3 Balance equations

Classical continuum mechanics of closed systems

in classical closed system continuum mechanics (here), \( r = 0 \) and \( \mathcal{R} = 0 \), such that the mass density \( \rho \) is constant in time, i.e. \( D_t (\rho v) = \rho D_t v \) and thus

\[
\rho D_t v = \text{div} (\sigma^t) + b
\]  (3.3.12)

the above equation is referred to as ‘Cauchy’s first equation of motion’, Cauchy [1827]

3.4 Balance of angular momentum

total angular momentum \( l \) of a body \( \mathcal{B} \)

\[
l := \int_{\mathcal{B}} x \times D_t u \, d m = \int_{\mathcal{B}} x \times v \, d m = \int_{\mathcal{B}} \rho x \times v \, dV \quad (3.4.1)
\]

angular momentum exchange of body \( \mathcal{B} \) with environment through momentum from contact forces \( l^\text{sur} \) and momentum from at-a-distance forces \( l^\text{vol} \)

\[
l^\text{sur} := \int_{\partial \mathcal{B}} x \times t_\sigma \, d A \quad l^\text{vol} := \int_{\mathcal{B}} x \times b \, dV \quad (3.4.2)
\]

with momentum due to contact/surface forces \( x \times t_\sigma = x \times \sigma^t \cdot n \) and volume forces \( x \times b \), assumption: no additional external torques

3.4.1 Global form of balance of angular momentum

“The time rate of change of the total angular momentum \( l \) of a body \( \mathcal{B} \) is balanced with the angular momentum exchange
due to contact momentum flux / surface force \( l^\text{sur} \) and due to the at-a-distance momentum exchange / volume force \( l^\text{vol} \).”

\[
D_t l = l^\text{sur} + l^\text{vol} \quad (3.4.3)
\]

and thus

\[
D_t \int_{\Bar{B}} \rho \mathbf{x} \times \mathbf{v} \, dV = \int_{\partial \Bar{B}} \mathbf{x} \times \mathbf{t}_\sigma \, dA + \int_{\Bar{B}} \mathbf{x} \times \mathbf{b} \, dV \quad (3.4.4)
\]

### 3.4.2 Local form of balance of angular momentum

modification of rate term \( D_t p \)

\[
D_t l = D_t \int_{\Bar{B}} \rho \mathbf{x} \times \mathbf{v} \, dV \overset{\text{fixed}}{=} \int_{\Bar{B}} D_t (\rho \mathbf{x} \times \mathbf{v}) \, dV \quad (3.4.5)
\]

modification of surface term \( l^\text{sur} \)

\[
l^\text{sur} = \int_{\partial \Bar{B}} \mathbf{x} \times \mathbf{t}_\sigma \, dA \overset{\text{Cauchy}}{=} \int_{\partial \Bar{B}} \mathbf{x} \times \sigma^t \cdot \mathbf{n} \, dA \overset{\text{Gauss}}{=} \int_{\Bar{B}} \text{div} (\mathbf{x} \times \sigma^t) \, dV
\]

\[
= \int_{\Bar{B}} \mathbf{x} \times \text{div} (\sigma^t) \, dV + \int_{\Bar{B}} \nabla \mathbf{x} \times \sigma^t \, dV
\]

and thus

\[
\int_{\Bar{B}} D_t (\rho \mathbf{x} \times \mathbf{v}) \, dV = \int_{\Bar{B}} \mathbf{x} \times \text{div} (\sigma^t) \, dV
\]

\[
+ \int_{\Bar{B}} \mathbf{I} \times \sigma^t \, dV + \int_{\Bar{B}} \mathbf{x} \times \mathbf{b} \, dV \quad (3.4.7)
\]

for arbitrary bodies \( \Bar{B} \rightarrow \) local form of ang. mom. balance

\[
D_t (\rho \mathbf{x} \times \mathbf{v}) = \mathbf{x} \times \text{div} (\sigma^t) + \mathbf{I} \times \sigma^t + \mathbf{x} \times \mathbf{b} \quad (3.4.8)
\]
3.4.3 Reduction with lower order balance equations

Modification with the help of lower order balance equations

\[ D_t (\rho \mathbf{x} \times \mathbf{v}) = D_t \mathbf{x} \times (\rho \mathbf{v}) + \mathbf{x} \times D_t (\rho \mathbf{v}) \]

\[ = \mathbf{x} \times \text{div} (\mathbf{\sigma}^t) + \mathbf{I} \times \mathbf{\sigma}^t + \mathbf{x} \times \mathbf{b} \]  

(3.4.9)

Position vector \( \mathbf{x} \) fixed, thus \( D_t \mathbf{x} = 0 \), with balance of linear momentum multiplied by position \( \mathbf{x} \)

\[ \mathbf{x} \times D_t (\rho \mathbf{v}) = \mathbf{x} \times \text{div} (\mathbf{\sigma}^t) + \mathbf{x} \times \mathbf{b} \]  

(3.4.10)

Local angular momentum balance in reduced format

\[ \mathbf{I} \times \mathbf{\sigma}^t = 3 \mathbf{e}: \mathbf{\sigma} = -2 \text{axl} (\mathbf{\sigma}) = 0 \quad \mathbf{\sigma} = \mathbf{\sigma}^t \]  

(3.4.11)

The above equation is referred to as ‘Cauchy’s second equation of motion’, Cauchy [1827]

In the absence of couple stresses, surface and body couples, the stress tensor is symmetric, \( \mathbf{\sigma} = \mathbf{\sigma}^t \), else: micropolar / Cosserat continua

