me 338a
contiuum mechanics

lecture notes

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ME338A
CONTINUUM MECHANICS

lecture notes 01

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1 Tensor calculus

1.1 Tensor algebra

1.1.1 Vector algebra

Notation

• Einstein's summation convention

$$u_i = \sum_{j=1}^{3} A_{ij}\, x_j + b_i = A_{ij}\, x_j + b_i \tag{1.1.1}$$

summation over indices that appear twice in a term or symbol, with silent (dummy) index j and free index i, and thus

$$\begin{aligned} u_1 &= A_{11}\, x_1 + A_{12}\, x_2 + A_{13}\, x_3 + b_1 \\ u_2 &= A_{21}\, x_1 + A_{22}\, x_2 + A_{23}\, x_3 + b_2 \\ u_3 &= A_{31}\, x_1 + A_{32}\, x_2 + A_{33}\, x_3 + b_3 \end{aligned} \tag{1.1.2}$$

• Kronecker symbol δ_{ij}

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{1.1.3}$$

multiplication with Kronecker symbol corresponds to exchange of silent index with free index of Kronecker symbol

$$u_i = \delta_{ij}\, u_j \tag{1.1.4}$$

• permutation symbol $\overset{3}{e}_{ijk}$

$$\overset{3}{e}_{ijk} = \begin{cases} 1 & \text{for} \quad \{i,j,k\} \text{ ... even permutation} \\ -1 & \text{for} \quad \{i,j,k\} \text{ ... odd permutation} \\ 0 & \text{for} \quad \text{ ... else} \end{cases} \tag{1.1.5}$$

Euclidian vector space

• consider linear vector space $\mathcal{V}^3$ characterized through addition of its elements $\boldsymbol{u}, \boldsymbol{v}$ and multiplication with real scalars α, β

$$\alpha, \beta \in \mathcal{R} \qquad \mathcal{R} \text{ ... real numbers}$$
$$\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}^3 \qquad \mathcal{V}^3 \text{ ... linear vector space}$$

definition of linear vector space $\mathcal{V}^3$ through the following axioms

$$\begin{aligned} \alpha\,(\boldsymbol{u}+\boldsymbol{v}) &= \alpha\,\boldsymbol{u} + \alpha\,\boldsymbol{v} \\ (\alpha+\beta)\,\boldsymbol{u} &= \alpha\,\boldsymbol{u} + \beta\,\boldsymbol{u} \\ (\alpha\,\beta)\,\boldsymbol{u} &= \alpha\,(\beta\,\boldsymbol{u}) \end{aligned} \tag{1.1.6}$$

zero element and identity

$$0\,\boldsymbol{u} = \boldsymbol{0} \qquad 1\,\boldsymbol{u} = \boldsymbol{u} \tag{1.1.7}$$

linear independence of elements $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathcal{V}^3$ if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is the only (trivial) solution to

$$\alpha_i\,\boldsymbol{e}_i = 0 \tag{1.1.8}$$

• consider linear vector space $\mathcal{V}^3$ equipped with a norm $n(\boldsymbol{u})$ mapping elements of the linear vector space $\mathcal{V}^3$ to the space of real numbers $\mathcal{R}$

$$n: \ \mathcal{V}^3 \to \mathcal{R} \qquad \text{norm} \tag{1.1.9}$$

definition of norm through the following axioms

$$\begin{aligned}
& n(\boldsymbol{u}) \geq 0 \qquad\qquad n(\boldsymbol{u}) = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0} \\
& n(\alpha\, \boldsymbol{u}) = |\alpha|\, n(\boldsymbol{u}) \\
& n(\boldsymbol{u} + \boldsymbol{v}) \leq n(\boldsymbol{u}) + n(\boldsymbol{v}) \\
& n^2(\boldsymbol{u} + \boldsymbol{v}) + n^2(\boldsymbol{u} - \boldsymbol{v}) = 2\,[n^2(\boldsymbol{u}) + n^2(\boldsymbol{v})]
\end{aligned} \tag{1.1.10}$$

