ME338A
CONTINUUM MECHANICS

lecture notes 09

tuesday, april 29, 2008

2.10 Some problems...

Problem: Let $\boldsymbol{Q}$ be a proper orthogonal tensor such that

$$\boldsymbol{Q}\cdot\boldsymbol{Q}^{\mathrm{t}} = \boldsymbol{I} \qquad \det(\boldsymbol{Q}) = 1$$

possessing the real eigenvalue $\lambda = 1$ with the associated eigenvector $\boldsymbol{n}$.

$$\boldsymbol{Q}\cdot\boldsymbol{n} = \lambda\boldsymbol{n} = \boldsymbol{n}$$

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be unit vectors forming an orthonormal basis with $\boldsymbol{n}$ such that

$$\det|\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}| = [\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}] = 1.$$

Use the following representation of a proper orthogonal tensor

$$\boldsymbol{Q} = \boldsymbol{n}\otimes\boldsymbol{n} + [\boldsymbol{p}\otimes\boldsymbol{p} + \boldsymbol{q}\otimes\boldsymbol{q}]\,\cos(\theta) - [\boldsymbol{p}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{p}]\,\sin(\theta)$$

to show that the first and second invariant of $\boldsymbol{Q}$ take the following form.

$$I_{\mathrm{Q}} = II_{\mathrm{Q}} = 1 + 2\,\cos(\theta)$$

Hint: Make use of the following identities.

$$\begin{aligned}
[\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}] \;+\; [\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{q}] \;+\; [\boldsymbol{n},\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}] &= I_{\mathrm{Q}}[\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}] \\
[\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}] + [\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}] + [\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{q}] &= II_{\mathrm{Q}}[\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}] \\
[\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}] &= III_{\mathrm{Q}}[\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}]
\end{aligned}$$

First rewrite first and second invariant!
First invariant with $\boldsymbol{Q}\cdot\boldsymbol{n} = \boldsymbol{n}$ and $\det\,|\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}| = 1$

$$\begin{aligned}
I_{\mathrm{Q}} &= [\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}] + [\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{q}] + [\boldsymbol{n},\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}] \\
&= [\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}] + [\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{q}] + [\boldsymbol{n},\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}] \\
&= 1 + [\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{q}] + [\boldsymbol{n},\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}]
\end{aligned}$$

Second invariant with $\boldsymbol{Q}\cdot\boldsymbol{n}=\boldsymbol{n}$
and third invariant $[\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}]=\det\boldsymbol{Q}=III_Q$

$$\begin{aligned} II_Q &= [\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}]+[\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}]+[\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{q}] \\ &= [\boldsymbol{Q}\cdot\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}]+[\boldsymbol{n},\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}]+[\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{q}] \\ &= 1+[\boldsymbol{n},\boldsymbol{p},\boldsymbol{Q}\cdot\boldsymbol{q}]+[\boldsymbol{n},\boldsymbol{Q}\cdot\boldsymbol{p},\boldsymbol{q}] \end{aligned}$$

Now, evaluate $\boldsymbol{Q}\cdot\boldsymbol{p}$ and $\boldsymbol{Q}\cdot\boldsymbol{q}$

$$\begin{aligned} \boldsymbol{Q}\cdot\boldsymbol{p} &= \boldsymbol{n}\otimes\boldsymbol{n}\cdot\boldsymbol{p} + [\boldsymbol{p}\otimes\boldsymbol{p}\cdot\boldsymbol{p}+\boldsymbol{q}\otimes\boldsymbol{q}\cdot\boldsymbol{p}]\cos(\theta) \\ &\quad - [\boldsymbol{p}\otimes\boldsymbol{q}\cdot\boldsymbol{p}-\boldsymbol{q}\otimes\boldsymbol{p}\cdot\boldsymbol{p}]\sin(\theta) \\ \boldsymbol{Q}\cdot\boldsymbol{q} &= \boldsymbol{n}\otimes\boldsymbol{n}\cdot\boldsymbol{q} + [\boldsymbol{p}\otimes\boldsymbol{p}\cdot\boldsymbol{q}+\boldsymbol{q}\otimes\boldsymbol{q}\cdot\boldsymbol{q}]\cos(\theta) \\ &\quad - [\boldsymbol{p}\otimes\boldsymbol{q}\cdot\boldsymbol{q}-\boldsymbol{q}\otimes\boldsymbol{p}\cdot\boldsymbol{q}]\sin(\theta) \end{aligned}$$

