ME338A
CONTINUUM MECHANICS

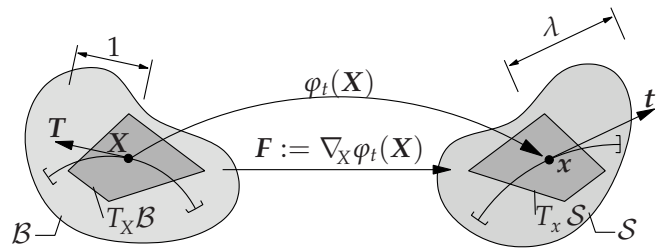
lecture notes 07

Thursday, April 22, 2008

2.4 Deformation and Strain Tensors

This section outlines the fundamental deformation and strain tensors that often enter the strain energy functions.

2.4.1 Stretch Vector and Stretch



Let $\boldsymbol{T} \in T_X\mathcal{B}$ be a material tangent vector emanating from $\boldsymbol{X} \in \mathcal{B}$. We define the associated *stretch vector* $\boldsymbol{t} \in T_x\mathcal{S}$ as the Gateaux derivative of the deformation $\varphi_t(\boldsymbol{X})$ in the direction $\boldsymbol{T}$

$$\begin{aligned} \boldsymbol{t} &:= \tfrac{d}{d\epsilon}\big|_{\epsilon=0}\, \varphi_t(\boldsymbol{X} + \epsilon \boldsymbol{T}) \\ &= \tfrac{d}{d\epsilon}\big|_{\epsilon=0}\, [\varphi_t(\boldsymbol{X}) + \epsilon \boldsymbol{F} \cdot \boldsymbol{T} + O(\epsilon^2)] \\ &= \boldsymbol{F} \cdot \boldsymbol{T}\,. \end{aligned} \tag{2.4.1}$$

The ratio of the length of the spatial tangent vector $\boldsymbol{t}$ to the length of the corresponding reference tangent vector $\boldsymbol{T}$ is called the *stretch*

$$\lambda := \frac{||\boldsymbol{t}||}{||\boldsymbol{T}||} = \frac{\sqrt{\boldsymbol{t} \cdot \boldsymbol{t}}}{\sqrt{\boldsymbol{T} \cdot \boldsymbol{T}}} > 0\,. \tag{2.4.2}$$

By choosing $||\boldsymbol{T}|| = 1$ as the *reference* value, the stretch can be expressed as

$$\begin{aligned} \lambda &= \sqrt{\boldsymbol{t}\cdot\boldsymbol{t}} = \sqrt{(\boldsymbol{F}\cdot\boldsymbol{T})\cdot(\boldsymbol{F}\cdot\boldsymbol{T})} \\ &= \sqrt{\boldsymbol{T}\cdot(\boldsymbol{F}^T\cdot\boldsymbol{F})\cdot\boldsymbol{T}} = \sqrt{\boldsymbol{T}\cdot\boldsymbol{C}\cdot\boldsymbol{T}} =: ||\boldsymbol{T}||_{\boldsymbol{C}} \,, \end{aligned} \tag{2.4.3}$$

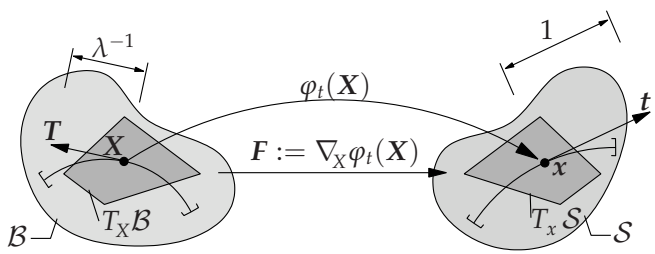
where we already introduced the *right Cauchy-Green tensor*

$$\boldsymbol{C} := \boldsymbol{F}^T\cdot\boldsymbol{F} \,, \qquad C_{AB} = F_{aA}F_{aB} \,. \tag{2.4.4}$$

It is important to observe that the right Cauchy-Green tensor is symmetric and positive-definite; that is,

$$\boldsymbol{C} = \boldsymbol{C}^T = (\boldsymbol{F}^T\cdot\boldsymbol{F})^T = \boldsymbol{F}^T\cdot\boldsymbol{F} \quad \text{and} \quad \boldsymbol{T}\cdot\boldsymbol{C}\cdot\boldsymbol{T} \geq 0 \quad \forall \boldsymbol{T}\in\mathcal{R}^3 \,. \tag{2.4.5}$$

Thus, its eigenvalues $\lambda^2_{\alpha=1,2,3}$ are positive and real numbers. The stretch formulae $\lambda = ||\boldsymbol{t}||_{\boldsymbol{1}}$ and $\lambda = ||\boldsymbol{T}||_{\boldsymbol{C}}$ are often referred to as the Eulerian and Lagrangean descriptions, respectively. Observe that the latter allows us to compute the stretch in terms of material quantities.



