2.10 Some problems...

**Problem:** Let $Q$ be a proper orthogonal tensor such that

\[
Q \cdot Q^t = I \quad \det(Q) = 1
\]

possessing the real eigenvalue $\lambda = 1$ with the associated eigenvector $n$.

\[
Q \cdot n = \lambda n = n
\]

Let $p$ and $q$ be unit vectors forming an orthonormal basis with $n$ such that

\[
\det|n, p, q| = |n, p, q| = 1.
\]

Use the following representation of a proper orthogonal tensor

\[
Q = n \otimes n + [p \otimes p + q \otimes q] \cos(\theta) - [p \otimes q - q \otimes p] \sin(\theta)
\]

to show that the first and second invariant of $Q$ take the following form.

\[
I_Q = II_Q = 1 + 2 \cos(\theta)
\]

Hint: Make use of the following identities.

\[
\begin{align*}
[Q \cdot n, p, q] + [n, Q \cdot p, q] + [n, p, Q \cdot q] &= I_Q[n, p, q] \\
[n, Q \cdot p, Q \cdot q] + [Q \cdot n, p, Q \cdot q] + [Q \cdot n, Q \cdot p, q] &= II_Q[n, p, q] \\
[Q \cdot n, Q \cdot p, Q \cdot q] &= III_Q[n, p, q]
\end{align*}
\]

First rewrite first and second invariant!

First invariant with $Q \cdot n = n$ and $\det|n, p, q| = 1$

\[
I_Q = [Q \cdot n, p, q] + [n, Q \cdot p, q] + [n, p, Q \cdot q]
\]

\[
= [n, p, q] + [n, Q \cdot p, q] + [n, p, Q \cdot q]
\]

\[
= 1 + [n, Q \cdot p, q] + [n, p, Q \cdot q]
\]
Second invariant with \( Q \cdot n = n \)

and third invariant \([Q \cdot n, Q \cdot p, Q \cdot q] = \det Q = lll_Q\)

\[
ll_Q = [n, Q \cdot p, Q \cdot q] + [Q \cdot n, p, Q \cdot q] + [Q \cdot n, Q \cdot p, q]
= [Q \cdot n, Q \cdot p, Q \cdot q] + [n, p, Q \cdot q] + [n, Q \cdot p, q]
= 1 + [n, p, Q \cdot q] + [n, Q \cdot p, q]
\]

Now, evaluate \( Q \cdot p \) and \( Q \cdot q \)

\[
Q \cdot p = n \otimes n \cdot p + [p \otimes p \cdot q + q \otimes q \cdot p] \cos(\theta)
- [p \otimes q \cdot q - q \otimes p \cdot q] \sin(\theta)
\]

\[
Q \cdot q = n \otimes n \cdot q + [p \otimes p \cdot q + q \otimes q \cdot q] \cos(\theta)
- [p \otimes q \cdot q - q \otimes p \cdot q] \sin(\theta)
\]

with \( n, p \) and \( q \) being orthogonal unit vectors

\[
Q \cdot p = \cos(\theta)p + \sin(\theta)q
\]

\[
Q \cdot q = \cos(\theta)q - \sin(\theta)p
\]

thus

\[
l_Q = 1 + [n, [\cos(\theta)p + \sin(\theta)q], q] + [n, p, [\cos(\theta)q - \sin(\theta)p]]
= 1 + 2 \cos(\theta) [n, p, q] = 1 + 2 \cos(\theta) lll_Q.
\]

Show that if \( \theta \neq 0 \), \( Q \) has only one real eigenvalue!

Rewrite the characteristic equation

\[
\lambda^3 - l_Q \lambda^2 + ll_Q \lambda - lll_Q = 0
\]

in factorized form.

\[
(\lambda - 1) (\lambda^2 - 2\lambda \cos(\theta) + 1) = 0
\]

Since \( |\cos(\theta)| \leq 1 \), \( Q \) has only one eigenvalue \( \lambda = 1 \).

**Problem:** Show that \((A^{adj})^{adj} = \det(A) A\)

\[
(A^{adj})^{adj} = \left(\frac{1}{\det(A)} A^{-1}\right)^{adj} = \det(A) (A^{-1})^{-t} = \det(A) A
\]

**Problem:** The following expression

\[
v = c [n \otimes m] \cdot x = c [x \cdot m] \cdot n
\]

characterizes a simple shearing motion in the spatial description. Here \( c \) is a positive constant and \( n \) and \( m \) are orthogonal unit vectors. Determine the principal stretches, the principal axes of stretching, and the angular velocity!

The above equation describes a steady motion in which the particle paths and streamlines are straight lines in the direction \( n \) with material planes orthogonal to \( m \) sliping over one another without being distorted. The spatial velocity gradient can be expressed as follows.

\[
l = \text{grad}(v) = \frac{\partial}{\partial x} c [n \otimes m] \cdot x = c n \otimes m
\]

Its symmetric part takes the following format

\[
d = l^{sym} = \frac{1}{2} [l + l^t] = \frac{1}{2} c [n \otimes m + m \otimes n].
\]

Reformulation with

\[
d = \frac{1}{4} c [n \otimes n + n \otimes m + m \otimes n + m \otimes m]
- \frac{1}{4} c [n \otimes n - n \otimes m - m \otimes n + m \otimes m].
\]

yields spectral representation

\[
d = \frac{1}{2} c \frac{1}{\sqrt{2}} [n + m] \otimes \frac{1}{\sqrt{2}} [n + m] - \frac{1}{2} c \frac{1}{\sqrt{2}} [n - m] \otimes \frac{1}{\sqrt{2}} [n - m]
\]

\[
d = \frac{1}{2} c \frac{1}{\sqrt{2}} [n + m] \otimes \frac{1}{\sqrt{2}} [n + m] - \frac{1}{2} c \frac{1}{\sqrt{2}} [n - m] \otimes \frac{1}{\sqrt{2}} [n - m]
\]
principal stretches: \{-\frac{1}{2}c; 0; +\frac{1}{2}c\}
principal axes of stretching: \{\frac{1}{\sqrt{2}}[n + m]; r; +\frac{1}{\sqrt{2}}[n - m]\}
where \(r\) is orthogonal to \(n\) and \(m\)

How does the volume change upon this motion?
Elaborate first invariant \(\text{tr}(d) = I_d\)!

\[ I_d = -\frac{1}{2}c + 0 + \frac{1}{2}c = 0 \]

The motion is isochoric, i.e., volume preserving!
Typical example: crystallographic sliding / plasticity

For the angular velocity, consider the spin tensor \(w\), i.e., the skew symmetric part of spatial velocity gradient.

\[ w = \Gamma^{skw} = \frac{1}{2}[I - I'] = \frac{1}{2}c[n \otimes m - m \otimes n] \]

Let \(x\) be an arbitrary vector.

\[ w \cdot x = \frac{1}{2}c[n \otimes m - m \otimes n] \cdot x \]
\[ = \frac{1}{2}c[(m \cdot x)n - (n \cdot x)m] \]
\[ = -\frac{1}{2}cr \times [(x \cdot m)m + (x \cdot n)n + (x \cdot r)r] \]
\[ = -\frac{1}{2}cr \times x = \omega \times x \]

Accordingly, \(\omega = -\frac{1}{2}cr\) is the axial vector of the spin tensor \(w\), and \(||\omega|| = -\frac{1}{2}c\) is the angular velocity.