3 ME338A - Homework

due 02/18/10, 12:15pm, 530-127

late homework can be dropped off in a box in front of durand 217, please mark clearly with date and time @drop off, we will take off 1/10 of points for each 24 hours late

3.1 Balance equations

Given the stress tensor $\boldsymbol{\sigma}^{\mathrm{t}}$ and the volume force vector $\boldsymbol{b}$ with

$$\boldsymbol{\sigma}^{\mathrm{t}} = [\,\boldsymbol{A} \cdot \boldsymbol{x}\,] \otimes \boldsymbol{x} \qquad \text{and} \qquad \boldsymbol{b} = -4\,\boldsymbol{A} \cdot \boldsymbol{x}$$

where $\boldsymbol{A}$ is a constant second order tensor and $\boldsymbol{x}$ is the position vector.

1.1 Show that $\boldsymbol{\sigma}^{\mathrm{t}}$ and $\boldsymbol{b}$ satisfy the balance of linear momentum $\mathrm{div}(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{b} = \boldsymbol{0}$.

1.2 Determine $\boldsymbol{A}$ such that the stress tensor satisfies the balance of angular momentum $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$.

3.2 Constitutive equations (isotropic)

In homework 2, you have calculated the linear and nonlinear cardiac strains based on measured marker coordinates.

Given the linear strains $\boldsymbol{\epsilon} = \epsilon_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ from homework 2,

$$[\,\epsilon_{ij}\,] = \begin{bmatrix} +0.100 & -0.121 & -0.067 \\ -0.121 & -0.100 & +0.066 \\ -0.067 & +0.066 & -0.014 \end{bmatrix}$$

determine the stress tensor $\boldsymbol{\sigma}$ assuming Hooke's law

$$\boldsymbol{\sigma} = 2\,\mu\boldsymbol{\epsilon} + \lambda\,[\,\boldsymbol{\epsilon} : \boldsymbol{I}\,]\,\boldsymbol{I}$$

in terms of the Lamé parameters $\lambda = 0.577\,\mathrm{N/mm^2}$ and $\mu = 0.385\,\mathrm{N/mm^2}$.

3.3 Constitutive equations (isotropic)

Given the constitutive relation according to Hooke's law,

$$\boldsymbol{\sigma} = 2\,\mu\boldsymbol{\epsilon} + \lambda\,[\,\boldsymbol{\epsilon} : \boldsymbol{I}\,]\,\boldsymbol{I}$$

show that if you wanted to calculate the cardiac strains for given cardiac stresses, you could use the following equation.

$$\boldsymbol{\epsilon} = \frac{1}{2\,\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\,\mu\,[2\,\mu + 3\,\lambda]}\,[\,\boldsymbol{\sigma} : \boldsymbol{I}\,]\,\boldsymbol{I}.$$

3.4 Constitutive equations (isotropic)

Sometimes it's inconvenient to use the Lamé parameters λ and μ. Engineers usually prefer Young's modulus E, Poisson's ratio ν, and the bulk modulus K.

$$E = \frac{\mu\,[3\,\lambda + 2\,\mu]}{\lambda + \mu} \qquad \nu = \frac{\lambda}{2\,[\lambda + \mu]} \qquad K = \lambda + \frac{2}{3}\,\mu$$

4.1 Reparameterize Hooke's law,

$$\boldsymbol{\sigma} = 2\,\mu\boldsymbol{\epsilon} + \lambda\,[\,\boldsymbol{\epsilon} : \boldsymbol{I}\,]\,\boldsymbol{I}$$

to show that

$$\boldsymbol{\epsilon} = \frac{1}{E}[[1+\nu]\boldsymbol{\sigma} - \nu[\sigma : \boldsymbol{I}]\boldsymbol{I}].$$

4.2 Biological tissues such as the heart are usually assumed to be incompressible. Show that for incompressible tissues with $k/E \to \infty$, Poisson's ratio tends to one half, $\nu \to 1/2$.

