ME338A
CONTINUUM MECHANICS

lecture notes 10

thursday, may 01, 2008

3 Balance equations

Basic ideas

• until now: kinematics, i.e., general statements that characterize deformation of a material body $\mathcal{B}$ without studying its physical cause

• now: balance equations, i.e., general statements that characterize the cause of motion of material body $\mathcal{B}$

Basic strategy / recipe

[1] isolation of arbitrary subset $\mathcal{P}_\mathcal{S}$ of the body $\mathcal{S}$

[2] characterization of the influence of the remaining body $\mathcal{B}\backslash\mathcal{P}_\mathcal{S}$ on $\mathcal{P}_\mathcal{S}$ through phenomenological quantities, i.e., the contact stress $\boldsymbol{t}$ and the contact heat flux q

[3] definition of basic physical quantities, i.e., mass $\mathcal{M}$, linear momentum I, moment of momentum D, and energy E of subset $\mathcal{P}_\mathcal{S}$

[4] postulate of balance of these quantities renders global balance equations for subset $\mathcal{P}_\mathcal{S}$

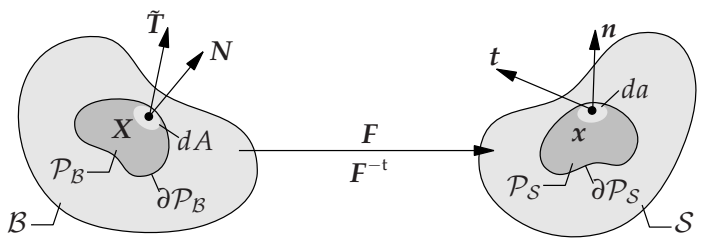
[5] localization of global balance equations renders local balance equations at point $\boldsymbol{x} \in \mathcal{P}_\mathcal{S}$

3.1 Concept of stress

Consider a part $\mathcal{P}_\mathcal{B} \subset \mathcal{B}$ cut off from the reference body $\mathcal{B}$ and its spatial counterpart $\mathcal{P}_\mathcal{S} \subset \mathcal{S}$ closed by the respective boundaries $\partial\mathcal{P}_\mathcal{B}$ and $\partial\mathcal{P}_\mathcal{S}$. In the deformed configuration, we introduce the traction vector $\boldsymbol{t} \in T_x\mathcal{S}$ such that

$$\boldsymbol{t} = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{f}}{\Delta a} = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}a} \tag{3.1.1}$$

that acts on the surface element $\mathrm{d}a$ of $\partial\mathcal{P}_\mathcal{S}$ and represents the mechanical action $\mathrm{d}\boldsymbol{f}$ of the rest of the body at the vicinity $\mathcal{P}_\mathcal{S} \setminus \mathcal{S}$ on $\partial\mathcal{P}_\mathcal{S}$.



Cauchy's postulate

The traction vector $\boldsymbol{t}$ at point $\boldsymbol{x}$ and time t can be expressed exclusively in terms of the point $\boldsymbol{x}$, the time t, and the normal $\boldsymbol{n} \in T_x^*\mathcal{S}$ to the tangent plane to an imaginary surface $\partial\mathcal{P}_\mathcal{S}$ passing through this point.

$$\boldsymbol{t} = \boldsymbol{t}\,(\boldsymbol{x}, t; \boldsymbol{n}) \tag{3.1.2}$$

Cauchy's lemma

The traction vectors acting on opposite sides of a surface are

equal in magnitude and opposite in sign.

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x}, -\boldsymbol{n}) \tag{3.1.3}$$

Cauchy's theorem

The spatial traction vector $\boldsymbol{t}$ depends linearly on the spatial normal $\boldsymbol{n}$ to the surface $\partial\mathcal{P}_\mathcal{S}$ through the Cauchy or true stress tensor $\boldsymbol{\sigma}$,

$$\boldsymbol{t}(\boldsymbol{x}, t; \boldsymbol{n}) := \boldsymbol{\sigma}(\boldsymbol{x}, t) \cdot \boldsymbol{n} \tag{3.1.4}$$