Vector product of second order tensors \( \mathbf{A} \times \mathbf{B} = 3 \mathbf{e}: [\mathbf{A} \cdot \mathbf{B}^t] \)
3.5 Balance of energy

total energy $E$ of a body $\bar{B}$ as the sum of the kinetic energy $K$ and the internal energy $I$

$$
E := \int_{\bar{B}} e \, dV = \int_{\bar{B}} k + i \, dV = K + I
$$

$$
K := \int_{\bar{B}} k \, dV = \int_{\bar{B}} \frac{1}{2} \rho \, v \cdot v \, dV
$$

$$
I := \int_{\bar{B}} i \, dV
$$

energy exchange of body $\bar{B}$ with environment $e^{\text{sur}}$ and $e^{\text{vol}}$

$$
E^{\text{sur}} := \int_{\partial \bar{B}} v \cdot t_{\sigma} \, dA - \int_{\partial \bar{B}} q_n \, dA
$$

$$
E^{\text{vol}} := \int_{\bar{B}} v \cdot b \, dV + \int_{\partial \bar{B}} Q \, dV
$$

with contact heat flux $q_n = r \cdot n$ and heat source $Q$

3.5.1 Global form of balance of energy

"The time rate of change of the total energy $E$ of a body $\bar{B}$ is balanced with the energy exchange due to contact energy flux $E^{\text{sur}}$ and the at-a-distance energy exchange $E^{\text{vol}}$.”

$$
D_t E = E^{\text{sur}} + E^{\text{vol}}
$$

and thus

$$
D_t \int_{\bar{B}} e \, dV = \int_{\partial \bar{B}} v \cdot t_{\sigma} - q_n \, dA + \int_{\bar{B}} v \cdot b + Q \, dV
$$
3.5.2 Local form of balance of energy

modification of rate term $D_t E$

$$D_t E = D_t \int_{\bar{B}} e \, dV = \int_{\bar{B}} D_t e \, dV$$  \hspace{1cm} (3.5.5)

modification of surface term $E_{\text{sur}}$

$$E_{\text{sur}} = \int_{\partial \bar{B}} \left( v \cdot t_{\sigma} - q_n \right) \, dA$$  \hspace{1cm} (3.5.6)

and thus

$$\int_{\bar{B}} D_t e \, dV = \int_{\bar{B}} \text{div} \left( v \cdot \sigma^t - q \right) \, dV + \int_{\bar{B}} v \cdot b + Q \, dV$$ \hspace{1cm} (3.5.7)

for arbitrary bodies $\bar{B} \rightarrow$ local form of energy balance

$$D_t e = \text{div} \left( v \cdot \sigma^t - q \right) + v \cdot b + Q$$ \hspace{1cm} (3.5.8)

3.5.3 Reduction with lower order balance equations

modification with the help of lower order balance equations with

$$D_t e = D_t (k + i) = v D_t (\rho v) + D_t i$$

$$= \text{div} \left( v \cdot \sigma^t - q \right) + v \cdot b + Q$$ \hspace{1cm} (3.5.9)

with

$$D_t k = D_t (\frac{1}{2} \rho v \cdot v) = \frac{1}{2} D_t (\rho v) \cdot v + \frac{1}{2} v \cdot D_t (\rho v) = v \cdot D_t (\rho v)$$
with balance of momentum multiplied by velocity $v$

\[ v \, D_t (\rho \, v) = v \, \text{div} (\sigma^t) + v \, b = \text{div} (v \cdot \sigma^t) - \nabla v : \sigma^t + v \cdot b \]  

(3.5.11)

with stress power

\[ \nabla v : \sigma^t = \sigma^t : \nabla v = \sigma^t : \nabla D_t \varphi \]

(3.5.12)

\[ = \sigma^t : D_t \nabla \varphi = \sigma^t : D_t \nabla^{\text{sym}} \varphi = \sigma^t : D_t \varepsilon \]

local energy balance in reduced format / balance of int. energy

\[ D_t i = \sigma^t : D_t \varepsilon - \text{div} (q) + Q \]  

(3.5.13)

### 3.5.4 First law of thermodynamics

alternative definitions

\[ p^{\text{ext}} := \text{div} (v \cdot \sigma^t) + v \cdot b \ldots \text{external mechanical power} \]

\[ p^{\text{int}} := \sigma^t : \nabla v = \sigma^t : D_t \varepsilon \ldots \text{internal mechanical power} \]

\[ q^{\text{ext}} := \text{div} (-q) + Q \ldots \text{external thermal power} \]  

(3.5.14)

**Balance of total energy / first law of thermodynamics**

the rate of change of the total energy $e$, i.e. the sum of the kinetic energy $k$ and the potential energy $i$ is in equilibrium with the external mechanical power $p^{\text{ext}}$ and the external thermal power $q^{\text{ext}}$
\[ D_t e = D_t k + D_t i = p^{\text{ext}} + q^{\text{ext}} \] (3.5.15)

The above equation is typically referred to as "principle of interconvertibility of heat and mechanical work", Carnot [1832], Joule [1843], Duhem [1892] first law of thermodynamics does not provide any information about the direction of a thermodynamic process

**balance of kinetic energy**

\[ D_t k = p^{\text{ext}} - p^{\text{int}} \] (3.5.16)

The balance of kinetic energy is an alternative statement of the balance of linear momentum

**balance of internal energy**

\[ D_t i = q^{\text{ext}} + p^{\text{int}} \] (3.5.17)

Kinetic energy \( k \) and internal energy \( i \) are no conservation properties, they exchange the internal mechanical energy \( p^{\text{int}} \)