensor $\overset{3}{\boldsymbol{a}}$

$$\overset{3}{\boldsymbol{a}} = \overset{3}{a}_{ijk}\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{1.1.38}$$

with $\overset{3}{a}_{ijk} = A_{ij}\, u_k$ coordinates (components) of $\overset{3}{\boldsymbol{a}}$ relative to the tensor basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$

third order permutation tensor $\overset{3}{\boldsymbol{e}}$ in terms of permutation symbol $\overset{3}{e}_{ijk}$

$$\overset{3}{\boldsymbol{e}} = \overset{3}{e}_{ijk}\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{1.1.39}$$

Fourth order tensors

tensor (dyadic) product $\boldsymbol{A} \otimes \boldsymbol{B}$ of two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ introduces a fourth order tensor $\mathbb{A}$

$$\mathbb{A} = \boldsymbol{A} \otimes \boldsymbol{B} \tag{1.1.40}$$

introducing $\boldsymbol{A} = A_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\boldsymbol{B} = B_{kl}\,\boldsymbol{e}_k \otimes \boldsymbol{e}_l$ yields index representation of three-dimensional fourth order tensor $\mathbb{A}$

$$\mathbb{A} = A_{ijkl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.41}$$

with $A_{ijkl} = A_{ij}\,B_{kl}$ coordinates (components) of $\mathbb{A}$ relative to the tensor basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$

fourth order unit tensor $\mathbb{I}$

$$\mathbb{I} = \delta_{ik}\,\delta_{jl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.42}$$

transpose fourth order unit tensor $\mathbb{I}^{\mathrm{t}}$

$$\mathbb{I}^{\mathrm{t}} = \delta_{il}\,\delta_{jk}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.43}$$

symmetric fourth order unit tensor $\mathbb{I}^{\mathrm{sym}}$

$$\mathbb{I}^{\mathrm{sym}} = \tfrac{1}{2}\,[\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}]\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.44}$$

skew-symmetric fourth order unit tensor $\mathbb{I}^{\mathrm{skw}}$

$$\mathbb{I}^{\mathrm{skw}} = \tfrac{1}{2}\,[\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}]\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.45}$$

volumetric fourth order unit tensor $\mathbb{I}^{\mathrm{vol}}$

$$\mathbb{I}^{\mathrm{vol}} = \tfrac{1}{3}\,\delta_{ij}\,\delta_{kl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.46}$$

deviatoric fourth order unit tensor $\mathbb{I}^{\mathrm{dev}}$

$$\mathbb{I}^{\mathrm{dev}} = [\tfrac{1}{2}\,\delta_{ik}\,\delta_{jl} + \tfrac{1}{2}\,\delta_{il}\,\delta_{jk} - \tfrac{1}{3}\,\delta_{ij}\,\delta_{kl}]\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.47}$$

Scalar products

• scalar product $\boldsymbol{A} \cdot \boldsymbol{u}$ between second order tensor $\boldsymbol{A}$ and vector $\boldsymbol{u}$ defines a new vector $\boldsymbol{v} \in \mathcal{E}^3$

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{u} &= (A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (u_k \, \boldsymbol{e}_k) \\ &= A_{ij} \, u_k \, \delta_{jk} \, \boldsymbol{e}_i = A_{ij} \, u_j \, \boldsymbol{e}_i = v_i \boldsymbol{e}_i = \boldsymbol{v} \end{aligned} \tag{1.1.48}$$

second order zero tensor $\mathbf{0}$, second order identity tensor $\boldsymbol{I}$

$$\mathbf{0} \cdot \boldsymbol{a} = \mathbf{0} \qquad \boldsymbol{I} \cdot \boldsymbol{a} = \boldsymbol{a} \tag{1.1.49}$$

positive semi-definiteness of second order tensor $\boldsymbol{A}$

$$\boldsymbol{a} \cdot \boldsymbol{A} \cdot \boldsymbol{a} \geq 0 \tag{1.1.50}$$

positive definiteness of second order tensor $\boldsymbol{A}$

$$\boldsymbol{a} \cdot \boldsymbol{A} \cdot \boldsymbol{a} > 0 \tag{1.1.51}$$

properties of scalar product

$$\begin{aligned} &\boldsymbol{A} \cdot (\alpha \boldsymbol{a} + \beta \boldsymbol{b}) = \alpha \, (\boldsymbol{A} \cdot \boldsymbol{a}) + \beta \, (\boldsymbol{A} \cdot \boldsymbol{b}) \\ &(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{a} = \boldsymbol{A} \cdot \boldsymbol{a} + \boldsymbol{B} \cdot \boldsymbol{a} \\ &(\alpha \, \boldsymbol{A}) \cdot \boldsymbol{a} = \alpha \, (\boldsymbol{A} \cdot \boldsymbol{a}) \end{aligned} \tag{1.1.52}$$

