Volumetric–deviatoric decomposition

in analogy to the strain tensor $\varepsilon$, the stress tensor $\sigma$ can be additively decomposed into a volumetric part $\sigma^{\text{vol}}$ and a traceless deviatoric part $\sigma^{\text{dev}}$

volumetric – deviatoric decomposition of stress tensor $\sigma$

$$\sigma = \sigma^{\text{vol}} + \sigma^{\text{dev}}$$  (3.1.21)

with volumetric and deviatoric stress tensor $\sigma^{\text{vol}}$ and $\sigma^{\text{dev}}$

$$\text{tr}(\sigma^{\text{vol}}) = \text{tr}(\sigma) \quad \text{tr}(\sigma^{\text{dev}}) = 0$$  (3.1.22)

• volumetric second order tensor $\sigma^{\text{vol}}$

$$\sigma^{\text{vol}} = \frac{1}{3} [\sigma : I] I = \Pi^{\text{vol}} : \sigma$$  (3.1.23)

upon double contraction volumetric fourth order unit tensor $\Pi^{\text{vol}}$ extracts volumetric part $\sigma^{\text{vol}}$ of stress tensor

$$\Pi^{\text{vol}} = \frac{1}{3} I \otimes I$$

$$\Pi^{\text{vol}} = \frac{1}{3} \delta_{ij} \delta_{kl} e_i \otimes e_j \otimes e_k \otimes e_l$$  (3.1.24)

• deviatoric second order tensor $\sigma^{\text{dev}}$

$$\sigma^{\text{dev}} = \sigma - \frac{1}{3} [\sigma : I] I = \Pi^{\text{dev}} : \sigma$$  (3.1.25)

upon double contraction deviatoric fourth order unit tensor $\Pi^{\text{dev}}$ extracts deviatoric part of stress tensor

$$\Pi^{\text{dev}} = \Pi^{\text{sym}} - \Pi^{\text{vol}} = \Pi^{\text{sym}} - \frac{1}{3} I \otimes I$$

$$\Pi^{\text{dev}} = \left[ \frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{3} \delta_{ij} \delta_{kl} \right] e_i \otimes e_j \otimes e_k \otimes e_l$$  (3.1.26)
Volumetric stress

volumetric part $\sigma^{\text{vol}}$ of stress tensor $\sigma$

$$\sigma^{\text{vol}} = \frac{1}{3} [\sigma : I] I = \frac{1}{3} [I \otimes I] : \sigma = \mathbf{I}^{\text{vol}} : \sigma$$  \hspace{1cm} (3.1.27)

interpretation of trace as hydrostatic pressure

$$p = \frac{1}{3} \text{tr}(\sigma) = \frac{1}{3} \sigma : I = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})$$  \hspace{1cm} (3.1.28)

index representation

$$\sigma^{\text{vol}} = \sigma_{ij}^{\text{vol}} e_i \otimes e_j$$  \hspace{1cm} (3.1.29)

matrix representation of coordinates $[\sigma_{ij}^{\text{vol}}]$

$$[\sigma_{ij}^{\text{vol}}] = p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad p = \frac{1}{3} \text{tr}(\sigma)$$  \hspace{1cm} (3.1.30)

volumetric stress tensor $\sigma^{\text{vol}}$ is a spherical second order tensor as $\sigma^{\text{vol}} = p \mathbf{I}$

volumetric stress tensor $\sigma^{\text{vol}}$ contains the hydrostatic pressure part of the total stress tensor $\sigma$
Deviatoric stress

deviatoric stress tensor \( \sigma^{\text{dev}} \) preserves the volume and contains the remaining part of the total stress tensor \( \sigma \)
deviatoric part \( \sigma^{\text{dev}} \) of the stress tensor \( \sigma \)

\[
\sigma^{\text{dev}} = \sigma - \sigma^{\text{vol}} = \sigma - \frac{1}{3} [\sigma : I] I = \mathbb{I}^{\text{dev}} : \sigma
\]  

(3.1.31)

index representation

\[
\sigma^{\text{dev}} = \sigma_{ij}^{\text{dev}} e_i \otimes e_j
\]  