4.3 Show that for incompressible tissue with $\nu = 1/2$

$$\boldsymbol{\sigma} = 2\,\mu\boldsymbol{\epsilon} + \frac{1}{3}\,[\,\boldsymbol{\sigma} : \boldsymbol{I}\,]\,\boldsymbol{I}.$$

3.5 Constitutive equations (isotropic)

Given the stress tensor $\boldsymbol{\sigma} = \sigma_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ at a particular point $\boldsymbol{x}$ close to the transmural beat set in the heart wall,

$$[\,\sigma_{ij}\,] = \begin{bmatrix} 0.062 & -0.021 & -0.044 \\ -0.021 & -0.064 & 0.056 \\ -0.044 & 0.056 & -0.068 \end{bmatrix}$$

and two planar cuts S_1 and S_2 with unit normals

$$\boldsymbol{n}_1 = 1\,/\,\sqrt{2}\,\boldsymbol{e}_2 + 1\,/\,\sqrt{2}\,\boldsymbol{e}_3 \qquad \text{and} \qquad \boldsymbol{n}_2 = \boldsymbol{e}_1$$

5.1 Determine the stress vectors $\boldsymbol{t}_\sigma$ on S_1 and S_2.

5.2 Determine the normal stress vectors $\boldsymbol{\sigma}_n$ and the magnitude of the normal stress σ_n on S_1 and S_2.

5.3 Determine the shear stress vectors $\boldsymbol{\sigma}_t$ and the amount of shear stress τ_n on S_1 and S_2.

5.4 Determine the principal stresses $\lambda_{\sigma i}$ and the corresponding principal directions $\boldsymbol{n}_{\sigma i}$ for $i = 1, 2, 3$.

3.6 Constitutive equations (transversely isotropic)

Assume the heart is transversely isotropic with a measured muscle fiber direction $\boldsymbol{n}$ introducing the second order structural tensor $\boldsymbol{N} = \boldsymbol{n} \otimes \boldsymbol{n}$. Given the general representation of the free energy function ψ for a transversely isotropic material

$$\psi = \psi(\bar{I}_1^\epsilon, \bar{I}_2^\epsilon, \bar{I}_3^\epsilon, \bar{I}_4^\epsilon, \bar{I}_5^\epsilon)$$

expressed in terms of the five invariants $\bar{I}_1^\epsilon = \boldsymbol{\epsilon} : \boldsymbol{I}$, $\bar{I}_2^\epsilon = \boldsymbol{\epsilon}^2 : \boldsymbol{I}$, $\bar{I}_3^\epsilon = \boldsymbol{\epsilon}^3 : \boldsymbol{I}$, $\bar{I}_4^\epsilon = \boldsymbol{\epsilon} : \boldsymbol{N}$, and $\bar{I}_5^\epsilon = \boldsymbol{\epsilon}^2 : \boldsymbol{N}$.

6.1 Show that the stress tensor

$$\boldsymbol{\sigma} = \frac{\partial \psi(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}}$$

takes the following general representation.

$$\boldsymbol{\sigma} = c_1\, \boldsymbol{I} + c_2\, \boldsymbol{\epsilon} + c_3\, \boldsymbol{\epsilon}^2 + c_4\, \boldsymbol{N} + c_5\, [\boldsymbol{N} \cdot \boldsymbol{\epsilon}]^{\mathrm{sym}}$$

6.2 Determine the corresponding elasticity tensor

$$\mathbb{E} = \frac{\partial \psi^2(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}} = \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}}$$

and show that it is symmetric. For this problem, you can assume that $c_3 = 0$ and $c_5 = 0$.

hint: Rewrite the derivatives using the chain rule!

$$\frac{\partial \psi(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}} = \sum_i \frac{\partial \psi(\boldsymbol{\epsilon})}{\partial \bar{I}_i^{\epsilon}} \frac{\partial \bar{I}_i^{\epsilon}}{\partial \boldsymbol{\epsilon}}$$

3.7 Midterm preparation

For the midterm exam, you are allowed to use one letter format cheat sheet, either handwritten or printed. Prepare your notes as part of this homework and attach the original or a copy.

You can use MATLAB or any other computer program to solve the matrix and vector operations. If you choose to do so, you must deliver a printout of your code together with the homework.