• scalar product $\boldsymbol{A} \cdot \boldsymbol{B}$ between two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ defines a second order tensor $\boldsymbol{C}$

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= (A_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (B_{kl} \, \boldsymbol{e}_k \otimes \boldsymbol{e}_l) \\ &= A_{ij} \, B_{kl} \, \delta_{jk} \boldsymbol{e}_i \otimes \boldsymbol{e}_l \\ &= A_{ij} \, B_{jl} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_l = C_{il} \boldsymbol{e}_i \otimes \boldsymbol{e}_l = \boldsymbol{C} \end{aligned} \tag{1.1.53}$$

second order zero tensor $\mathbf{0}$, second order identity tensor $\boldsymbol{I}$

$$\mathbf{0} \cdot \boldsymbol{A} = \mathbf{0} \qquad \boldsymbol{I} \cdot \boldsymbol{A} = \boldsymbol{A} \tag{1.1.54}$$

properties of scalar product

$$\begin{aligned}
&\alpha\,(\boldsymbol{A}\cdot\boldsymbol{B}) = (\alpha\,\boldsymbol{A})\cdot\boldsymbol{B} = \boldsymbol{A}\cdot(\alpha\,\boldsymbol{B})\\
&\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}\\
&(\boldsymbol{A}+\boldsymbol{B})\cdot\boldsymbol{C} = \boldsymbol{A}\cdot\boldsymbol{C}+\boldsymbol{B}\cdot\boldsymbol{C}
\end{aligned} \tag{1.1.55}$$

properties in terms of transpose $\boldsymbol{A}^{\mathrm{t}}$ of a tensor $\boldsymbol{A}$

$$\begin{aligned}
&\boldsymbol{a}\cdot(\boldsymbol{A}^{\mathrm{t}}\cdot\boldsymbol{b}) = \boldsymbol{b}\cdot(\boldsymbol{A}\cdot\boldsymbol{a})\\
&(\alpha\,\boldsymbol{A}+\beta\,\boldsymbol{B})^{\mathrm{t}} = \alpha\,\boldsymbol{A}^{\mathrm{t}}+\beta\,\boldsymbol{B}^{\mathrm{t}}\\
&(\boldsymbol{A}\cdot\boldsymbol{B})^{\mathrm{t}} = \boldsymbol{B}^{\mathrm{t}}\cdot\boldsymbol{A}^{\mathrm{t}}
\end{aligned} \tag{1.1.56}$$

• scalar product $\boldsymbol{A}:\boldsymbol{B}$ between two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ defines a scalar $\alpha\in\mathcal{R}$

$$\begin{aligned}
\boldsymbol{A}:\boldsymbol{B} &= (A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j):(B_{kl}\,\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\
&= A_{ij}\,B_{kl}\,\delta_{ik}\,\delta_{jl} = A_{ij}\,B_{ij} = \alpha
\end{aligned} \tag{1.1.57}$$

• scalar product $\mathbb{A}:\boldsymbol{B}$ between fourth order tensor $\mathbb{A}$ and second order tensor $\boldsymbol{B}$ defines a new second order tensor $\boldsymbol{C}$

$$\begin{aligned}
\mathbb{A}:\boldsymbol{B} &= (A_{ijkl}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l):(B_{mn}\,\boldsymbol{e}_m\otimes\boldsymbol{e}_n)\\
&= A_{ijkl}\,B_{mn}\,\delta_{km}\,\delta_{ln}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\\
&= A_{ijkl}\,B_{kl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = C_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \boldsymbol{C}
\end{aligned} \tag{1.1.58}$$

