ME338A
CONTINUUM MECHANICS

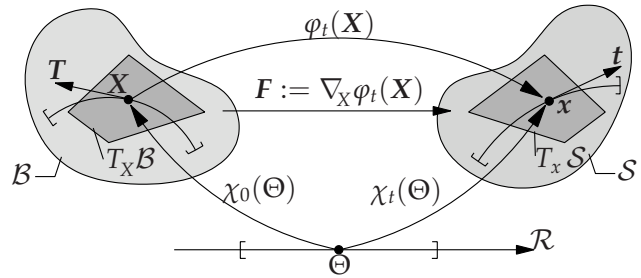
lecture notes 06

Thursday, April 17, 2008

2.3 Fundamental Geometric Maps

This section is devoted to the geometric mapping of basic *tangential, normal* and *volumetric* objects.

2.3.1 Deformation Gradient. Tangent Map



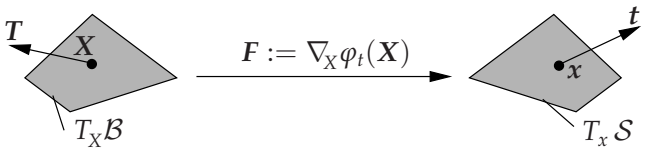
Probably the most fundamental deformation measure used in kinematics of finite deformation is the *deformation gradient*. It can be considered as a linear map of the referential *tangent vectors* onto the spatial counterparts. To this end, let $\chi_0(\Theta)$ and $\chi_t(\Theta)$ be the material and spatial curves parameterized by a common variable $\Theta \in \mathcal{R}$ on $\mathcal{B}$ and $\mathcal{S}$, respectively. It is important to observe that the spatial curve is related to the reference curve by the non-linear deformation map $\chi_t(\Theta) = \varphi_t(\chi_0(\Theta))$. Tangents of the curves belonging to the respective tangent spaces are defined as

$$\boldsymbol{T} := \frac{d\chi_0(\Theta)}{d\Theta} \in T_X\mathcal{B} \quad \text{and} \quad \boldsymbol{t} := \frac{d\chi_t(\Theta)}{d\Theta} \in T_x\mathcal{S} \tag{2.3.1}$$

Through the chain rule, we obtain

$$\boldsymbol{t} = \frac{d\chi_t(\Theta)}{d\Theta} = \nabla_X \varphi_t(\boldsymbol{X}) \cdot \frac{d\chi_0(\Theta)}{d\Theta} = \boldsymbol{F} \cdot \boldsymbol{T}, \tag{2.3.2}$$

where we introduced the *deformation gradient* $\boldsymbol{F} := \nabla_X \varphi_t(\boldsymbol{X})$ as the Fréchet derivative of the deformation.



The deformation gradient is also called the *tangent map*

$$\boldsymbol{F}(\boldsymbol{X}, t) : \begin{cases} T_X\mathcal{B} \;\rightarrow\; T_x\mathcal{S}\,, \\ \quad \boldsymbol{T} \;\mapsto\; \boldsymbol{t} = \boldsymbol{F} \cdot \boldsymbol{T}\,, \end{cases} \tag{2.3.3}$$

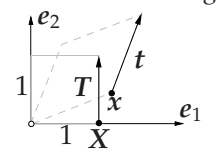
between the tangent spaces $T_X\mathcal{B}$ and $T_x\mathcal{S}$ of the configurations (manifolds) $\mathcal{B}$ and $\mathcal{S}$, respectively. Note that the deformation gradient is a *two-point* tensor that involves points from the both configurations; that is, in index notation

$$F_{aA} := \partial\varphi_a / \partial X_A \quad \text{and} \quad t_a = F_{aA} T_A\,.$$

The determinant of the deformation gradient $\det(\boldsymbol{F})$ is restricted to non-zero ($\det(\boldsymbol{F}) \neq 0$), positive ($\det(\boldsymbol{F}) > 0$) values due to the facts that $\varphi_t(\boldsymbol{X})$ is one-to-one and we rule out interpenetration of the material in the course of a motion.

Examples:

Pure shear (Homogeneous)

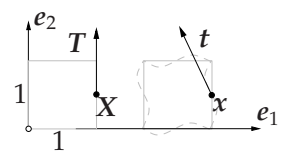


$$\begin{aligned} x_1 &= \tfrac{\lambda+\lambda^{-1}}{2} X_1 + \tfrac{\lambda-\lambda^{-1}}{2} X_2 \\ x_2 &= \tfrac{\lambda-\lambda^{-1}}{2} X_1 + \tfrac{\lambda+\lambda^{-1}}{2} X_2 \\ x_3 &= X_3 \end{aligned}$$

$$\boldsymbol{F} = \tfrac{1}{2} \begin{bmatrix} \lambda+\lambda^{-1} & \lambda-\lambda^{-1} & 0 \\ \lambda-\lambda^{-1} & \lambda+\lambda^{-1} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{T} = \boldsymbol{e}_2\,, \qquad \boldsymbol{t} = \boldsymbol{F} \cdot \boldsymbol{T} = \tfrac{1}{2}\left[(\lambda-\lambda^{-1})\boldsymbol{e}_1 + (\lambda+\lambda^{-1})\boldsymbol{e}_2\right]$$