The Cauchy stress tensor $\boldsymbol{\sigma}$ can thus be understood as a mapping transforming normals $\boldsymbol{n} \in T_x^*\mathcal{S}$ onto tangent vectors $\boldsymbol{t} \in T_x\mathcal{S}$.

$$\boldsymbol{\sigma} := \begin{cases} T_x^*\mathcal{S} \;\to\; T_x\mathcal{S}\,, \\ \quad \boldsymbol{n} \;\mapsto\; \boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \end{cases} \tag{3.1.5}$$

Cauchy tetrahedron

The above represenation can be illustrated in terms of the Cauchy tetrahedron. The equilibrium of all traction vectors acting on the tetrahedron yields the following expression.

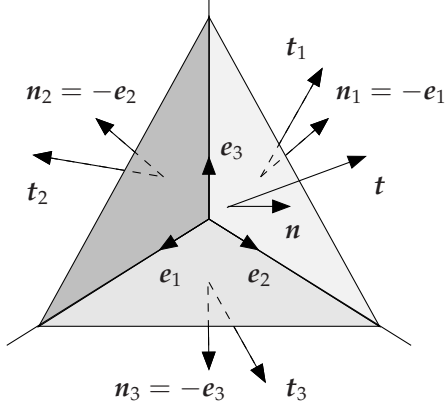
$$\boldsymbol{t}(\boldsymbol{n})\,\mathrm{d}a = -\boldsymbol{t}(\boldsymbol{n}_i)\,\mathrm{d}a_i = \boldsymbol{t}(\boldsymbol{e}_i)\,\mathrm{d}a_i = \boldsymbol{t}_i\,\mathrm{d}a_i \tag{3.1.6}$$

With the help of the surface theorem, the area fractions can be related as follows.

$$\boldsymbol{n}\mathrm{d}a = -\boldsymbol{n}_i\mathrm{d}a_i = \boldsymbol{e}_i\mathrm{d}a_i \qquad \frac{\mathrm{d}a_i}{\mathrm{d}a} = \boldsymbol{e}_i \cdot \boldsymbol{n} = \cos\angle(\boldsymbol{e}_i, \boldsymbol{n}) \tag{3.1.7}$$

The traction vectors $\boldsymbol{t}$ then take the natural interpretation as a linear map of the corresponding surface normals $\boldsymbol{n}$.

$$\boldsymbol{t}(\boldsymbol{n}) = \boldsymbol{t}_i\frac{\mathrm{d}a_i}{\mathrm{d}a} = \boldsymbol{t}_i\cos\angle(\boldsymbol{e}_i, \boldsymbol{n}) = \boldsymbol{t}_i\,[\boldsymbol{e}_i \cdot \boldsymbol{n}] = [\boldsymbol{t}_i \otimes \boldsymbol{e}_i] \cdot \boldsymbol{n} \tag{3.1.8}$$



A comparison with $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$ yields the interpretation of second order stress tensor as $\boldsymbol{\sigma} = \boldsymbol{t}_i \otimes \boldsymbol{e}_i$. For Barry, we can express the Cauchy stress as $\boldsymbol{\sigma} = \boldsymbol{t}_i \otimes \boldsymbol{e}_j = \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and rewrite the Cauchy theorem (3.1.4) in index representation.

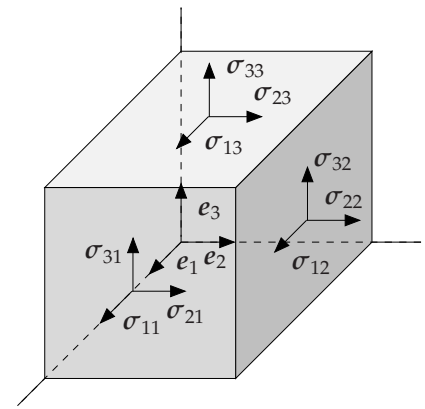
$$\boldsymbol{t} = \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot n_k\, \boldsymbol{e}_k = \sigma_{ij}\, n_k\, \delta_{jk}\, \boldsymbol{e}_i = \sigma_{ij}\, n_j\, \boldsymbol{e}_i = t_i\, \boldsymbol{e}_i. \quad (3.1.9)$$

