2.3.4 Volumetric–deviatoric decomposition

A material volume element can deform volumetrically and deviatorically, volumetric deformation conserves the shape (i.e. no changes in angles, no sliding) while deviatoric (isochoric) deformation conserves the volume.

Volumetric–deviatoric decomposition of strain tensor $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \varepsilon^\text{vol} + \varepsilon^\text{dev} \quad (2.3.24)$$

With volumetric and deviatoric strain tensor $\varepsilon^\text{vol}$ and $\varepsilon^\text{dev}$

$$\text{tr}(\varepsilon^\text{vol}) = \text{tr}(\varepsilon) \quad \text{tr}(\varepsilon^\text{dev}) = 0 \quad (2.3.25)$$

- **Volumetric second order tensor $\boldsymbol{\varepsilon}^\text{vol}$**

  $$\varepsilon^\text{vol} = \frac{1}{3} [\varepsilon : \mathbf{I}] \mathbf{I} = \mathbf{II}^\text{vol} : \varepsilon \quad (2.3.26)$$

  Upon double contraction, volumetric fourth order unit tensor $\mathbf{II}^\text{vol}$ extracts volumetric part $\varepsilon^\text{vol}$ of strain tensor

  $$\mathbf{II}^\text{vol} = \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \quad (2.3.27)$$

- **Deviatoric second order tensor $\boldsymbol{\varepsilon}^\text{dev}$**

  $$\varepsilon^\text{dev} = \varepsilon - \frac{1}{3} [\varepsilon : \mathbf{I}] \mathbf{I} = \mathbf{II}^\text{dev} : \varepsilon \quad (2.3.28)$$

  Upon double contraction, deviatoric fourth order unit tensor $\mathbf{II}^\text{dev}$ extracts deviatoric part of strain tensor

  $$\mathbf{II}^\text{dev} = \mathbf{II}^\text{sym} - \mathbf{II}^\text{vol} = \mathbf{II}^\text{sym} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \quad (2.3.29)$$
Volumetric strain

volumetric deformation is characterized through the volume dilatation \( e \in \mathcal{R} \), i.e. difference of deformed volume and original volume \( dv - dV \) scaled by original volume \( dV \)

\[
e = \frac{dv - dV}{dV} = (1 + \epsilon_{11})(1 + \epsilon_{22})(1 + \epsilon_{33}) - 1 = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + \mathcal{O}(\epsilon_{ij}^2)
\]

neglection of higher order terms: trace of strain tensor \( \text{tr} (\epsilon) = e : I \in \mathcal{R} \) as characteristic measure for volume changes

\[
e = \text{div} \ u = \nabla u : I = e : I = \text{tr} (\epsilon)
\]

volumetric part \( \epsilon^{\text{vol}} \) of strain tensor \( \epsilon \)

\[
\epsilon^{\text{vol}} = \frac{1}{3} e I = \frac{1}{3} [e : I] I = \frac{1}{3} [I \otimes I] e = \mathbb{II}^{\text{vol}} : e
\]

index representation

\[
\epsilon^{\text{vol}} = \epsilon_{ij}^{\text{vol}} e_i \otimes e_j
\]

matrix representation of coordinates \([\epsilon_{ij}^{\text{vol}}]\)

\[
[\epsilon_{ij}^{\text{vol}}] = \frac{1}{3} e \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
e = \text{tr}(\epsilon)
\]

• incompressibility is characterized through \( \text{div} \ u = 0 \)

• volumetric strain tensor \( \epsilon^{\text{vol}} \) is a spherical second order tensor as \( \epsilon^{\text{vol}} = \frac{1}{3} e I \)

• volumetric strain tensor \( \epsilon^{\text{vol}} \) contains the volume changing, shape preserving part of the total strain tensor \( \epsilon \)
Deviatoric strain

deviatoric strain tensor $\epsilon^{\text{dev}}$ preserves the volume and contains the remaining part of the total strain tensor $\epsilon$

deviatoric part $\epsilon^{\text{dev}}$ of the strain tensor $\epsilon$

$$\epsilon^{\text{dev}} = \epsilon - \epsilon^{\text{vol}} = \epsilon - \frac{1}{3} [\epsilon : I] I = \mathbb{I}^{\text{dev}} : \epsilon$$  \hfill (2.3.35)