Now, let us consider a dual Eulerian (spatial) approach by setting $||\boldsymbol{t}||_{\boldsymbol{1}} = 1$ as a reference value. We then express the inverse stretch

$$\begin{aligned} \lambda^{-1} &= \sqrt{\boldsymbol{T}\cdot\boldsymbol{T}} = \sqrt{(\boldsymbol{F}^{-1}\cdot\boldsymbol{t})\cdot(\boldsymbol{F}^{-1}\cdot\boldsymbol{t})} \\ &= \sqrt{\boldsymbol{t}\cdot(\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1})\cdot\boldsymbol{t}} = \sqrt{\boldsymbol{t}\cdot\boldsymbol{b}^{-1}\boldsymbol{t}} =: ||\boldsymbol{t}||_{\boldsymbol{b}^{-1}} \end{aligned} \tag{2.4.6}$$

in terms of the *inverse left Cauchy-Green tensor* $\boldsymbol{b}^{-1}$

$$\boldsymbol{b}^{-1} := \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1} \,, \qquad b^{-1}_{ab} = F^{-1}_{Aa}F^{-1}_{Ab} \,. \tag{2.4.7}$$

The *left Cauchy-Green tensor* $\boldsymbol{b}$ is then defined as

$$\boldsymbol{b} := \boldsymbol{F}\cdot\boldsymbol{F}^T \,, \qquad b_{ab} = F_{aA}F_{bA} \,. \tag{2.4.8}$$

From this definition, we readily note that the left Cauchy-Green (Finger) tensor is also symmetric and positive-definite, i.e.

$$\boldsymbol{b} = \boldsymbol{b}^T = (\boldsymbol{F}\cdot\boldsymbol{F}^T)^T = \boldsymbol{F}\cdot\boldsymbol{F}^T \quad \text{and} \quad \boldsymbol{t}\cdot\boldsymbol{b}\cdot\boldsymbol{t} \geq 0 \quad \forall \boldsymbol{t}\in\mathcal{R}^3 \,. \tag{2.4.9}$$

In (2.4.3) and (2.4.6), we observe that $\boldsymbol{C}$ and $\boldsymbol{b}^{-1}$ act as metric tensors in the respective Lagrangean and Eulerian description of the length deformation.

2.4.2 Strain Tensors

Having the stretch defined in (2.4.3), we are now in a position to define the *Green-Lagrange strain tensor*. The Green-Lagrange strain measure ε_{GL}, Lagrangean strain measure, compares the squared lengths of the spatial vector $||\boldsymbol{t}||^2_{\boldsymbol{1}} = ||\boldsymbol{T}||^2_{\boldsymbol{C}}$ and the reference vector $||\boldsymbol{T}||^2_{\boldsymbol{1}} = 1$ in an *additive* manner

$$\varepsilon_{GL} := \tfrac{1}{2}\left[\lambda^2 - 1\right] \,. \tag{2.4.10}$$

Insertion of the above definitions yields

$$\begin{aligned} \varepsilon_{GL} &:= \tfrac{1}{2}\left[||\boldsymbol{T}||^2_{\boldsymbol{C}} - ||\boldsymbol{T}||^2_{\boldsymbol{1}}\right] \\ &= \tfrac{1}{2}\left[\boldsymbol{T}\cdot\boldsymbol{C}\cdot\boldsymbol{T} - \boldsymbol{T}\cdot\boldsymbol{1}\cdot\boldsymbol{T}\right] \\ &= \boldsymbol{T}\cdot\tfrac{1}{2}\left[\boldsymbol{C}-\boldsymbol{1}\right]\cdot\boldsymbol{T} =: \boldsymbol{T}\cdot\boldsymbol{E}\cdot\boldsymbol{T}\,. \end{aligned} \tag{2.4.11}$$

The Green-Lagrange strain tensor $\boldsymbol{E}$ is then defined as

$$\boldsymbol{E} := \tfrac{1}{2}\left[\boldsymbol{C}-\boldsymbol{1}\right]\,, \qquad E_{AB} = \tfrac{1}{2}\left[C_{AB} - \delta_{AB}\right]\,. \tag{2.4.12}$$