The matrix representation of tensor coordinates of σ_{ij} can then be expressed as

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{t}_1^{\mathrm{t}} \\ \boldsymbol{t}_2^{\mathrm{t}} \\ \boldsymbol{t}_3^{\mathrm{t}} \end{bmatrix} \quad (3.1.10)$$

giving rise to the following geometric interpretation in terms of the traction vectors on the tetrahedral surfaces.

$$\begin{aligned} \boldsymbol{t}_1 &= [\ \sigma_{11}\ \ \sigma_{12}\ \ \sigma_{13}\ ]^{\mathrm{t}} \\ \boldsymbol{t}_2 &= [\ \sigma_{21}\ \ \sigma_{22}\ \ \sigma_{23}\ ]^{\mathrm{t}} \\ \boldsymbol{t}_3 &= [\ \sigma_{31}\ \ \sigma_{32}\ \ \sigma_{33}\ ]^{\mathrm{t}} \end{aligned} \quad (3.1.11)$$



In our notation, the first index of the stress tensor represents the direction of the corresponding traction vector and the second index indicates the surface normal. The diagonal entries of the matrix of the stress components represent the normal stresses, the off-diagonal terms are related to the shear stress. The pressure p corrensponds to the average of all normal stresses.

$$p = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}) = \tfrac{1}{3}\,[\,\sigma_{11} + \sigma_{22} + \sigma_{33}\,] \quad (3.1.12)$$

Kirchhoff stress

Another spatial stress measure, the Kirchhoff stress tensor, also known as the weighted Cauchy stress tensor, is defined as

$$\boldsymbol{\tau} := J\, \boldsymbol{\sigma} \quad (3.1.13)$$

and widely used in the spatial description of stress power terms in the reference volume.

First Piola Kirchhoff stress

Now let us consider another spatial traction vector $\boldsymbol{T} \in T_x\mathcal{S}$ defined through the force equality $\boldsymbol{T}\, dA := \boldsymbol{t}\, da$ by scaling the spatial force term $(\boldsymbol{t}\, da)$ through the reference area element dA. Based on this definition, we introduce the Piola stress, sometimes also referred to as the first Piola-Kirchhoff stress, by the reference Cauchy theorem

$$\boldsymbol{T} := \boldsymbol{P} \cdot \boldsymbol{N} \tag{3.1.14}$$

leading to $\boldsymbol{P} \cdot \boldsymbol{N}\, \mathrm{d}A = \boldsymbol{\sigma} \cdot \boldsymbol{n}\, \mathrm{d}a$. Using the area map, we obtain the relation

$$\boldsymbol{P} = \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathrm{t}} = J\, \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-\mathrm{t}} \tag{3.1.15}$$

between the Piola stress and the spatial Cauchy stress and Kirchhoff stress. Notice that similar to the deformation gradient $\boldsymbol{F}$, $\boldsymbol{P}$ is a two point tensor. It possesses the following geometrical mapping properties.

$$\boldsymbol{P} := \begin{cases} T_X^*\mathcal{B} \;\rightarrow\; T_x\mathcal{S}\,, \\ \quad \boldsymbol{N} \;\mapsto\; \boldsymbol{T} = \boldsymbol{P} \cdot \boldsymbol{N}. \end{cases} \tag{3.1.16}$$

Piola transform / Piola identity

The transformation

$$(\bullet)_{\mathcal{B}} = J(\bullet)_{\mathcal{S}} \cdot \boldsymbol{F}^{-\mathrm{t}} \tag{3.1.17}$$

is called the Piola transform. It can be used to transform the spatial Cauchy stress tensor into the two point Piola stress. It is widely employed in transforming the objects acting on a spatial surface onto their material counterparts. The immediate outcome of the Piola transformation is the Piola identity that also implies the equality $\mathrm{Div}\,(J\boldsymbol{F}^{-\mathrm{t}}) = \boldsymbol{0}$.

