ME338A
CONTINUUM MECHANICS

lecture notes 07

tuesday, january 27th, 2009
3 Balance equations

3.1 Basic ideas

• until now:
  kinematics, i.e. characterization of deformation of a material body \( B \) without studying its physical cause

• now:
  balance equations, i.e. general statements that characterize the cause of cause of the motion of any body \( B \)
basic strategy

• isolation of an arbitrary subset $\bar{B}$ of the body $B$

• characterization of the influence of the remaining body $B \setminus \bar{B}$ on $\bar{B}$ through phenomenological quantities, i.e. the contact mass flux $r$, the contact stress $t_\sigma$, the contact heat flux $q$

• definition of basic physical quantities, i.e. the mass $m$, the linear momentum $I$, the moment of momentum $D$ and the energy $E$ of subset $\bar{B}$

• postulate of balance of these quantities renders global balance equations for subset $\bar{B}$

• localization of global balance equations renders local balance equations at point $x \in \bar{B}$
3.1.1 Concept of mass flux

the contact mass flux \( r_n \) at a point \( x \) is a scalar of the unit [mass/time/surface area]

the contact mass flux \( r_n \) characterizes the transport of matter normal to the tangent plane to an imaginary surface passing through this point with normal vector \( n \)

definition of contact heat flux \( q_n \) in analogy to Cauchy’s postulate, lemma and theorem originally introduced for the momentum flux in §3.1.2

**Cauchy’s postulate**

\[
   r_n = r_n(x, n) \quad (3.1.1)
\]

**Cauchy’s lemma**

\[
   r_n(x, n) = -r_n(x, -n) \quad (3.1.2)
\]

**Cauchy’s theorem**

the contact mass flux \( r_n \) can be expressed as linear function of the surface normal \( n \) and the mass flux vector \( r \)

\[
   r_n = r \cdot n \quad (3.1.3)
\]
Mass flux vector

the vector field $\mathbf{r}$ is called mass flux vector

$$\mathbf{r} = r_i \mathbf{e}_i$$  \hspace{1cm} (3.1.4)

Cauchy’s theorem

$$r_n = \mathbf{r} \cdot \mathbf{n}$$  \hspace{1cm} (3.1.5)

index representation

$$r_n = (r_i \mathbf{e}_i) \cdot (n_j \mathbf{e}_j) = r_i n_j \delta_{ij} = r_i n_i$$  \hspace{1cm} (3.1.6)

gameometric interpretation

the coordinates $r_i$ characterize the transport of matter through the planes parallel to the coordinate planes

in classical closed system continuum mechanics (here) the mass flux vector vanishes identically

examples of mass flux: transport of chemical reactants in chemomechanics or cell migration in biomechanics
3.1.2 Concept of stress

traction vector

\[ t_\sigma = \lim_{\Delta a \to 0} \frac{\Delta f}{\Delta a} = \frac{df}{da} \]  \hspace{1cm} (3.1.7)

interpretation as surface force per unit surface area

**Cauchy’s postulate**

the traction vector \( t_\sigma \) at a point \( x \) can be expressed exclusively in terms of the point \( x \) and the normal \( n \) to the tangent plane to an imaginary surface passing through this point

traction vector

\[ t_\sigma = t_\sigma(x, n) \]  \hspace{1cm} (3.1.8)

**Cauchy’s lemma**

the traction vectors acting on opposite sides of a surface are equal in magnitude and opposite in sign

\[ t_{\sigma_1}(x, n_1) = -t_{\sigma_2}(x, n_2) \]  \hspace{1cm} (3.1.9)
generalization with $n = n_1 = -n_2$ and $t_\sigma = t_{\sigma 1}$

$$t_\sigma (x, n) = -t_\sigma (x, -n) \quad (3.1.10)$$

**Cauchy's theorem**

the traction vector $t_\sigma$ can be expressed as a linear map of the surface normal $n$ mapped via the transposed stress tensor $\sigma^t$

$$t_\sigma = \sigma^t \cdot n \quad (3.1.11)$$

accordingly with $n = n_1 = -n_2$ and $t_\sigma = t_{\sigma 1}$

$$t_{\sigma 1} = \sigma^t \cdot n_1 = \sigma^t \cdot n = t_\sigma$$
$$t_{\sigma 2} = \sigma^t \cdot n_2 = -\sigma^t \cdot n = -t_\sigma \quad (3.1.12)$$

Cauchy tetraeder

balance of momentum (pointwise)

$$t_\sigma (n) \, da = -t_\sigma (n_i) \, da_i = t_\sigma (e_i) \, da_i = t_{\sigma i} \, da_i \quad (3.1.13)$$

surface theorem, area fractions from Gauss theorem

$$nda = -n_i da_i = e_i da_i \quad \frac{da_i}{da} = e_i \cdot n = \cos \angle (e_i, n) \quad (3.1.14)$$
traction vector as linear map of surface normal

\[ t_\sigma(n) = t_\sigma \frac{da_i}{da} = t_\sigma i \cos \angle(e_i, n) = t_\sigma i [e_i \cdot n] = [t_\sigma i \otimes e_i] \cdot n \]  

(3.1.15)

compare \( t_\sigma(n) = \sigma^t \cdot n \)

interpretation of second order stress tensor as \( \sigma^t = t_\sigma i \otimes e_i \)

**Stress tensor**

Cauchy stress (true stress)

\[ \sigma^t = t_\sigma i \otimes e_i = \sigma_{ji} e_j \otimes e_i \quad \sigma = e_i \otimes t_\sigma i = \sigma_{ij} e_i \otimes e_j \]  

(3.1.16)

Cauchy theorem

\[ t_\sigma = \sigma^t \cdot n \]  

(3.1.17)

index representation

\[ t_\sigma = \sigma_{ji} e_j \otimes e_i \cdot n_k e_k = \sigma_{ji} n_k \delta_{ik} e_j = \sigma_{ji} n_i e_j = t_j e_j \]  

(3.1.18)
matrix representation of tensor coordinates of $\sigma_{ij}$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} t_{\sigma_1}^t \\ t_{\sigma_2}^t \\ t_{\sigma_3}^t \end{bmatrix} \quad (3.1.19)$$

diagram: geometric interpretation

with traction vectors on surfaces

$$t_{\sigma_1} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \end{bmatrix}^t$$

$$t_{\sigma_2} = \begin{bmatrix} \sigma_{21} & \sigma_{22} & \sigma_{23} \end{bmatrix}^t \quad (3.1.20)$$

$$t_{\sigma_3} = \begin{bmatrix} \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}^t$$

first index ... surface normal
second index ... direction (coordinate of traction vector)
diagonal entries ... normal stresses
non–diagonal entries .. shear stresses