Analogous to the dual approach we considered for the inverse stretch, the Eulerian strain measure, the so-called *Almansi strain* measure is defined as

$$\varepsilon_A := \tfrac{1}{2}\left[1-\lambda^{-2}\right]\,, \tag{2.4.13}$$

where $||\boldsymbol{t}||^2_{\boldsymbol{1}} = 1$ and $||\boldsymbol{t}||^2_{\boldsymbol{b}^{-1}} = \lambda^{-2}$. Incoporating these definitions, we have

$$\begin{aligned} \varepsilon_A &:= \tfrac{1}{2}\left[||\boldsymbol{t}||^2_{\boldsymbol{1}} - ||\boldsymbol{t}||^2_{\boldsymbol{b}^{-1}}\right] \\ &= \tfrac{1}{2}\left[\boldsymbol{t}\cdot\boldsymbol{1}\cdot\boldsymbol{t} - \boldsymbol{t}\cdot\boldsymbol{b}^{-1}\cdot\boldsymbol{t}\right] \\ &= \boldsymbol{t}\cdot\tfrac{1}{2}\left[\boldsymbol{1}-\boldsymbol{b}^{-1}\right]\cdot\boldsymbol{t} =: \boldsymbol{t}\cdot\boldsymbol{e}\cdot\boldsymbol{t}\,. \end{aligned} \tag{2.4.14}$$

with

$$\boldsymbol{e} := \tfrac{1}{2}\left[\boldsymbol{1}-\boldsymbol{b}^{-1}\right]\,, \qquad e_{ab} = \tfrac{1}{2}\left[\delta_{ab} - b^{-1}_{ab}\right]\,. \tag{2.4.15}$$

denoting the Eulerian *Almansi strain tensor*.
It is important to note that linearization of the both strain tensors, $\boldsymbol{E}$ and $\boldsymbol{e}$ about the undeformed state leads to the strain tensor ε=sym$(\nabla_x \boldsymbol{u})$ in the geometrically linear theory.

$$\varepsilon = \mathrm{sym}(\nabla_x \boldsymbol{u}) = \mathrm{Lin}|_{\boldsymbol{F}=\boldsymbol{1}}\boldsymbol{E} = \mathrm{Lin}|_{\boldsymbol{F}=\boldsymbol{1}}\boldsymbol{e} \tag{2.4.16}$$

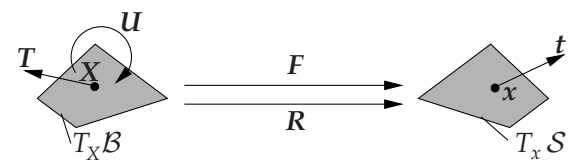
2.5 Polar Decomposition of the Deformation Gradient

The *polar decomposition theorem* states that any non-singular, second-order tensor –here the deformation gradient $\boldsymbol{F}$– has two unique multiplicative decompositions

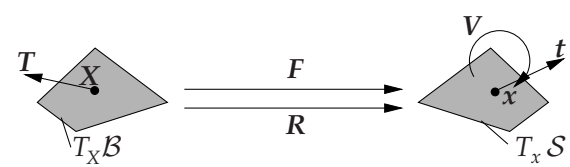
$$\boldsymbol{F} = \boldsymbol{R}\cdot\boldsymbol{U}\,, \qquad F_{aA} = R_{aB}U_{BA}\,, \tag{2.5.1}$$

$$\boldsymbol{F} = \boldsymbol{V}\cdot\boldsymbol{R}\,, \qquad F_{aA} = V_{ab}R_{bA}\,, \tag{2.5.2}$$

where $\boldsymbol{R}$ is a proper orthogonal *rotation* tensor, $\boldsymbol{U}$ and $\boldsymbol{V}$ stand for the positive-definite, symmetric *right* and *left stretch* tensors, respectively.



The *right* polar decomposition of $\boldsymbol{F}$ into $\boldsymbol{U}$ and $\boldsymbol{R}$ splits the tangent map $\boldsymbol{F} : T_X\mathcal{B} \to T_x\mathcal{S}$ into $\boldsymbol{U} : T_X\mathcal{B} \to T_X\mathcal{B}$ and $\boldsymbol{R} : T_X\mathcal{B} \to T_x\mathcal{S}$.



On the other hand, the *left* polar decomposition decomposes the tangent map $\boldsymbol{F} : T_X\mathcal{B} \to T_x\mathcal{S}$ into $\boldsymbol{R} : T_X\mathcal{B} \to T_x\mathcal{S}$ and $\boldsymbol{V} : T_x\mathcal{S} \to T_x\mathcal{S}$.