index representation

$$\epsilon^{\text{dev}} = \epsilon_{ij}^{\text{dev}} e_i \otimes e_j$$  \hfill (2.3.36)

matrix representation of coordinates $[\epsilon_{ij}^{\text{dev}}]$

$$[\epsilon_{ij}^{\text{dev}}] = \frac{1}{3} \begin{bmatrix} 2\epsilon_{11} - \epsilon_{22} - \epsilon_{33} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & 2\epsilon_{22} - \epsilon_{11} - \epsilon_{33} & \epsilon_{13} \\ \epsilon_{31} & \epsilon_{32} & 2\epsilon_{33} - \epsilon_{11} - \epsilon_{22} \end{bmatrix}$$  \hfill (2.3.37)

trace of deviatoric strains $\text{tr} (\epsilon^{\text{dev}})$

$$\text{tr} (\epsilon^{\text{dev}}) = \frac{1}{3} [2\epsilon_{11} - \epsilon_{22} - \epsilon_{33}]$$

$$+ \frac{1}{3} [2\epsilon_{22} - \epsilon_{11} - \epsilon_{33}]$$

$$+ \frac{1}{3} [2\epsilon_{33} - \epsilon_{11} - \epsilon_{22}] = 0$$  \hfill (2.3.38)

- deviatoric strain tensor $\epsilon^{\text{dev}}$ is a traceless second order tensor as $\text{tr} (\epsilon^{\text{dev}}) = 0$

- deviatoric strain tensor $\epsilon^{\text{dev}}$ contains the shape changing, volume preserving part of the total strain tensor $\epsilon$
Volumetric–deviatoric decomposition

- examples of purely volumetric deformation

\[ \epsilon^{\text{vol}} = \frac{1}{3}[\epsilon : I] I = \mathbb{I}^{\text{vol}} : \epsilon \quad \text{tr}(\epsilon^{\text{vol}}) = \text{tr}(\epsilon) \]  

\[ e > 0 \text{ and } \epsilon^{\text{dev}} = 0 \quad e < 0 \text{ and } \epsilon^{\text{dev}} = 0 \]

- examples of purely deviatoric deformation

\[ \epsilon^{\text{dev}} = \epsilon - \frac{1}{3}[\epsilon : I] I = \mathbb{I}^{\text{dev}} : \epsilon \quad \text{tr}(\epsilon^{\text{dev}}) = 0 \]  

\[ e = 0 \text{ and } \epsilon^{\text{dev}} \neq 0 \quad e = 0 \text{ and } \epsilon^{\text{dev}} \neq 0 \]
2.3.5 Strain vector

assume we are interested in strain on a plane characterized through its normal \( n \), strain vector \( t_\varepsilon \) acting on plane given through normal projection of strain tensor \( \varepsilon \)

\[
t_\varepsilon = \varepsilon \cdot n \tag{2.3.41}
\]

index representation

\[
t_\varepsilon = (\varepsilon_{ij} e_i \otimes e_j) \cdot (n_k e_k)
\]

\[
= \varepsilon_{ij} n_k \delta_{jk} e_i = \varepsilon_{ij} n_j e_i = t_{\varepsilon i} e_i
\tag{2.3.42}
\]

representation of coordinates \([t_{\varepsilon i}]\)

\[
\begin{bmatrix}
t_{\varepsilon 1} \\
t_{\varepsilon 2} \\
t_{\varepsilon 3}
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_{11} n_1 + \varepsilon_{12} n_2 + \varepsilon_{13} n_3 \\
\varepsilon_{21} n_1 + \varepsilon_{22} n_2 + \varepsilon_{23} n_3 \\
\varepsilon_{31} n_1 + \varepsilon_{32} n_2 + \varepsilon_{33} n_3
\end{bmatrix}
\tag{2.3.43}
\]

alternative interpretation: assume we are interested in strains along a particular material direction, i.e. the stretch of a fiber at \( x \in \mathcal{B} \) characterized through its normal \( n \) with \( ||n|| = 1 \)

stretch as change of displacement vector \( u \) in the direction of \( n \) given through the Gateaux derivative \( \frac{\partial u(x)}{\partial \varepsilon} \)

\[
D u(x) \cdot n = \left. \frac{d}{d\varepsilon} u(x + \varepsilon n) \right|_{\varepsilon=0}
= \left. \nabla u(x + \varepsilon n) \cdot n \right|_{\varepsilon=0} = \nabla u(x) \cdot n
\tag{2.3.44}
\]