• consider Euclidian vector space $\mathcal{E}^3$ equipped with the Euclidian norm

$$n(\boldsymbol{u}) = ||\boldsymbol{u}|| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = [u_1^2 + u_2^2 + u_3^2]^{1/2} \tag{1.1.11}$$

mapping elements of the Euclidian vector space $\mathcal{E}^3$ to the space of real numbers $\mathcal{R}$

$$n: \ \mathcal{E}^3 \to \mathcal{R} \qquad \text{Euclidian norm} \tag{1.1.12}$$

representation of three–dimensional vector $\boldsymbol{a} \in \mathcal{E}^3$

$$\boldsymbol{a} = a_i\, \boldsymbol{e}_i = a_1\, \boldsymbol{e}_1 + a_2\, \boldsymbol{e}_2 + a_3\, \boldsymbol{e}_3 \tag{1.1.13}$$

with a_1, a_2, a_3 coordinates (components) of $\boldsymbol{a}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

$$\boldsymbol{a} = [\, a_1, a_2, a_3\,]^{\text{t}} \tag{1.1.14}$$

Scalar product

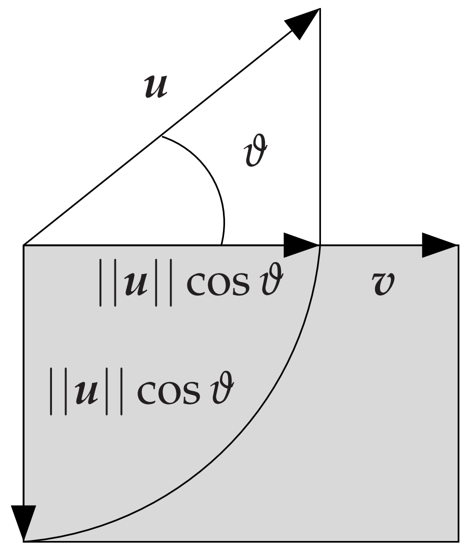
Euclidian norm enables the definition of scalar (inner) product between two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and introduces a scalar $\alpha \in \mathcal{R}$

$$\boldsymbol{u} \cdot \boldsymbol{v} = \alpha \tag{1.1.15}$$

geometric interpretation with $0 \leq \vartheta \leq \pi$ being the angle enclosed by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then $||\boldsymbol{u}|| \cos\vartheta$ can be interpreted as the projection of $\boldsymbol{u}$ onto the direction of $\boldsymbol{v}$ and

$$\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}|| \, ||\boldsymbol{v}|| \cos\vartheta$$

corresponds to the grey area in the picture



with the above interpretation with $0 \leq \vartheta \leq \pi$, obviously

$$||\boldsymbol{u} \cdot \boldsymbol{v}|| \leq ||\boldsymbol{u}|| \, ||\boldsymbol{v}|| \tag{1.1.16}$$

properties of scalar product

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= \boldsymbol{v} \cdot \boldsymbol{u} \\ (\alpha \, \boldsymbol{u} + \beta \, \boldsymbol{v}) \cdot \boldsymbol{w} &= \alpha \, (\boldsymbol{u} \cdot \boldsymbol{w}) + \beta \, (\boldsymbol{v} \cdot \boldsymbol{w}) \\ \boldsymbol{w} \cdot (\alpha \, \boldsymbol{u} + \beta \, \boldsymbol{v}) &= \alpha \, (\boldsymbol{w} \cdot \boldsymbol{u}) + \beta \, (\boldsymbol{w} \cdot \boldsymbol{v}) \end{aligned} \tag{1.1.17}$$

positive definiteness of scalar product

$$\boldsymbol{u} \cdot \boldsymbol{u} \geq 0, \quad \boldsymbol{u} \cdot \boldsymbol{u} = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0} \tag{1.1.18}$$