with $\boldsymbol{n}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ being orthogonal unit vectors

$$\begin{aligned} \boldsymbol{Q}\cdot\boldsymbol{p} &= \cos(\theta)\boldsymbol{p}+\sin(\theta)\boldsymbol{q} \\ \boldsymbol{Q}\cdot\boldsymbol{q} &= \cos(\theta)\boldsymbol{q}-\sin(\theta)\boldsymbol{p} \end{aligned}$$

thus

$$\begin{aligned} I_Q &= 1+[\boldsymbol{n},[\cos(\theta)\boldsymbol{p}+\sin(\theta)\boldsymbol{q}],\boldsymbol{q}]+[\boldsymbol{n},\boldsymbol{p},[\cos(\theta)\boldsymbol{q}-\sin(\theta)\boldsymbol{p}]] \\ &= 1+2\cos(\theta)\,[\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}] = 1+2\cos(\theta) = II_Q. \end{aligned}$$

Show that if $\theta\neq 0$, $\boldsymbol{Q}$ has only one real eigenvalue!
Rewrite the characteristic equation

$$\lambda^3 - I_Q\,\lambda^2 + II_Q\,\lambda - III_Q = 0$$

in factorized form.

$$(\lambda-1)\,(\lambda^2-2\lambda\cos(\theta)+1)=0$$

Since $|\cos(\theta)|\leq 1$, $\boldsymbol{Q}$ has only one eigenvalue $\lambda=1$.

Problem: Show that $(\boldsymbol{A}^{\mathrm{adj}})^{\mathrm{adj}} = \det(\boldsymbol{A})\,\boldsymbol{A}$

$$(\boldsymbol{A}^{\mathrm{adj}})^{\mathrm{adj}} = (\tfrac{1}{\det(\boldsymbol{A})}\,\boldsymbol{A}^{-\mathrm{t}})^{\mathrm{adj}} = \det(\boldsymbol{A})\,(\boldsymbol{A}^{-\mathrm{t}})^{-\mathrm{t}} = \det(\boldsymbol{A})\,\boldsymbol{A}$$

Problem: The following expression

$$\boldsymbol{v} = c\,[\boldsymbol{n}\otimes\boldsymbol{m}]\cdot\boldsymbol{x} = c\,[\boldsymbol{x}\cdot\boldsymbol{m}]\;\cdot\boldsymbol{n}$$

characterizes a simple shearing motion in the spatial description. Here c is a positive constant and $\boldsymbol{n}$ and $\boldsymbol{m}$ are orthogonal unit vectors. Determine the principal stretches, the principal axes of stretching, and the angular velocity!

The above equation describes a steady motion in which the particle paths and streamlines are straight lines in the direction $\boldsymbol{n}$ with material planes orthogonal to $\boldsymbol{m}$ sliping over one another without being distorted. The spatial velocity gradient can be expressed as follows.

$$\boldsymbol{l} = \mathrm{grad}(\boldsymbol{v}) = \frac{\partial}{\partial\boldsymbol{x}}c\,[\boldsymbol{n}\otimes\boldsymbol{m}]\cdot\boldsymbol{x} = c\,\boldsymbol{n}\otimes\boldsymbol{m}$$

Its symmetric part takes the following format

$$\boldsymbol{d} = \boldsymbol{l}^{\mathrm{sym}} = \tfrac{1}{2}[\boldsymbol{l}+\boldsymbol{l}^{\mathrm{t}}] = \tfrac{1}{2}c[\boldsymbol{n}\otimes\boldsymbol{m}+\boldsymbol{m}\otimes\boldsymbol{n}].$$