recall that \( \nabla u = \nabla^\text{sym} u + \nabla^\text{skw} u = \varepsilon + \omega \) whereby rotation \( \omega = \nabla^\text{skw} u \) does not induce strain, thus

\[
t_\varepsilon = \nabla^\text{sym} u \cdot n = \varepsilon \cdot n
\tag{2.3.45}
\]
2.3.6 Normal–shear decomposition

assume we are interested in strain along a particular fiber characterized through its normal \( n \), stretch of fiber \( \epsilon_n \) given through normal projection of strain vector \( t_\epsilon \)

\[
\epsilon_n = t_\epsilon \cdot n \tag{2.3.46}
\]

alternative interpretation: stretch of a line element can be understood as the projection of change of displacement in the direction of \( n \) as \( Du \cdot n = \nabla u \cdot n \) onto the direction \( n \)

\[
\epsilon_n = n \cdot \nabla u \cdot n = n \cdot \epsilon \cdot n = \epsilon : [n \otimes n] \tag{2.3.47}
\]

normal-shear (tangential) decomposition of strain vector \( t_\epsilon \)

\[
t_\epsilon = \epsilon_n + \epsilon_t \tag{2.3.48}
\]

normal strain vector – stretch of fibers in direction of \( n \)

\[
\epsilon_n = \epsilon : [n \otimes n] n \tag{2.3.49}
\]

shear (tangential) strain vector – sliding of fibers parallel to \( n \)

\[
\epsilon_t = t_\epsilon - \epsilon_n = \epsilon : [\Pi^{\text{sym}} \cdot n - n \otimes n \otimes n] \tag{2.3.50}
\]

amount of sliding \( \gamma_n \)

\[
\gamma_n = 2 ||\epsilon_t|| = 2 \sqrt{\epsilon_t \cdot \epsilon_t} = 2 \sqrt{t_\epsilon \cdot t_\epsilon - \epsilon_n^2} \tag{2.3.51}
\]

in general, i.e. for an arbitrary direction \( n \), we have normal and shear contributions to the strain vector, however, three particular directions \( \{n_{\epsilon i}\}_{i=1,2,3} \) can be identified, for which \( t_\epsilon = \epsilon_n \) and thus \( \epsilon_t = 0 \), the corresponding \( \{n_{\epsilon i}\}_{i=1,2,3} \) are called principal strain directions and \( \{\epsilon_{ni}\}_{i=1,2,3} = \{\lambda_{ei}\}_{i=1,2,3} \) are the principal strains or stretches
2.3.7 Principal strains – stretches

assume strain tensor \( \varepsilon \) to be known at \( x \in B \), principal strains \( \{ \lambda_{ei} \}_{i=1,2,3} \) and principal strain directions \( \{ n_{ei} \}_{i=1,2,3} \) can be derived from solution of special eigenvalue problem according to §1.1.3

\[
\varepsilon \cdot n_{ei} = \lambda_{ei} n_{ei} \quad [\varepsilon - \lambda_{ei}] \cdot n_{ei} = 0
\]  

(2.3.52)
solution

\[
\det (\varepsilon - \lambda I) = 0
\]  

(2.3.53)

or in terms of roots of characteristic equation

\[
\lambda^{3} - I \lambda^{2} + II \lambda - III = 0
\]  

(2.3.54)

roots of characteristic equations in terms of principal invariants of \( \varepsilon \)

\[
I_{\varepsilon} = \text{tr} (\varepsilon) = \lambda_{e1} + \lambda_{e2} + \lambda_{e3}
\]

\[
II_{\varepsilon} = \frac{1}{2} [\text{tr}^{2}(\varepsilon) - \text{tr}(\varepsilon^{2})] = \lambda_{e2}\lambda_{e3} + \lambda_{e3}\lambda_{e1} + \lambda_{e1}\lambda_{e2}
\]

\[
III_{\varepsilon} = \det (\varepsilon) = \lambda_{e1}\lambda_{e2}\lambda_{e3}
\]  

(2.3.55)
spectral representation of \( \varepsilon \)

\[
\varepsilon = \sum_{i=1}^{3} \lambda_{ei} n_{ei} \otimes n_{ei}
\]  

