3 ME338A - Homework

due 02/24/09, 12pm

3.1 Balance equations

Given the stress tensor $\sigma^t$ and the volume force vector $b$ with

$$\sigma^t = [A \cdot x] \otimes x$$

and

$$b = -4A \cdot x$$

where $A$ is a constant second order tensor and $x$ is the position vector.

1.1 Show that $\sigma^t$ and $b$ satisfy the balance of linear momentum

$$\text{div}(\sigma^t) + b = 0.$$ 

1.2 Determine $A$ such that the stress tensor satisfies the balance of angular momentum $\sigma = \sigma^t$.

3.2 Constitutive equations (isotropic)

In homework 2, you have calculated the linear and nonlinear cardiac strains based on measured marker coordinates. Given the linear strains $\epsilon = \epsilon_{ij} e_i \otimes e_j$ from problem 9,

$$[\epsilon_{ij}] = \begin{bmatrix} +0.100 & -0.023 & -0.063 \\ -0.023 & -0.063 & +0.080 \\ -0.063 & +0.080 & -0.064 \end{bmatrix}$$

(3.2.1)

determine the stress tensor $\sigma$ assuming Hooke’s law

$$\sigma = 2\mu \epsilon + \lambda [\epsilon : I] I$$
in terms of the Lamé parameters $\lambda = 0.577$ N/mm$^2$ and $\mu = 0.385$ N/mm$^2$.

### 3.3 Constitutive equations (isotropic)

Given the constitutive relation according to Hooke’s law,

$$\sigma = 2\mu\epsilon + \lambda \left[ \epsilon : I \right] I$$

show that if you wanted to calculate the cardiac strains for given cardiac stresses, you could use the following equation.

$$\epsilon = \frac{1}{2\mu} \sigma - \frac{\lambda}{2\mu [2\mu + 3\lambda]} \left[ \sigma : I \right] I.$$

### 3.4 Constitutive equations (isotropic)

Sometimes it’s inconvenient to use the Lamé parameters $\lambda$ and $\mu$. Engineers usually prefer Young’s modulus $E$, Poisson’s ratio $\nu$, and the bulk modulus $K$.

$$E = \frac{\mu [3\lambda + 2\mu]}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2[\lambda + \mu]}, \quad K = \lambda + \frac{2}{3}\mu$$

4.1 Reparameterize Hooke’s law,

$$\sigma = 2\mu\epsilon + \lambda \left[ \epsilon : I \right] I$$

to show that

$$\epsilon = \frac{1}{E}[[1 + \nu]\sigma - \nu[\sigma : I]I].$$

4.2 Biological tissues such as the heart are usually assumed to be incompressible. Show that for incompressible tissues with $k/E \to \infty$, Poisson’s ratio tends to one half, $\nu \to 1/2$. 
4.3 Show that for incompressible tissue with $\nu = 1/2$

$$\sigma = 2\mu \varepsilon + \frac{1}{3} [\sigma : I] I.$$ 

3.5 Constitutive equations (isotropic)

Given the stress tensor $\sigma = \sigma_{ij} e_i \otimes e_j$ at a particular point $x$ close to the transmural beat set in the heart wall,

$$[\sigma_{ij}] = \begin{bmatrix}
0.062 & -0.018 & -0.049 \\
-0.018 & -0.064 & 0.061 \\
-0.049 & 0.061 & -0.065
\end{bmatrix}$$

and two planar cuts $S_1$ and $S_2$ with unit normals

$$n_1 = 1 / \sqrt{2} e_2 + 1 / \sqrt{2} e_3$$

and

$$n_2 = e_1$$
cut with a scalpel as Wolf has shown in class.

5.1 Determine the stress vectors $t_\sigma$ on $S_1$ and $S_2$.

5.2 Determine the normal stress vectors $\sigma_n$ and the magnitude of the normal stress $\sigma_n$ on $S_1$ and $S_2$.

5.3 Determine the shear stress vectors $\sigma_t$ and the amount of shear stress $\tau_n$ on $S_1$ and $S_2$.

5.4 Determine the principal stresses $\lambda_{\sigma i}$ and the corresponding principal directions $n_{\sigma i}$ for $i = 1, 2, 3$.

3.6 Constitutive equations (transversely isotropic)

Assume the heart is transversely isotropic with a measured muscle fiber direction $n$ introducing the second order structural tensor $N = n \otimes n$. Given the general representation of
the free energy function $\psi$ for a transversely isotropic material

$$\psi = \psi(\bar{I}_1^e, \bar{I}_2^e, \bar{I}_3^e, \bar{I}_4^e, \bar{I}_5^e)$$

expressed in terms of the five invariants $\bar{I}_1^e = \epsilon : I$, $\bar{I}_2^e = \epsilon^2 : I$, $\bar{I}_3^e = \epsilon^3 : I$, $\bar{I}_4^e = \epsilon : N$, and $\bar{I}_5^e = \epsilon^2 : N$.

6.1 Show that the stress tensor

$$\sigma = \frac{\partial \psi(\epsilon)}{\partial \epsilon}$$

takes the following general representation.

$$\sigma = c_1 I + c_2 \epsilon + c_3 \epsilon^2 + c_4 N + c_5 [N \cdot \epsilon]^\text{sym}$$

6.2 Determine the corresponding elasticity tensor and show that it is symmetric. For this problem, you can assume that $c_3 = 0$ and $c_5 = 0$.

$$\mathbf{IE} = \frac{\partial \psi^2(\epsilon)}{\partial \epsilon \otimes \epsilon} = \frac{\partial \sigma(\epsilon)}{\partial \epsilon}$$

hint: Rewrite the derivatives using the chain rule!

$$\frac{\partial \psi(\epsilon)}{\partial \epsilon} = \sum_i \frac{\partial \psi(\epsilon)}{\partial \bar{I}_i^e} \frac{\partial \bar{I}_i^e}{\partial \epsilon}$$

3.7 Midterm preparation

For the midterm exam, you are allowed to use one letter format page of notes, either handwritten or printed. Prepare your notes as part of this homework.