Since $\boldsymbol{R}$ is orthogonal, i.e. $\boldsymbol{R} \cdot \boldsymbol{R}^T = \mathbf{1}$, it can be readily shown that

$$\boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T \quad \text{and} \quad \boldsymbol{U} = \boldsymbol{R}^T \cdot \boldsymbol{V} \cdot \boldsymbol{R} \,. \tag{2.5.3}$$

We further observe that the right Cauchy-Green tensor

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{U}^T \cdot (\boldsymbol{R}^T \cdot \boldsymbol{R}) \cdot \boldsymbol{U} = \boldsymbol{U}^2 \tag{2.5.4}$$

and the left Cauchy-Green tensor

$$\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{V} \cdot (\boldsymbol{R} \cdot \boldsymbol{R}^T) \cdot \boldsymbol{V}^T = \boldsymbol{V}^2 \tag{2.5.5}$$

are identical to the squares of the respective stretch tensors. In general, computation of the stretch tensors and the rotation tensor requires the eigenvalue analysis. For a given deformation gradient $\boldsymbol{F}$, the following steps describe the procedure:

i) Compute $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{U}^2$ or $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{V}^2$

ii) Solve the eigenvalue problem for $\boldsymbol{C}$ or $\boldsymbol{b}$ to obtain

$$\boldsymbol{C} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \quad \text{or} \quad \boldsymbol{b} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \boldsymbol{n}_\alpha \otimes \boldsymbol{n}_\alpha \tag{2.5.6}$$

iii) Compute $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ or $\boldsymbol{V}$ and $\boldsymbol{V}^{-1}$

$$\begin{aligned} \boldsymbol{U} &= \sqrt{\boldsymbol{C}} = \textstyle\sum_{\alpha=1}^{3} \lambda_\alpha \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \quad \text{and} \quad \boldsymbol{U}^{-1} = \textstyle\sum_{\alpha=1}^{3} \lambda_\alpha^{-1} \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \\ &\qquad\qquad\qquad\qquad \text{or} \\ \boldsymbol{V} &= \sqrt{\boldsymbol{b}} = \textstyle\sum_{\alpha=1}^{3} \lambda_\alpha \boldsymbol{n}_\alpha \otimes \boldsymbol{n}_\alpha \quad \text{and} \quad \boldsymbol{V}^{-1} = \textstyle\sum_{\alpha=1}^{3} \lambda_\alpha^{-1} \boldsymbol{n}_\alpha \otimes \boldsymbol{n}_\alpha \end{aligned} \tag{2.5.7}$$

iv) Compute $\boldsymbol{R}$ through

$$\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1} \quad \text{or} \quad \boldsymbol{R} = \boldsymbol{V}^{-1}\boldsymbol{F}$$

Since $\boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T$ and $\boldsymbol{U} = \boldsymbol{R}^T \cdot \boldsymbol{V} \cdot \boldsymbol{R}$, we observe that

$$\boldsymbol{n}_\alpha = \boldsymbol{R} \cdot \boldsymbol{N}_\alpha \quad \text{and} \quad \boldsymbol{N}_\alpha = \boldsymbol{R}^T \cdot \boldsymbol{n}_\alpha \,. \tag{2.5.8}$$

Furthermore, since eigenvectors of a symmetric second-order tensor are orthogonal, i.e. $\boldsymbol{n}_\alpha \cdot \boldsymbol{n}_\beta = \delta_{\alpha\beta}$ and $\boldsymbol{N}_\alpha \cdot \boldsymbol{N}_\beta = \delta_{\alpha\beta}$, we can deduce that

$$\boldsymbol{R} = \sum_{\alpha=1}^{3} \boldsymbol{n}_\alpha \otimes \boldsymbol{N}_\alpha \,. \tag{2.5.9}$$

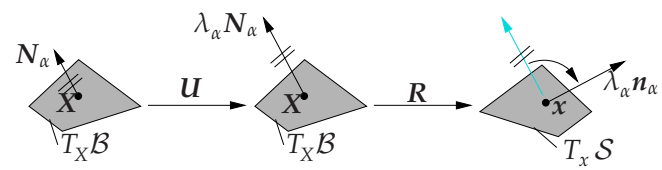
Then, the multiplication of the spectral representation of $\boldsymbol{U} = \sum_{\alpha=1}^{3} \lambda_\alpha \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha$ with $\boldsymbol{R}$ yields the spectral representation of the deformation gradient

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \sum_{\alpha=1}^{3} \lambda_\alpha \boldsymbol{n}_\alpha \otimes \boldsymbol{N}_\alpha \tag{2.5.10}$$

corresponding to the eigenvalue problem

$$\boldsymbol{F} \cdot \boldsymbol{N}_\alpha = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{N}_\alpha = \boldsymbol{R} \cdot (\lambda_\alpha \boldsymbol{N}_\alpha) = \lambda_\alpha \boldsymbol{n}_\alpha \,. \tag{2.5.11}$$

This makes the geometric interpretation of the polar decomposition more transparent



Analogous sketch can also be done for the left polar decomposition.