Periodic deformation (Inhomogeneous)

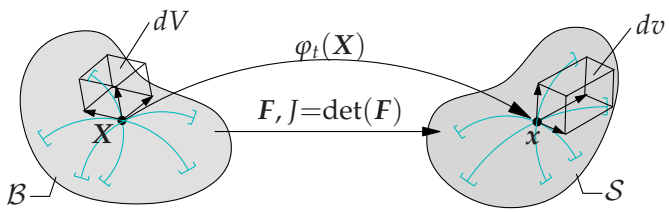


$$\begin{aligned} x_1 &= X_1 + \alpha(t)\sin(2\pi X_2) + a \\ x_2 &= X_2 - \alpha(t)\sin(2\pi X_1) \\ x_3 &= X_3 \end{aligned}$$

$$\begin{aligned} \boldsymbol{F} &= \begin{bmatrix} 1 & 2\pi\alpha\cos(2\pi X_2) & 0 \\ -2\pi\alpha\cos(2\pi X_1) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\boldsymbol{X}=(1,0.5,0)} \\ &= \begin{bmatrix} 1 & -2\pi\alpha & 0 \\ -2\pi\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\boldsymbol{T} = \boldsymbol{e}_2\,, \qquad \boldsymbol{t} = \boldsymbol{F} \cdot \boldsymbol{T} = -2\pi\alpha\boldsymbol{e}_1 + \boldsymbol{e}_2$$

2.3.2 Volume Transformation. Jacobi Map



Let dV and dv denote the infinitesimal volumes of parallelepipeds

$$dV := d\boldsymbol{X}_1 \cdot (d\boldsymbol{X}_2 \times d\boldsymbol{X}_3) \quad \text{and} \quad dv := d\boldsymbol{x}_1 \cdot (d\boldsymbol{x}_2 \times d\boldsymbol{x}_3) \tag{2.3.4}$$

defined as the scalar triple product of vectors $d\boldsymbol{X}_{i=1,2,3} \in T_X\mathcal{B}$ and $d\boldsymbol{x}_{i=1,2,3} \in T_x\,\mathcal{S}$, respectively. Each spatial tangent vector $d\boldsymbol{x}_i$ is defined as a tangential map of its material counterpart, i.e. $d\boldsymbol{x}_i := \boldsymbol{F} \cdot d\boldsymbol{X}_i$ for $i = 1, 2, 3$. We define the *volume map*

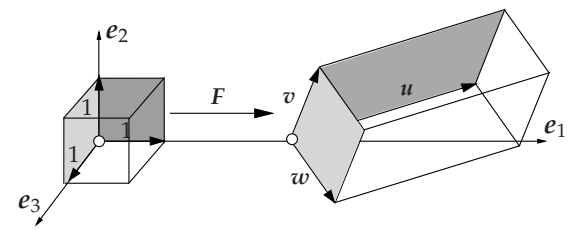
$$\begin{aligned} dv &= (\boldsymbol{F} \cdot d\boldsymbol{X}_1) \cdot [(\boldsymbol{F} \cdot d\boldsymbol{X}_2) \times (\boldsymbol{F} \cdot d\boldsymbol{X}_3)] \\ &= \det(\boldsymbol{F})\; d\boldsymbol{X}_1 \cdot (d\boldsymbol{X}_2 \times d\boldsymbol{X}_3) = \det\boldsymbol{F}\; dV =: J\; dV \end{aligned} \tag{2.3.5}$$

through the conventional coordinate-free definition of the determinant of a second order tensor. Recall that the value of the Jacobian J is restricted to positive real numbers $\mathcal{R}_+$ in order to ensure the one-to-one relation between $\boldsymbol{x}$ and $\boldsymbol{X}$ and the impenetrability of a material. Then, we say that the volume map, $\det\boldsymbol{F}$, maps the reference volume elements onto

their spatial counterparts

$$J = \det\boldsymbol{F} := \begin{cases} \mathcal{R}_+ \;\rightarrow\; \mathcal{R}_+\,, \\ dV \;\mapsto\; dv = \det\boldsymbol{F} dV\,. \end{cases} \tag{2.3.6}$$

Example:



Assume that vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are defined as

$$\boldsymbol{u} := \boldsymbol{F} \cdot \boldsymbol{e}_1,\; \boldsymbol{v} := \boldsymbol{F} \cdot \boldsymbol{e}_2 \quad \text{and} \quad \boldsymbol{w} := \boldsymbol{F} \cdot \boldsymbol{e}_3\,.$$

Components of the deformation gradient are then

$$\begin{aligned} u_i &= F_{i1}, \\ v_i &= F_{i2}, \\ w_i &= F_{i3}, \end{aligned} \qquad \rightsquigarrow \boldsymbol{F} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