(3.1.32)

matrix representation of coordinates \( [\sigma_{ij}^{\text{dev}}] \)

\[
[\sigma_{ij}^{\text{dev}}] = \frac{1}{3} \begin{bmatrix}
2\sigma_{11} - \sigma_{22} - \sigma_{33} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & 2\sigma_{22} - \sigma_{11} - \sigma_{33} & \sigma_{13} \\
\sigma_{31} & \sigma_{32} & 2\sigma_{33} - \sigma_{11} - \sigma_{22}
\end{bmatrix}
\]  

(3.1.33)

trace of deviatoric stresss \( \text{tr} (\sigma^{\text{dev}}) \)

\[
\text{tr} (\sigma^{\text{dev}}) = \frac{1}{3} [2\sigma_{11} - \sigma_{22} - \sigma_{33}] \\
+ \frac{1}{3} [2\sigma_{22} - \sigma_{11} - \sigma_{33}] \\
+ \frac{1}{3} [2\sigma_{33} - \sigma_{11} - \sigma_{22}] = 0
\]  

(3.1.34)

deviatoric stress tensor \( \sigma^{\text{dev}} \) is a traceless second order tensor as \( \text{tr} (\sigma^{\text{dev}}) = 0 \)
deviatoric stress tensor \( \sigma^{\text{dev}} \) contains the hydrostatic pressure free part of the total stress tensor \( \sigma \)
Normal–shear decomposition

assume we are interested in the stress \( \sigma_n \) normal to a particular plane characterized through its normal \( n \), i.e. the normal projection of the stress vector \( t_\sigma \nabla \)

\[
\sigma_n = t_\sigma \cdot n = [\sigma^t \cdot n] \cdot n = \sigma^t : [n \otimes n] = \sigma^t : N \quad (3.1.35)
\]

normal–shear (tangential) decomposition of stress vector \( t_\sigma \)

\[
t_\sigma = \sigma_n + \sigma_t \quad (3.1.36)
\]

normal stress vector – stress in direction of \( n \)

\[
\sigma_n = [\sigma^t : [n \otimes n]] n = \sigma^t : [n \otimes n \otimes n] \quad (3.1.37)
\]

shear (tangential) stress vector – stress in the plane

\[
\sigma_t = t_\sigma - \sigma_n = \sigma^t \cdot n - \sigma^t : [n \otimes n \otimes n] \\
= \sigma^t : [\mathbb{I}^{\text{sym}} \cdot n - n \otimes n \otimes n] = \sigma^t : T \quad (3.1.38)
\]

amount of shear stress \( \tau_n \)

\[
||\tau_n||^2 = (t_\sigma - \sigma_n) \cdot (t_\sigma - \sigma_n) = t_\sigma \cdot t_\sigma - 2t_\sigma \cdot \sigma_n + \sigma_n^2 n \cdot n \\
\quad (3.1.39)
\]

and thus

\[
\tau_n = ||\sigma_t|| = \sqrt{\sigma_t \cdot \sigma_t} = \sqrt{t_\sigma \cdot t_\sigma - \sigma_n^2} \quad (3.1.40)
\]

in general, i.e. for an arbitrary direction \( n \), we have normal and shear contributions to the stress vector, however, three particular directions \( \{n_{\sigma i}\}_{i=1,2,3} \) can be identified, for which \( t_\sigma = \sigma_n \) and thus \( \sigma_t = 0 \), the corresponding \( \{n_{\sigma i}\}_{i=1,2,3} \) are called principal stress directions and \( \{\sigma_{n i}\}_{i=1,2,3} = \{\lambda_{\sigma i}\}_{i=1,2,3} \) are the principal stresses
**Principal stresses**

assume stress tensor $\sigma^t$ to be known at $x \in B$, principal stresses $\{\lambda_{\sigma i}\}_{i=1,2,3}$ and principal stress directions $\{n_{\sigma i}\}_{i=1,2,3}$ can be derived from solution of special eigenvalue problem according to §1.1.3

\[
\sigma^t \cdot n_{\sigma i} = \lambda_{\sigma i} n_{\sigma i} \quad [\sigma^t - \lambda_{\sigma i}] \cdot n_{\sigma i} = 0 \quad (3.1.41)
\]
solution

\[
\det (\sigma^t - \lambda_{\sigma} I) = 0 \quad (3.1.42)
\]
or in terms of roots of characteristic equation