orthogonal vectors $\boldsymbol{u}$ and $\boldsymbol{v}$

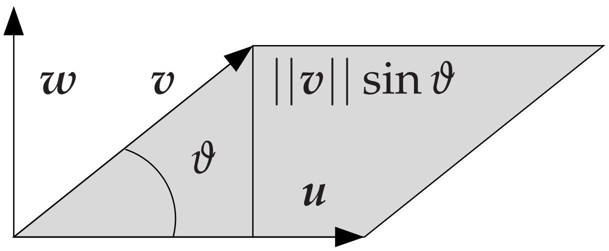
$$\boldsymbol{u} \cdot \boldsymbol{v} = 0 \quad \Leftrightarrow \quad \boldsymbol{u} \perp \boldsymbol{v} \tag{1.1.19}$$

Vector product

vector product of two vectors $\boldsymbol{u}, \boldsymbol{v}$ defines a new vector $\boldsymbol{w} \in \mathcal{E}^3$

$$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w} \tag{1.1.20}$$

geometric interpretation with $0 \le \vartheta \le \pi$ being the angle enclosed by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then $||\boldsymbol{v}||\,\sin\vartheta$ can be interpreted as the height of the grey polygon and



$$\boldsymbol{u} \times \boldsymbol{v} = ||\boldsymbol{u}||\;||\boldsymbol{v}||\;\sin\vartheta\,\boldsymbol{n}$$

introduces the vector $\boldsymbol{w}$ orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$ whereby its length corresponds to the grey area

with the above interpretation, obviously $\boldsymbol{u}$ parallel to $\boldsymbol{v}$ if

$$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0} \qquad \Leftrightarrow \qquad \boldsymbol{u} \parallel \boldsymbol{v} \tag{1.1.21}$$

index representation of $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_2\,v_3 & - & u_3\,v_2 \\ u_3\,v_1 & - & u_1\,v_3 \\ u_1\,v_2 & - & u_2\,v_1 \end{bmatrix} \tag{1.1.22}$$

properties of vector product

$$\begin{aligned} &\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u} \\ &(\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v}) \times \boldsymbol{w} = \alpha\,(\boldsymbol{u} \times \boldsymbol{w}) + \beta\,(\boldsymbol{v} \times \boldsymbol{w}) \\ &\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0 \\ &(\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{u} \cdot \boldsymbol{u})\,(\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})^2 \end{aligned} \tag{1.1.23}$$

Scalar triple vector product

scalar triple vector product of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ introduces a scalar $\alpha \in \mathcal{R}$

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \alpha \tag{1.1.24}$$

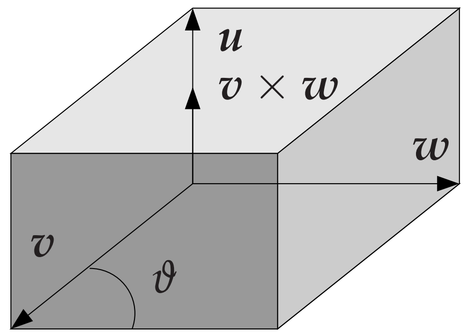
geometric interpretation
with vector product

$$\boldsymbol{v} \times \boldsymbol{w} = ||\boldsymbol{v}||\,||\boldsymbol{w}||\,\sin\vartheta\,\boldsymbol{n}$$

defining aera of ground surface

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$$

defines volume of parallelepiped



obviously

$$\begin{aligned} \alpha &= \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) \\ -\alpha &= \boldsymbol{u} \cdot (\boldsymbol{w} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{w}) = \boldsymbol{w} \cdot (\boldsymbol{v} \times \boldsymbol{u}) \end{aligned} \tag{1.1.25}$$

index representation of $\alpha = [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$

$$\alpha = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_3 w_1) \tag{1.1.26}$$

properties of scalar triple product

$$\begin{aligned} [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] &= [\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}] = [\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}] \\ &= -[\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}] = -[\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}] \\ [\alpha\boldsymbol{u} + \beta\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{d}] &= \alpha\,[\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{d}] + \beta\,[\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{d}] \end{aligned} \tag{1.1.27}$$

three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent if

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \neq 0 \tag{1.1.28}$$