3 ME338A - Homework

due 02/24/09, 12pm

3.1 Balance equations

Given the stress tensor $\boldsymbol{\sigma}^{\mathrm{t}}$ and the volume force vector $\boldsymbol{b}$ with

$$\boldsymbol{\sigma}^{\mathrm{t}} = [\,\boldsymbol{A}\cdot\boldsymbol{x}\,]\otimes\boldsymbol{x} \qquad \text{and} \qquad \boldsymbol{b} = -4\,\boldsymbol{A}\cdot\boldsymbol{x}$$

where $\boldsymbol{A}$ is a constant second order tensor and $\boldsymbol{x}$ is the position vector.

1.1 Show that $\boldsymbol{\sigma}^{\mathrm{t}}$ and $\boldsymbol{b}$ satisfy the balance of linear momentum $\operatorname{div}(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{b} = \boldsymbol{0}$.

1.2 Determine $\boldsymbol{A}$ such that the stress tensor satisfies the balance of angular momentum $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$.

3.2 Constitutive equations (isotropic)

In homework 2, you have calculated the linear and nonlinear cardiac strains based on measured marker coordinates. Given the linear strains $\boldsymbol{\epsilon} = \epsilon_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ from problem 9,

$$[\,\epsilon_{ij}\,] = \begin{bmatrix} +0.100 & -0.023 & -0.063 \\ -0.023 & -0.063 & +0.080 \\ -0.063 & +0.080 & -0.064 \end{bmatrix} \tag{3.2.1}$$

determine the stress tensor $\boldsymbol{\sigma}$ assuming Hooke's law

$$\boldsymbol{\sigma} = 2\,\mu\boldsymbol{\epsilon} + \lambda\,[\,\boldsymbol{\epsilon} : \boldsymbol{I}\,]\,\boldsymbol{I}$$

in terms of the Lamé parameters $\lambda = 0.577\,\mathrm{N/mm^2}$ and $\mu = 0.385\,\mathrm{N/mm^2}$.

3.3 Constitutive equations (isotropic)

Given the constitutive relation according to Hooke's law,

$$\boldsymbol{\sigma} = 2\,\mu\boldsymbol{\epsilon} + \lambda\,[\,\boldsymbol{\epsilon} : \boldsymbol{I}\,]\,\boldsymbol{I}$$

show that if you wanted to calculate the cardiac strains for given cardiac stresses, you could use the following equation.

$$\boldsymbol{\epsilon} = \frac{1}{2\,\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\,\mu\,[2\,\mu + 3\,\lambda]}\,[\,\boldsymbol{\sigma} : \boldsymbol{I}\,]\,\boldsymbol{I}.$$

3.4 Constitutive equations (isotropic)

Sometimes it's inconvenient to use the Lamé parameters λ and μ. Engineers usually prefer Young's modulus E, Poisson's ratio ν, and the bulk modulus K.

$$E = \frac{\mu\,[3\,\lambda + 2\,\mu]}{\lambda + \mu} \qquad \nu = \frac{\lambda}{2\,[\lambda + \mu]} \qquad K = \lambda + \frac{2}{3}\,\mu$$

4.1 Reparameterize Hooke's law,

$$\boldsymbol{\sigma} = 2\,\mu\boldsymbol{\epsilon} + \lambda\,[\,\boldsymbol{\epsilon} : \boldsymbol{I}\,]\,\boldsymbol{I}$$

to show that

$$\boldsymbol{\epsilon} = \frac{1}{E}[[1+\nu]\boldsymbol{\sigma} - \nu[\sigma : \boldsymbol{I}]\boldsymbol{I}].$$

4.2 Biological tissues such as the heart are usually assumed to be incompressible. Show that for incompressible tissues with $k/E \to \infty$, Poisson's ratio tends to one half, $\nu \to 1/2$.