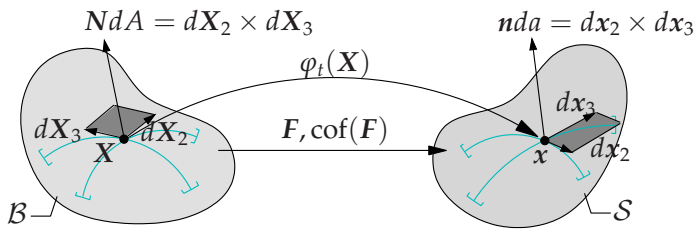
Volume of the deformed parallelpiped is identical to $\mathrm{def}(\boldsymbol{F})$:

$$\begin{aligned} dv &:= \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{F} \cdot \boldsymbol{e}_1) \cdot [(\boldsymbol{F} \cdot \boldsymbol{e}_2) \times (\boldsymbol{F} \cdot \boldsymbol{e}_3)] \\ &= \det(\boldsymbol{F})\boldsymbol{e}_1 \cdot [\boldsymbol{e}_2 \times \boldsymbol{e}_3] \\ &= \det(\boldsymbol{F}) \end{aligned}$$

2.3.3 Adjoint Transformation. Normal Map

The reference and spatial *area normals* are defined as

$$\boldsymbol{N}dA := d\boldsymbol{X}_2 \times d\boldsymbol{X}_3 \quad \text{and} \quad \boldsymbol{n}da := d\boldsymbol{x}_2 \times d\boldsymbol{x}_3 \,.$$



With these definitions, we can recast (2.3.5) into the following from

$$d\boldsymbol{x}_1 \cdot \boldsymbol{n}da = J d\boldsymbol{X}_1 \cdot \boldsymbol{N}dA \,. \tag{2.3.7}$$

If we incorporate the identity $d\boldsymbol{x}_1 = \boldsymbol{F}d\boldsymbol{X}_1$ in (2.3.7)

$$(\boldsymbol{F} \cdot d\boldsymbol{X}_1) \cdot \boldsymbol{n}da = d\boldsymbol{X}_1 \cdot (\boldsymbol{F}^T \cdot \boldsymbol{n}da) = J d\boldsymbol{X}_1 \cdot \boldsymbol{N}dA \,.$$

and solve this equality for $\boldsymbol{n}da$ for an arbitrary tangent vector $d\boldsymbol{X}_1$, we end up with the *area (normal) map*

$$\boldsymbol{n}da = \mathrm{cof}(\boldsymbol{F}) \cdot \boldsymbol{N}dA \quad \text{with} \quad \mathrm{cof}(\boldsymbol{F}) := J\boldsymbol{F}^{-T} \,. \tag{2.3.8}$$

It transforms the normals of material surfaces onto the normal vectors of spatial surfaces. Furthermore, we observe that the tensorial quantity carrying out the mapping operation in (2.3.8) is none other than $\boldsymbol{F}^{-T}$. Thus, we consider $\boldsymbol{F}^{-T}$ as the *normal map* transforming the reference normals $\boldsymbol{N}$ onto the spatial normals (co-vectors) $\boldsymbol{n}$ belonging to the

respective *co-tangent* spaces $T^*_X\mathcal{B}$ and $T^*_x\mathcal{S}$. The *normal map* is then defined as

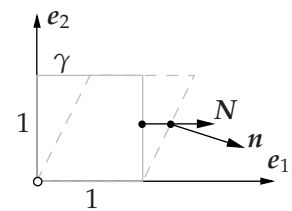
$$\boldsymbol{F}^{-T} := \begin{cases} T^*_X\mathcal{B} \;\rightarrow\; T^*_x\mathcal{S} \,, \\ \quad \boldsymbol{N} \;\mapsto\; \boldsymbol{n} = \boldsymbol{F}^{-T} \cdot \boldsymbol{N} \,. \end{cases} \tag{2.3.9}$$

The *co-factor* of the deformation gradient $\mathrm{cof}(\boldsymbol{F})$ is also defined as the derivative of the volume map $J := \det\boldsymbol{F}$ with respect to the deformation gradient $\boldsymbol{F}$

$$\mathrm{cof}\,\boldsymbol{F} := \partial_{\boldsymbol{F}}\det\boldsymbol{F} = \det(\boldsymbol{F})\boldsymbol{F}^{-T} \,. \tag{2.3.10}$$

Example:

Simple shear



$$\begin{aligned} x_1 &= X_1 + \gamma(t)X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned}$$

$$\boldsymbol{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \,, \quad \boldsymbol{F}^{-T} = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \,, \quad \begin{aligned} \boldsymbol{N} &= \boldsymbol{e}_1 \\ \boldsymbol{n} &= \boldsymbol{F}^{-T} \cdot \boldsymbol{N} \\ &= \boldsymbol{e}_1 - \gamma\boldsymbol{e}_2 \end{aligned}$$

Observe that the area change of the side surface is given by $||\boldsymbol{n}|| = \sqrt{1+\gamma^2}$.