$$J\,\mathrm{div}(\bullet)_{\mathcal{S}} = \mathrm{Div}\,(\bullet)_{\mathcal{B}} = \mathrm{Div}\,(J(\bullet)_{\mathcal{S}} \cdot \boldsymbol{F}^{-\mathrm{t}}) \tag{3.1.18}$$

Second Piola Kirchhoff stress

The Lagrangean stress vector $\tilde{\boldsymbol{T}} \in T_X\mathcal{B}$ may be defined through the pull-back of the spatial stress vector $\boldsymbol{T} \in T_x\mathcal{S}$.

$$\tilde{\boldsymbol{T}} = \varphi^*(\boldsymbol{t}) = \boldsymbol{F}^{-1} \cdot \boldsymbol{T} \in T_X\mathcal{B} \quad , \quad \tilde{T}^A = (F^{-1})^A{}_a T^a$$

The last fundamental stress measure we will introduce for now is the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$, which is defined by

$$\tilde{\boldsymbol{T}} := \boldsymbol{S} \cdot \boldsymbol{N} \tag{3.1.19}$$

yielding

$$\boldsymbol{S} := \begin{cases} T_X^*\mathcal{B} \;\rightarrow\; T_X\mathcal{B}\,, \\ \quad \boldsymbol{N} \;\mapsto\; \tilde{\boldsymbol{T}} = \boldsymbol{S} \cdot \boldsymbol{N}\,. \end{cases} \tag{3.1.20}$$

The second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ does not possess a real physical interpretation, yet it is often used in computational applications since unlike $\boldsymbol{P}$, it has a symmetric structure and is thus easy to store computationally.

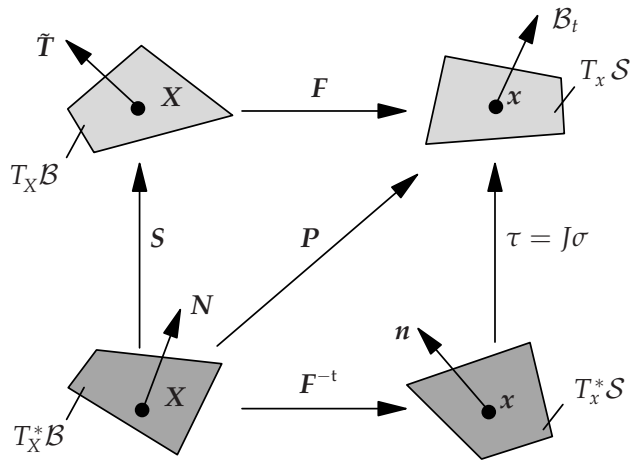
Pull back and push forward

We can express the second Piola-Kirchhoff stress tensor in terms of the other stress tensors

$$\begin{aligned} \boldsymbol{S} := \boldsymbol{\varphi}^*(\boldsymbol{P}) &= \boldsymbol{F}^{-1} \cdot \boldsymbol{P} \quad , \quad S^{AB} = (F^{-1})^A{}_a P^{aB}\,, \\ \boldsymbol{S} := \boldsymbol{\varphi}^*(\boldsymbol{\tau}) &= \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathrm{t}} \quad , \quad S^{AB} = (F^{-1})^A{}_a \tau^{ab} (F^{-1})^B{}_b \end{aligned}$$

as the pull-back of the two-point and spatial objects. Apparently the converse push-forward relations do also hold for the spatial stress tensors as shown in the diagram below.

$$\boldsymbol{\tau} = J\,\boldsymbol{\sigma} = \boldsymbol{\varphi}_*(\boldsymbol{P}) = \boldsymbol{P} \cdot \boldsymbol{F}^{\mathrm{t}} \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\varphi}_*(\boldsymbol{S}) = \boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^{\mathrm{t}}\,,$$



Physical interpretation of stress measures

Let $\mathrm{d}\boldsymbol{f}$ be a force element in the spatial configuration $\mathcal{S}$

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{t}\,\mathrm{d}a = \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}a \tag{3.1.21}$$

such that

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{\sigma}\cdot\mathrm{d}\boldsymbol{a}. \tag{3.1.22}$$

