

**me 338a**  
**continuum mechanics**

lecture notes

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**ME338A**  
**CONTINUUM MECHANICS**

lecture notes 01

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# 1 Tensor calculus

## 1.1 Tensor algebra

### 1.1.1 Vector algebra

#### Notation

- Einstein's summation convention

$$u_i = \sum_{j=1}^3 A_{ij} x_j + b_i = A_{ij} x_j + b_i \quad (1.1.1)$$

summation over indices that appear twice in a term or symbol, with silent (dummy) index  $j$  and free index  $i$ , and thus

$$\begin{aligned} u_1 &= A_{11} x_1 + A_{12} x_2 + A_{13} x_3 + b_1 \\ u_2 &= A_{21} x_1 + A_{22} x_2 + A_{23} x_3 + b_2 \\ u_3 &= A_{31} x_1 + A_{32} x_2 + A_{33} x_3 + b_3 \end{aligned} \quad (1.1.2)$$

- Kronecker symbol  $\delta_{ij}$

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad (1.1.3)$$

multiplication with Kronecker symbol corresponds to exchange of silent index with free index of Kronecker symbol

$$u_i = \delta_{ij} u_j \quad (1.1.4)$$

- permutation symbol  $e_{ijk}^3$

$$e_{ijk}^3 = \begin{cases} 1 & \text{for } \{i, j, k\} \dots \text{even permutation} \\ -1 & \text{for } \{i, j, k\} \dots \text{odd permutation} \\ 0 & \text{for } \dots \text{else} \end{cases} \quad (1.1.5)$$

## Euclidian vector space

- consider linear vector space  $\mathcal{V}^3$  characterized through addition of its elements  $\mathbf{u}, \mathbf{v}$  and multiplication with real scalars  $\alpha, \beta$

$$\alpha, \beta \in \mathcal{R} \quad \mathcal{R} \dots \text{real numbers}$$

$$\mathbf{u}, \mathbf{v} \in \mathcal{V}^3 \quad \mathcal{V}^3 \dots \text{linear vector space}$$

definition of linear vector space  $\mathcal{V}^3$  through the following axioms

$$\alpha (\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$$

$$(\alpha + \beta) \mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u} \quad (1.1.6)$$

$$(\alpha \beta) \mathbf{u} = \alpha (\beta \mathbf{u})$$

zero element and identity

$$0 \mathbf{u} = \mathbf{0} \quad 1 \mathbf{u} = \mathbf{u} \quad (1.1.7)$$

linear independence of elements  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathcal{V}^3$  if  $\alpha_1 = \alpha_2 = \alpha_3 = 0$  is the only (trivial) solution to

$$\alpha_i \mathbf{e}_i = \mathbf{0} \quad (1.1.8)$$

- consider linear vector space  $\mathcal{V}^3$  equipped with a norm  $n(\mathbf{u})$  mapping elements of the linear vector space  $\mathcal{V}^3$  to the space

of real numbers  $\mathcal{R}$

$$n : \mathcal{V}^3 \rightarrow \mathcal{R} \quad \text{norm} \quad (1.1.9)$$

definition of norm through the following axioms

$$\begin{aligned} n(\mathbf{u}) &\geq 0 & n(\mathbf{u}) = 0 &\Leftrightarrow \mathbf{u} = \mathbf{0} \\ n(\alpha \mathbf{u}) &= |\alpha| n(\mathbf{u}) \\ n(\mathbf{u} + \mathbf{v}) &\leq n(\mathbf{u}) + n(\mathbf{v}) \\ n^2(\mathbf{u} + \mathbf{v}) + n^2(\mathbf{u} - \mathbf{v}) &= 2 [n^2(\mathbf{u}) + n^2(\mathbf{v})] \end{aligned} \quad (1.1.10)$$

• consider Euclidian vector space  $\mathcal{E}^3$  equipped with the Euclidian norm

$$n(\mathbf{u}) = \|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = [u_1^2 + u_2^2 + u_3^2]^{1/2} \quad (1.1.11)$$

mapping elements of the Euclidian vector space  $\mathcal{E}^3$  to the space of real numbers  $\mathcal{R}$

$$n : \mathcal{E}^3 \rightarrow \mathcal{R} \quad \text{Euclidian norm} \quad (1.1.12)$$

representation of three-dimensional vector  $\mathbf{a} \in \mathcal{E}^3$

$$\mathbf{a} = a_i \mathbf{e}_i = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \quad (1.1.13)$$

with  $a_1, a_2, a_3$  coordinates (components) of  $\mathbf{a}$  relative to the basis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{a} = [a_1, a_2, a_3]^t \quad (1.1.14)$$