Reformulation with

$$\begin{aligned} \boldsymbol{d} &= \tfrac{1}{4}c[\boldsymbol{n}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{m}+\boldsymbol{m}\otimes\boldsymbol{n}+\boldsymbol{m}\otimes\boldsymbol{m}] \\ &\quad - \tfrac{1}{4}c[\boldsymbol{n}\otimes\boldsymbol{n}-\boldsymbol{n}\otimes\boldsymbol{m}-\boldsymbol{m}\otimes\boldsymbol{n}+\boldsymbol{m}\otimes\boldsymbol{m}]. \end{aligned}$$

yields spectral representation

$$\begin{aligned} \boldsymbol{d} &= \tfrac{1}{4}c[\boldsymbol{n}+\boldsymbol{m}]\otimes[\boldsymbol{n}+\boldsymbol{m}] - \tfrac{1}{4}c[\boldsymbol{n}-\boldsymbol{m}]\otimes[\boldsymbol{n}-\boldsymbol{m}] \\ &= \tfrac{1}{2}c\tfrac{1}{\sqrt{2}}[\boldsymbol{n}+\boldsymbol{m}]\otimes\tfrac{1}{\sqrt{2}}[\boldsymbol{n}+\boldsymbol{m}] - \tfrac{1}{2}c\tfrac{1}{\sqrt{2}}[\boldsymbol{n}-\boldsymbol{m}]\otimes\tfrac{1}{\sqrt{2}}[\boldsymbol{n}-\boldsymbol{m}] \end{aligned}$$

principal stretches: $\{-\frac{1}{2}c; 0; +\frac{1}{2}c\}$
principal axes of stretching: $\{\frac{1}{\sqrt{2}}[\boldsymbol{n}+\boldsymbol{m}]; \boldsymbol{r}; +\frac{1}{\sqrt{2}}[\boldsymbol{n}-\boldsymbol{m}]\}$
where $\boldsymbol{r}$ is orthogonal to $\boldsymbol{n}$ and $\boldsymbol{m}$

How does the volume change upon this motion?
Elaborate first invariant $\mathrm{tr}(\boldsymbol{d}) = I_d$!

$$I_d = -\tfrac{1}{2}c + 0 + \tfrac{1}{2}c = 0$$

The motion is isochoric, i.e., volume preserving!
Typical example: crystallographic sliding / plasticity

For the angular velocity, consider the spin tensor $\boldsymbol{w}$, i.e., the skew symmetric part of spatial velocity gradient.

$$\boldsymbol{w} = \boldsymbol{l}^{\mathrm{skw}} = \tfrac{1}{2}[\boldsymbol{l} - \boldsymbol{l}^{\mathrm{t}}] = \tfrac{1}{2}c[\boldsymbol{n}\otimes\boldsymbol{m} - \boldsymbol{m}\otimes\boldsymbol{n}]$$

Let $\boldsymbol{x}$ be an arbitrary vector.

$$\begin{aligned}
\boldsymbol{w}\cdot\boldsymbol{x} &= \tfrac{1}{2}c[\boldsymbol{n}\otimes\boldsymbol{m} - \boldsymbol{m}\otimes\boldsymbol{n}]\cdot\boldsymbol{x} \\
&= \tfrac{1}{2}c[[\boldsymbol{m}\cdot\boldsymbol{x}]\boldsymbol{n} - [\boldsymbol{n}\cdot\boldsymbol{x}]\boldsymbol{m}] \\
&= -\tfrac{1}{2}c\boldsymbol{r}\times[[\boldsymbol{x}\cdot\boldsymbol{m}]\boldsymbol{m} + [\boldsymbol{x}\cdot\boldsymbol{n}]\boldsymbol{n} + [\boldsymbol{x}\cdot\boldsymbol{r}]\boldsymbol{r}] \\
&= -\tfrac{1}{2}c\boldsymbol{r}\times\boldsymbol{x} = \boldsymbol{\omega}\times\boldsymbol{x}
\end{aligned}$$

Accordingly, $\boldsymbol{\omega} = -\frac{1}{2}c\boldsymbol{r}$ is the axial vector of the spin tensor $\boldsymbol{w}$, and $||\boldsymbol{\omega}|| = -\frac{1}{2}c$ is the angular velocity.