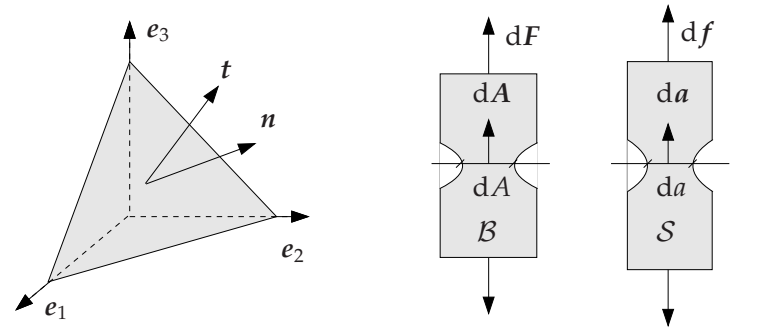
The Cauchy stress $\boldsymbol{\sigma}$ relates a force element $\mathrm{d}\boldsymbol{f}$ of the spatial configuration to a surface element $\mathrm{d}\boldsymbol{a}$ of the spatial configuration, it is thus a purely spatial quantity.

Let $\mathrm{d}\boldsymbol{f}$ be a force element in the spatial configuration $\mathcal{S}$

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{t}\,\mathrm{d}a = \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}a = \boldsymbol{\sigma}\cdot\mathrm{d}\boldsymbol{a} = J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-t}\cdot\mathrm{d}\boldsymbol{A} = \boldsymbol{P}\cdot\mathrm{d}\boldsymbol{A} \tag{3.1.23}$$

such that

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{P}\cdot\mathrm{d}\boldsymbol{A}. \tag{3.1.24}$$



The first Piola-Kirchhoff stress $\boldsymbol{P}$ relates a force element $\mathrm{d}\boldsymbol{f}$ of the spatial configuration to a surface element $\mathrm{d}\boldsymbol{A}$ of the material configuration, it is thus a two-point quantity.

Let $\mathrm{d}\boldsymbol{F}$ be a force element in the material configuration $\mathcal{B}$

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{F}^{-1}\cdot\mathrm{d}\boldsymbol{p} = \boldsymbol{F}^{-1}\cdot\boldsymbol{P}\cdot\mathrm{d}\boldsymbol{A} = J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-t}\cdot\mathrm{d}\boldsymbol{A} = \boldsymbol{S}\cdot\mathrm{d}\boldsymbol{A} \tag{3.1.25}$$

such that

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{S}\cdot\mathrm{d}\boldsymbol{A}. \tag{3.1.26}$$

The second Piola-Kirchhoff stress $\boldsymbol{S}$ relates a force element $\mathrm{d}\boldsymbol{F}$ of the material configuration to a surface element $\mathrm{d}\boldsymbol{A}$ of the material configuration, it is thus a purely material quantity.

The Cauchy stress is often called the true stress, because this is the stress that you can actually measure in an experiment. The first Piola-Kirchhoff stress relates to measuring the specimen geometry before the test whereas the second Piola-Kirchhoff stress can not be measured at all.

Example Holzapfel, p. 113, example 3.1

A deformation of a body is described by

$$x_1 = -6\,X_2 \qquad x_2 = \tfrac{1}{2}\,X_1 \qquad x_3 = \tfrac{1}{3}\,X_3 \tag{3.1.27}$$

The Cauchy stress tensor for a certain point of the body is given by its matrix representation as

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{kN/cm^2} \tag{3.1.28}$$

Determine the spatial / Cauchy traction vector $\boldsymbol{t}$ and the material / Piola traction vector $\boldsymbol{T}$ acting on the plane characterized by the outward normal $\boldsymbol{n} = \boldsymbol{e}_2$ in the current configuration.