(2.3.56)

principal strains (stretches) \( \lambda_{ei} \) are purely normal, no shear deformation (sliding) \( \gamma_{n} \) in principal directions, i.e. \( t_{ei} = \varepsilon_{n} = \lambda_{ei} n_{ei} \) and \( \varepsilon_{t} = 0 \) thus \( \gamma_{n} = 0 \)

due to symmetry of strains \( \varepsilon = \varepsilon^{t} \), strain tensor possesses three real eigenvalues \( \{ \lambda_{ei} \}_{i=1,2,3} \), corresponding eigendirections \( \{ n_{ei} \}_{i=1,2,3} \) are thus orthogonal \( n_{ei} \cdot n_{ej} = \delta_{ij} \)
2.3.8 Compatibility

until now, we have assumed the displacement field \( u(x,t) \) to be given, such that the strain field \( \epsilon = \nabla^\text{sym} u \) could have been derived uniquely as partial derivative of \( u \) with respect to the position \( x \) at fixed time \( t \)

assume now, that for a given strain field \( \epsilon(x,t) \), we want to know whether these strains \( \epsilon \) are compatible with a continuous single–valued displacement field \( u \)

symmetric second order incompatibility tensor

\[
\eta = \text{crl}(\text{crl}(\epsilon)) \quad \text{(2.3.57)}
\]

index representation of incompatibility tensor

\[
\eta = \eta_{ij} e_i \otimes e_j = \epsilon_{ikm} e_{kn,ml} e_{jl} \quad \text{(2.3.58)}
\]

coordinate representation of compatibility condition

\[
\epsilon_{kl,nn} + \epsilon_{mn,kl} - \epsilon_{ml,kn} - \epsilon_{kn,ml} = 0 \quad \text{(2.3.59)}
\]

valid \( \forall k,l,m,n \), thus 81 equations which are partly redundant, six independent conditions

St. Venant compatibility conditions

\[
\eta_{11} = \epsilon_{22,33} + \epsilon_{33,22} - 2\epsilon_{23,32} = 0
\]

\[
\eta_{22} = \epsilon_{33,11} + \epsilon_{11,33} - 2\epsilon_{31,13} = 0
\]

\[
\eta_{33} = \epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,21} = 0
\]

\[
\eta_{12} = \epsilon_{13,32} + \epsilon_{23,31} - \epsilon_{33,12} - \epsilon_{12,33} = 0
\]

\[
\eta_{23} = \epsilon_{21,13} + \epsilon_{31,12} - \epsilon_{11,23} - \epsilon_{23,11} = 0
\]

\[
\eta_{31} = \epsilon_{32,21} + \epsilon_{12,23} - \epsilon_{22,31} - \epsilon_{31,22} = 0
\]

incompatible displacement field, e.g. in dislocation theory
2.3.9 Special case of plane strain

dimensional reduction in case of plane strain with vanishing strains \( \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{31} = \varepsilon_{32} = \varepsilon_{33} = 0 \) in out of plane direction, e.g. in geomechanics

\[
\varepsilon = \varepsilon_{ij} e_i \otimes e_j
\]  
(2.3.61)

matrix representation of coordinates \([\varepsilon_{ij}]\)

\[
[\varepsilon_{ij}] = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & 0 \\
\varepsilon_{21} & \varepsilon_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  
(2.3.62)

2.3.10 Voigt representation of strain

three dimensional second order strain tensor \( \varepsilon \)

\[
\varepsilon = \varepsilon_{ij} e_i \otimes e_j
\]  
(2.3.63)

matrix representation of coordinates \([\varepsilon_{ij}]\)

\[
[\varepsilon_{ij}] = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
\]  
(2.3.64)

due to symmetry \([\varepsilon_{ij}] = [\varepsilon_{ji}]\) and thus \( \varepsilon_{12} = \varepsilon_{21}, \varepsilon_{23} = \varepsilon_{32}, \varepsilon_{31} = \varepsilon_{13}, \) strain tensor \( \varepsilon \) contains only six independent components \( \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31}, \) it proves convenient to represent second order tensor \( \varepsilon \) through a vector \( \underline{\varepsilon} \)

\[
\underline{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{23}, 2\varepsilon_{31}]^t
\]  
(2.3.65)

vector representation \( \underline{\varepsilon} \) of strain \( \varepsilon \) in case of plane strain

\[
\underline{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}]^t
\]  
(2.3.66)