2.4 Deformation and Strain Tensors

This section outlines the fundamental deformation and strain tensors that often enter the strain energy functions.

2.4.1 Stretch Vector and Stretch

Let \( T \in T_X B \) be a material tangent vector emanating from \( X \in B \). We define the associated stretch vector \( t \in T_X S \) as the Gateaux derivative of the deformation \( \varphi_i(X) \) in the direction \( T \)

\[
t := \frac{d}{d\epsilon} \big|_{\epsilon=0} \varphi_i(X + \epsilon T)
\]

\[
= \frac{d}{d\epsilon} \big|_{\epsilon=0} \left[ \varphi_i(X) + \epsilon F \cdot T + O(\epsilon^2) \right]
\]

\[
= F \cdot T .
\]

The ratio of the length of the spatial tangent vector \( t \) to the length of the corresponding reference tangent vector \( T \) is called the stretch

\[
\lambda := \frac{|t|}{|T|} = \frac{\sqrt{T \cdot T}}{\sqrt{T \cdot T}} > 0 .
\]
By choosing $||T|| = 1$ as the reference value, the stretch can be expressed as
\[
\lambda = \sqrt{t \cdot t} = \sqrt{(F \cdot T) \cdot (F \cdot T)} = \sqrt{T \cdot (F^T \cdot F) \cdot T} = \sqrt{T \cdot C \cdot T} =: ||T||_C,
\]
where we already introduced the right Cauchy-Green tensor
\[
C := F^T \cdot F, \quad C_{AB} = F_{aA} F_{bB}.
\] (2.4.4)

It is important to observe that the right Cauchy-Green tensor is symmetric and positive-definite; that is,
\[
C = C^T = (F^T \cdot F)^T = F^T \cdot F \quad \text{and} \quad T \cdot C \cdot T \geq 0 \quad \forall T \in \mathcal{R}^3.
\] (2.4.5)

Thus, its eigenvalues $\lambda^2 = 1, 2, 3$ are positive and real numbers. The stretch formulae $\lambda = ||t||_1$ and $\lambda = ||T||_C$ are often referred to as the Eulerian and Lagrangean descriptions, respectively. Observe that the latter allows us to compute the stretch in terms of material quantities.

Now, let us consider a dual Eulerian (spatial) approach by setting $||t||_1 = 1$ as a reference value. We then express the inverse stretch
\[
\lambda^{-1} = \sqrt{T \cdot T} = \sqrt{(F^{-1} \cdot t) \cdot (F^{-1} \cdot t)} = \sqrt{t \cdot (F^{-T} \cdot F^{-1}) \cdot t} = \sqrt{t \cdot b^{-1} \cdot t} =: ||t||_{b^{-1}}
\] (2.4.6)
in terms of the inverse left Cauchy-Green tensor $b^{-1}$
\[
b^{-1} := F^{-T} \cdot F^{-1}, \quad b^{-1}_{ab} = F_{Aa} F_{bA}.
\] (2.4.7)
The left Cauchy-Green tensor $b$ is then defined as
\[
b := F \cdot F^T, \quad b_{ab} = F_{aA} F_{bA}.
\] (2.4.8)

From this definition, we readily note that the left Cauchy-Green (Finger) tensor is also symmetric and positive-definite, i.e.
\[
b = b^T = (F \cdot F^T)^T = F \cdot F^T \quad \text{and} \quad t \cdot b \cdot t \geq 0 \quad \forall t \in \mathcal{R}^3.
\] (2.4.9)

In (2.4.3) and (2.4.6), we observe that $C$ and $b^{-1}$ act as metric tensors in the respective Lagrangean and Eulerian description of the length deformation.

### 2.4.2 Strain Tensors

Having the stretch defined in (2.4.3), we are now in a position to define the Green-Lagrange strain tensor. The Green-Lagrange strain measure $\varepsilon_{GL}$, Lagrangean strain measure, compares the squared lengths of the spatial vector $||t||_1^2 = ||T||_C^2$ and the reference vector $||T||_1^2 = 1$ in an additive manner
\[
\varepsilon_{GL} := \frac{1}{2} [\lambda^2 - 1].
\] (2.4.10)
Insertion of the above definitions yields

\[ \varepsilon_{GL} := \frac{1}{2} \left( \|T\|_C^2 - \|T\|_1^2 \right) \]
\[ = \frac{1}{2} \left[ T \cdot C \cdot T - T \cdot \mathbf{1} \cdot T \right] \quad \text{(2.4.11)} \]
\[ = T \cdot \frac{1}{2} [C - \mathbf{1}] \cdot T = : T \cdot E \cdot T . \]

The Green-Lagrange strain tensor \( E \) is then defined as

\[ E := \frac{1}{2} [C - \mathbf{1}], \quad E_{AB} = \frac{1}{2} [C_{AB} - \delta_{AB}] . \quad \text{(2.4.12)} \]