Solution From the given deformation, we find the components of the deformation gradient and its inverse.

$$\boldsymbol{F} = \begin{bmatrix} 0 & -6 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \qquad \boldsymbol{F}^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{3.1.29}$$

For this deformation $J = \det(\boldsymbol{F}) = 1$, i.e., the deformation is isochoric / volume preserving. The components of the Piola tensor read.

$$\boldsymbol{P} = J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-\mathrm{t}} = \begin{bmatrix} 0 & 0 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{kN/cm^2} \tag{3.1.30}$$

In order to find the outward normal $\boldsymbol{N}$ in the reference configuration, we recall Nanson's formula $\boldsymbol{N}\,\mathrm{d}A = 1/J\,\boldsymbol{F}_{\mathrm{t}}\cdot\boldsymbol{n}\,\mathrm{d}a$. Hence, with the transpose of $\boldsymbol{F}$ and $J = 1$ and knowing that $\boldsymbol{n} = \boldsymbol{e}_2$, we find that

$$\boldsymbol{N}\,\mathrm{d}A = \tfrac{1}{2}\,\boldsymbol{e}_1\,\mathrm{d}a \tag{3.1.31}$$

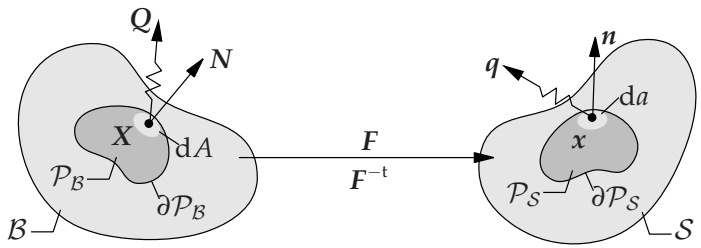
thus, $\boldsymbol{N} = \boldsymbol{e}_1$. Finally, with Cauchy's theorem,

$$\boldsymbol{t} = \boldsymbol{\sigma}\cdot\boldsymbol{n} = \begin{bmatrix} 0 \\ 50 \\ 0 \end{bmatrix} \mathrm{kN/cm^2} \qquad \boldsymbol{T} = \boldsymbol{P}\cdot\boldsymbol{N} = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} \mathrm{kN/cm^2} \tag{3.1.32}$$

i.e., $\boldsymbol{t} = 50\,\boldsymbol{e}_2$ and $\boldsymbol{T} = 100\,\boldsymbol{e}_2$, respectively. As can be seen, $\boldsymbol{t}$ and $\boldsymbol{T}$ have the same direction. The magnitude of $\boldsymbol{T}$ is twice that of $\boldsymbol{t}$, because the deformed area is half the undeformed area.

3.2 Concept of heat flux

Similar to the concept of stress, we consider a part $\mathcal{P}_\mathcal{B} \subset \mathcal{B}$ cut off from the reference body $\mathcal{B}$ and its spatial counterpart $\mathcal{P}_\mathcal{S} \subset \mathcal{S}$ closed by the respective boundaries $\partial\mathcal{P}_\mathcal{B}$ and $\partial\mathcal{P}_\mathcal{S}$. In the deformed configuration, we introduce the heat flux q that acts on the surface element $\mathrm{d}a$ of $\partial\mathcal{P}_\mathcal{S}$ and represents the thermal action of the rest of the body at the vicinity $\mathcal{P}_\mathcal{S} \setminus \mathcal{S}$ on $\partial\mathcal{P}_\mathcal{S}$.



Cauchy type theorem

The heat flux vector q depends linearly on the spatial normal $\boldsymbol{n}$ to the surface $\partial\mathcal{P}_\mathcal{S}$ through the Cauchy or true heat flux vector $\boldsymbol{q}$.

$$q(\boldsymbol{x}, t; \boldsymbol{n}) := \boldsymbol{q}(\boldsymbol{x}, t) \cdot \boldsymbol{n} \tag{3.2.1}$$

We can then use the Piola transform $(\bullet)_\mathcal{B} = J(\bullet)_\mathcal{S} \cdot \boldsymbol{F}^{-t}$ to obtain the material heat flux vector $\boldsymbol{Q}$

$$\boldsymbol{Q} = J\,\boldsymbol{q} \cdot \boldsymbol{F}^{-t} \tag{3.2.2}$$

such that

$$q := \boldsymbol{Q} \cdot \boldsymbol{N}. \tag{3.2.3}$$