4.3 Show that for incompressible tissue with $\nu = 1/2$

$$\boldsymbol{\sigma} = 2\,\mu\boldsymbol{\epsilon} + \frac{1}{3}\,[\,\boldsymbol{\sigma} : \boldsymbol{I}\,]\,\boldsymbol{I}.$$

3.5 Constitutive equations (isotropic)

Given the stress tensor $\boldsymbol{\sigma} = \sigma_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ at a particular point $\boldsymbol{x}$ close to the transmural beat set in the heart wall,

$$[\,\sigma_{ij}\,] = \begin{bmatrix} 0.062 & -0.018 & -0.049 \\ -0.018 & -0.064 & 0.061 \\ -0.049 & 0.061 & -0.065 \end{bmatrix}$$

and two planar cuts S_1 and S_2 with unit normals

$$\boldsymbol{n}_1 = 1\,/\,\sqrt{2}\,\boldsymbol{e}_2 + 1\,/\,\sqrt{2}\,\boldsymbol{e}_3 \qquad \text{and} \qquad \boldsymbol{n}_2 = \boldsymbol{e}_1$$

cut with a scalpel as Wolf has shown in class.

5.1 Determine the stress vectors $\boldsymbol{t}_\sigma$ on S_1 and S_2.

5.2 Determine the normal stress vectors $\boldsymbol{\sigma}_n$ and the magnitude of the normal stress σ_n on S_1 and S_2.

5.3 Determine the shear stress vectors $\boldsymbol{\sigma}_t$ and the amount of shear stress τ_n on S_1 and S_2.

5.4 Determine the principal stresses $\lambda_{\sigma i}$ and the corresponding principal directions $\boldsymbol{n}_{\sigma i}$ for $i = 1, 2, 3$.

3.6 Constitutive equations (transversely isotropic)

Assume the heart is transversely isotropic with a measured muscle fiber direction $\boldsymbol{n}$ introducing the second order structural tensor $\boldsymbol{N} = \boldsymbol{n} \otimes \boldsymbol{n}$. Given the general representation of

the free energy function ψ for a transversely isotropic material

$$\psi = \psi(\bar{I}_1^\epsilon, \bar{I}_2^\epsilon, \bar{I}_3^\epsilon, \bar{I}_4^\epsilon, \bar{I}_5^\epsilon)$$

expressed in terms of the five invariants $\bar{I}_1^\epsilon = \boldsymbol{\epsilon} : \boldsymbol{I}$, $\bar{I}_2^\epsilon = \boldsymbol{\epsilon}^2 : \boldsymbol{I}$, $\bar{I}_3^\epsilon = \boldsymbol{\epsilon}^3 : \boldsymbol{I}$, $\bar{I}_4^\epsilon = \boldsymbol{\epsilon} : \boldsymbol{N}$, and $\bar{I}_5^\epsilon = \boldsymbol{\epsilon}^2 : \boldsymbol{N}$.

6.1 Show that the stress tensor

$$\boldsymbol{\sigma} = \frac{\partial \psi(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}}$$

takes the following general representation.

$$\boldsymbol{\sigma} = c_1\, \boldsymbol{I} + c_2\, \boldsymbol{\epsilon} + c_3\, \boldsymbol{\epsilon}^2 + c_4\, \boldsymbol{N} + c_5\, [\boldsymbol{N} \cdot \boldsymbol{\epsilon}]^{\text{sym}}$$

6.2 Determine the corresponding elasticity tensor and show that it is symmetric. For this problem, you can assume that $c_3 = 0$ and $c_5 = 0$.

$$\mathbb{E} = \frac{\partial \psi^2(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}} = \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}}$$

hint: Rewrite the derivatives using the chain rule!

$$\frac{\partial \psi(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}} = \sum_i \frac{\partial \psi(\boldsymbol{\epsilon})}{\partial \bar{I}_i^\epsilon} \frac{\partial \bar{I}_i^\epsilon}{\partial \boldsymbol{\epsilon}}$$

3.7 Midterm preparation

For the midterm exam, you are allowed to use one letter format page of notes, either handwritten or printed. Prepare your notes as part of this homework.