## Scalar product

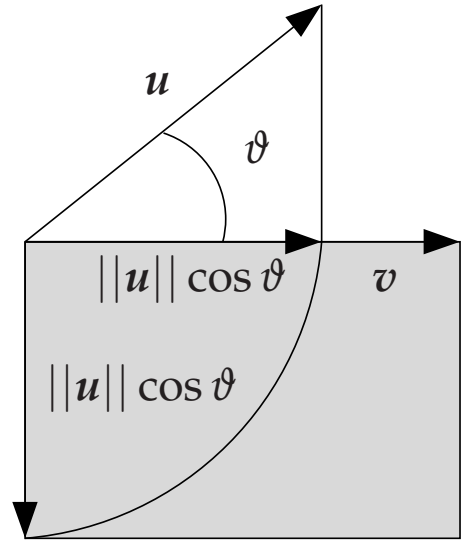
Euclidian norm enables the definition of scalar (inner) product between two vectors  $\mathbf{u}$ ,  $\mathbf{v}$  and introduces a scalar  $\alpha \in \mathcal{R}$

$$\mathbf{u} \cdot \mathbf{v} = \alpha \quad (1.1.15)$$

geometric interpretation with  $0 \leq \vartheta \leq \pi$  being the angle enclosed by the vectors  $\mathbf{u}$  and  $\mathbf{v}$ , then  $\|\mathbf{u}\| \cos \vartheta$  can be interpreted as the projection of  $\mathbf{u}$  onto the direction of  $\mathbf{v}$  and

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \vartheta$$

corresponds to the grey area in the picture



with the above interpretation with  $0 \leq \vartheta \leq \pi$ , obviously

$$\|\mathbf{u} \cdot \mathbf{v}\| \leq \|\mathbf{u}\| \|\mathbf{v}\| \quad (1.1.16)$$

properties of scalar product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

$$(\alpha \mathbf{u} + \beta \mathbf{v}) \cdot \mathbf{w} = \alpha (\mathbf{u} \cdot \mathbf{w}) + \beta (\mathbf{v} \cdot \mathbf{w}) \quad (1.1.17)$$

$$\mathbf{w} \cdot (\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha (\mathbf{w} \cdot \mathbf{u}) + \beta (\mathbf{w} \cdot \mathbf{v})$$

positive definiteness of scalar product

$$\mathbf{u} \cdot \mathbf{u} \geq 0, \quad \mathbf{u} \cdot \mathbf{u} = 0 \Leftrightarrow \mathbf{u} = \mathbf{0} \quad (1.1.18)$$

orthogonal vectors  $\mathbf{u}$  and  $\mathbf{v}$

$$\mathbf{u} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{u} \perp \mathbf{v} \quad (1.1.19)$$

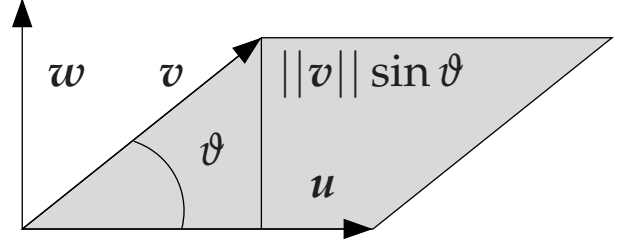


## Vector product

vector product of two vectors  $\mathbf{u}, \mathbf{v}$  defines a new vector  $\mathbf{w} \in \mathcal{E}^3$

$$\mathbf{u} \times \mathbf{v} = \mathbf{w} \quad (1.1.20)$$

geometric interpretation  
with  $0 \leq \vartheta \leq \pi$  being the  
angle enclosed by the vec-  
tors  $\mathbf{u}$  and  $\mathbf{v}$ , then  $\|\mathbf{v}\| \sin \vartheta$   
can be interpreted as the height of the grey polygon and



$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \sin \vartheta \mathbf{n}$$

introduces the vector  $\mathbf{w}$  orthogonal to  $\mathbf{u}$  and  $\mathbf{v}$  whereby its  
length corresponds to the grey area

with the above interpretation, obviously  $\mathbf{u}$  parallel to  $\mathbf{v}$  if

$$\mathbf{u} \times \mathbf{v} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{u} \parallel \mathbf{v} \quad (1.1.21)$$

index representation of  $\mathbf{w} = \mathbf{u} \times \mathbf{v}$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \quad (1.1.22)$$

properties of vector product

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$

$$(\alpha \mathbf{u} + \beta \mathbf{v}) \times \mathbf{w} = \alpha (\mathbf{u} \times \mathbf{w}) + \beta (\mathbf{v} \times \mathbf{w}) \quad (1.1.23)$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$$

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \mathbf{u}) (\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2$$

## Scalar triple vector product

scalar triple vector product of three vectors  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  introduces a scalar  $\alpha \in \mathcal{R}$

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \alpha \quad (1.1.24)$$

geometric interpretation  
with vector product

$$\mathbf{v} \times \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \sin \vartheta \mathbf{n}$$

defining area of ground surface

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

defines volume of parallelepiped

obviously

$$\begin{aligned} \alpha &= \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \\ -\alpha &= \mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) \end{aligned} \quad (1.1.25)$$

index representation of  $\alpha = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$

$$\alpha = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) \quad (1.1.26)$$

properties of scalar triple product

$$\begin{aligned} [\mathbf{u}, \mathbf{v}, \mathbf{w}] &= [\mathbf{v}, \mathbf{w}, \mathbf{u}] = [\mathbf{w}, \mathbf{u}, \mathbf{v}] \\ &= -[\mathbf{u}, \mathbf{w}, \mathbf{v}] = -[\mathbf{v}, \mathbf{u}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}, \mathbf{u}] \end{aligned} \quad (1.1.27)$$

$$[\alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w}, \mathbf{d}] = \alpha [\mathbf{u}, \mathbf{w}, \mathbf{d}] + \beta [\mathbf{v}, \mathbf{w}, \mathbf{d}]$$

three vectors  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  are linearly independent if

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] \neq 0 \quad (1.1.28)$$