\[
\lambda_{\sigma}^3 - I_{\sigma} \lambda_{\sigma}^2 + II_{\sigma} \lambda_{\sigma} - III_{\sigma} = 0 \quad (3.1.43)
\]
roots of characteristic equation in terms of principal invariants of $\sigma^t$

\[
I_{\sigma} = \text{tr} (\sigma^t) = \lambda_{\sigma 1} + \lambda_{\sigma 2} + \lambda_{\sigma 3}
\]
\[
II_{\sigma} = \frac{1}{2} [\text{tr}^2 (\sigma^t) - \text{tr} (\sigma^t)] = \lambda_{\sigma 2} \lambda_{\sigma 3} + \lambda_{\sigma 3} \lambda_{\sigma 1} + \lambda_{\sigma 1} \lambda_{\sigma 2}
\]
\[
III_{\sigma} = \det (\sigma^t) = \lambda_{\sigma 1} \lambda_{\sigma 2} \lambda_{\sigma 3} \quad (3.1.44)
\]
spectral representation of $\sigma$

\[
\sigma^t = \sum_{i=1}^{3} \lambda_{\sigma i} n_{\sigma i} \otimes n_{\sigma i} \quad (3.1.45)
\]
principal stresses $\lambda_{\sigma i}$ are purely normal, no shear stress $\tau_n$ in principal directions, i.e. $t_{\sigma i} = \sigma_n = \lambda_{\sigma i} n_{\sigma i}$ and $\sigma_t = 0$ thus $\tau_n = 0$

due to symmetry of stresses $\sigma = \sigma^t$, stress tensor posseses three real eigenvalues $\{\lambda_{\sigma i}\}_{i=1,2,3}$, corresponding eigendirections $\{n_{\sigma i}\}_{i=1,2,3}$ are thus orthogonal $n_{\sigma i} \cdot n_{\sigma j} = \delta_{ij}$
Special case of plane stress

dimensional reduction in case of plane stress with vanishing stresses $\sigma_{13} = \sigma_{23} = \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$ in out of plane direction, e.g. for flat sheets

$$\sigma = \sigma_{ij} e_i \otimes e_j$$

matrix representation of coordinates $[\sigma_{ij}]$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Voigt representation of stress

three dimensional second order stress tensor $\sigma$

$$\sigma = \sigma_{ij} e_i \otimes e_j$$

matrix representation of coordinates $[\sigma_{ij}]$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

due to symmetry $[\sigma_{ij}] = [\sigma_{ji}]$ and thus $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{31} = \sigma_{13}$, stress tensor $\sigma$ contains only six independent components $\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{12},\sigma_{23},\sigma_{31}$, it proves convenient to represent second order tensor $\sigma$ through a vector $\vec{\sigma}$

$$\vec{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^t$$

vector representation $\vec{\sigma}$ of stress $\sigma$ in case of plane stress

$$\vec{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}]^t$$
3.1.3 Concept of heat flux

the contact heat flux $q_n$ at a point $x$ is a scalar of the unit [energy/time/surface area]

the contact heat flux $q_n$ characterizes the energy transport normal to the tangent plane to an imaginary surface passing through this point with normal vector $n$

![Diagram showing contact heat flux](image)

definition of contact heat flux $q_n$ in analogy to Cauchy’s postulate, lemma and theorem originally introduced for the momentum flux in §3.1.2

**Cauchy’s postulate**

$$q_n = q_n(x, n)$$ (3.1.52)

**Cauchy’s lemma**

$$q_n(x, n) = -q_n(x, -n)$$ (3.1.53)

**Cauchy’s theorem**

the contact heat flux $q_n$ can be expressed as linear function of the surface normal $n$ and the heat flux vector $q$

$$q_n = q \cdot n$$ (3.1.54)
Heat flux vector

the vector field $q$ is called heat flux vector

$$ q = q_i e_i \quad (3.1.55) $$

Cauchy’s theorem

$$ q_n = q \cdot n \quad (3.1.56) $$

index representation

$$ q_n = (q_i e_i) \cdot (n_j e_j) = q_i n_j \delta_{ij} = q_i n_i \quad (3.1.57) $$

generic interpretation

the coordinates $q_i$ characterize the heat energy transport through the planes parallel to the coordinate planes

in continuum mechanics of adiabatic systems the heat flux vector vanishes identically