Analagous to the dual approach we considered for the inverse stretch, the Eulerian strain measure, the so-called Almansi strain measure is defined as

\[ \varepsilon_A := \frac{1}{2} \left[ 1 - \lambda^{-2} \right] , \quad \text{(2.4.13)} \]

where \( ||t||_1^2 = 1 \) and \( ||t||_{b-1}^2 = \lambda^{-2} \). Incorporating these definitions, we have

\[ \varepsilon_A := \frac{1}{2} \left[ ||t||_1^2 - ||t||_{b^{-1}}^2 \right] \]
\[ = \frac{1}{2} \left[ t \cdot \mathbf{1} \cdot t - t \cdot b^{-1} \cdot t \right] \quad \text{(2.4.14)} \]
\[ = t \cdot \frac{1}{2} [1 - b^{-1}] \cdot t = : t \cdot e \cdot t . \]

with

\[ e := \frac{1}{2} [1 - b^{-1}], \quad e_{ab} = \frac{1}{2} [\delta_{ab} - b_{ab}^{-1}] . \quad \text{(2.4.15)} \]

denoting the Eulerian Almansi strain tensor.

It is important to note that linearization of the both strain tensors, \( E \) and \( e \) about the undeformed state leads to the strain tensor \( \varepsilon = \text{sym}(\nabla u) \) in the geometrically linear theory.

\[ \varepsilon = \text{sym}(\nabla u) = \text{Lin}|_{F=1} E = \text{Lin}|_{F=1} e \quad \text{(2.4.16)} \]

### 2.5 Polar Decomposition of the Deformation Gradient

The polar decomposition theorem states that any non-singular, second-order tensor –here the deformation gradient \( F\)– has two unique multiplicative decompositions

\[ F = R \cdot U , \quad F_{aA} = R_{ab} U_{bA} , \quad \text{(2.5.1)} \]
\[ F = V \cdot R , \quad F_{aA} = V_{ab} R_{bA} , \quad \text{(2.5.2)} \]

where \( R \) is a proper orthogonal rotation tensor, \( U \) and \( V \) stand for the positive-definite, symmetric right and left stretch tensors, respectively.

The right polar decomposition of \( F \) into \( U \) and \( R \) splits the tangent map \( F : T_X B \to T_X S \) into \( U : T_X B \to T_X B \) and \( R : T_X B \to T_X S \).

On the other hand, the left polar decomposition decomposes the tangent map \( F : T_X B \to T_X S \) into \( R : T_X B \to T_X S \) and \( V : T_X S \to T_X S \).
Since $\mathbf{R}$ is orthogonal, i.e. $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{1}$, it can be readily shown that

$$V = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T \quad \text{and} \quad \mathbf{U} = \mathbf{R}^T \cdot V \cdot \mathbf{R} . \quad (2.5.3)$$

We further observe that the right Cauchy-Green tensor

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{U}^T \cdot (\mathbf{R}^T \cdot \mathbf{R}) \cdot \mathbf{U} = \mathbf{U}^2 \quad (2.5.4)$$

and the left Cauchy-Green tensor

$$\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^T = \mathbf{V} \cdot (\mathbf{R} \cdot \mathbf{R}^T) \cdot \mathbf{V}^T = \mathbf{V}^2 \quad (2.5.5)$$

are identical to the squares of the respective stretch tensors. In general, computation of the stretch tensors and the rotation tensor requires the eigenvalue analysis. For a given deformation gradient $\mathbf{F}$, the following steps describe the procedure:

i) Compute $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{U}^2$ or $\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^T = \mathbf{V}^2$

ii) Solve the eigenvalue problem for $\mathbf{C}$ or $\mathbf{b}$ to obtain

$$\mathbf{C} = \sum_{\alpha=1}^{3} \lambda_{\alpha}^2 \mathbf{N}_{\alpha} \otimes \mathbf{N}_{\alpha} \quad \text{or} \quad \mathbf{b} = \sum_{\alpha=1}^{3} \lambda_{\alpha}^2 \mathbf{n}_{\alpha} \otimes \mathbf{n}_{\alpha} \quad (2.5.6)$$

iii) Compute $\mathbf{U}$ and $\mathbf{U}^{-1}$ or $\mathbf{V}$ and $\mathbf{V}^{-1}$

$$\mathbf{U} = \sqrt[3]{\mathbf{C}} = \sum_{\alpha=1}^{3} \lambda_{\alpha} \mathbf{N}_{\alpha} \otimes \mathbf{N}_{\alpha} \quad \text{and} \quad \mathbf{U}^{-1} = \sum_{\alpha=1}^{3} \lambda_{\alpha}^{-1} \mathbf{N}_{\alpha} \otimes \mathbf{N}_{\alpha}$$

or

$$\mathbf{V} = \sqrt[3]{\mathbf{b}} = \sum_{\alpha=1}^{3} \lambda_{\alpha} \mathbf{n}_{\alpha} \otimes \mathbf{n}_{\alpha} \quad \text{and} \quad \mathbf{V}^{-1} = \sum_{\alpha=1}^{3} \lambda_{\alpha}^{-1} \mathbf{n}_{\alpha} \otimes \mathbf{n}_{\alpha} \quad (2.5.7)$$

iv) Compute $\mathbf{R}$ through

$$\mathbf{R} = \mathbf{F} \mathbf{U}^{-1} \quad \text{or} \quad \mathbf{R} = \mathbf{V}^{-1} \mathbf{F} \quad \text{62}$$