Dyadic product

• tensor (dyadic) product $\boldsymbol{u} \otimes \boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ introduces a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} = u_i\, \boldsymbol{e}_i \otimes v_j \boldsymbol{e}_j = u_i\, v_j\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.1.59}$$

properties of dyadic product

$$\begin{aligned}
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w})\, \boldsymbol{u} \\
&(\alpha\, \boldsymbol{u} + \beta\, \boldsymbol{v}) \otimes \boldsymbol{w} = \alpha\, (\boldsymbol{u} \otimes \boldsymbol{w}) + \beta\, (\boldsymbol{v} \otimes \boldsymbol{w}) \\
&\boldsymbol{u} \otimes (\alpha\, \boldsymbol{v} + \beta\, \boldsymbol{w}) = \alpha\, (\boldsymbol{u} \otimes \boldsymbol{v}) + \beta\, (\boldsymbol{u} \otimes \boldsymbol{w}) \\
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot (\boldsymbol{w} \otimes \boldsymbol{x}) = (\boldsymbol{v} \cdot \boldsymbol{w})\, (\boldsymbol{u} \otimes \boldsymbol{x}) \\
&\boldsymbol{A} \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{A} \cdot \boldsymbol{u}) \otimes \boldsymbol{v} \\
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{A} = \boldsymbol{u} \otimes (\boldsymbol{A}^{\mathrm{t}} \cdot \boldsymbol{v})
\end{aligned} \tag{1.1.60}$$

or in index notation

$$\begin{aligned}
&(u_i\, v_j)\, w_j = (v_j\, w_j)\, u_i \\
&(\alpha\, u_i + \beta\, v_i) w_j = \alpha\, (u_i\, w_j) + \beta\, (v_i\, w_j) \\
&u_i(\alpha\, v_j + \beta\, w_j) = \alpha\, (u_i\, v_j) + \beta\, (u_i\, w_j) \\
&(u_i\, v_j)(w_j\, x_k) = (v_j\, w_j)\, (u_i x_k) \\
&A_{ij}(u_j\, v_k) = (A_{ij}\, u_j) v_k \\
&(u_i\, v_j) A_{jk} = u_i (A_{kj}\, v_j)
\end{aligned} \tag{1.1.61}$$

Scalar triple vector product

consider the set of Cartesian base vectors $\{\boldsymbol{e}_i\}_{i=1,2,3}$ and an arbitrary second set of base vectors $\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}$ with scalar triple product $[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]$, with arbitrary second order tensor $\boldsymbol{A}$, evaluate

$$[\,[\boldsymbol{A}\cdot\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]+[\boldsymbol{u},\boldsymbol{A}\cdot\boldsymbol{v},\boldsymbol{w}]+[\boldsymbol{u},\boldsymbol{v},\boldsymbol{A}\cdot\boldsymbol{w}]\,]\;/\;[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \tag{1.1.62}$$

with index representation of each term according to

$$[\boldsymbol{A}\cdot\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]=[\boldsymbol{A}\cdot(u_i\,\boldsymbol{e}_i),(v_j\,\boldsymbol{e}_j),(w_k\,\boldsymbol{e}_k)]=u_i\,v_j\,w_k\,[\boldsymbol{A}\cdot\boldsymbol{e}_i,\boldsymbol{e}_j,\boldsymbol{e}_k] \tag{1.1.63}$$

expression (1.1.62) can be rewritten as

$$u_iv_jw_k[[\boldsymbol{A}\cdot\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{A}\cdot\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{A}\cdot\boldsymbol{e}_3]]\,/\,[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \tag{1.1.64}$$

term in brackets remains unchanged upon cyclic permutation of $\{\boldsymbol{e}_i\}_{i=1,2,3}$, its sign reverses upon non–cyclic permutations, thus

$$\begin{aligned} u_iv_jw_k\overset{3}{e}_{ijk}\;&[[\boldsymbol{A}\cdot\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{A}\cdot\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{A}\cdot\boldsymbol{e}_3]]\,/\,[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]\\ =\;&\pm[[\boldsymbol{A}\cdot\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{A}\cdot\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{A}\cdot\boldsymbol{e}_3]] \end{aligned} \tag{1.1.65}$$

the above expression according to (1.1.62) is thus invariant under the choice of base system, it yields the same scalar value I_A for arbitrary base systems

$$\begin{aligned} I_A&=[\,[\boldsymbol{A}\cdot\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]+[\boldsymbol{u},\boldsymbol{A}\cdot\boldsymbol{v},\boldsymbol{w}]+[\boldsymbol{u},\boldsymbol{v},\boldsymbol{A}\cdot\boldsymbol{w}]\,]\,/\,[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]\\ &=[[\boldsymbol{A}\cdot\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{A}\cdot\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{A}\cdot\boldsymbol{e}_3]]\,/\,[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3] \end{aligned} \tag{1.1.66}$$

I_A is called the first invariant of the second order tensor $\boldsymbol{A}$