2.3 Fundamental Geometric Maps

This section is devoted to the geometric mapping of basic tangential, normal and volumetric objects.

2.3.1 Deformation Gradient. Tangent Map

Probably the most fundamental deformation measure used in kinematics of finite deformation is the deformation gradient. It can be considered as a linear map of the referential tangent vectors onto the spatial counterparts. To this end, let \( \chi_0(\Theta) \) and \( \chi_l(\Theta) \) be the material and spatial curves parameterized by a common variable \( \Theta \in \mathcal{R} \) on \( \mathcal{B} \) and \( \mathcal{S} \), respectively. It is important to observe that the spatial curve is related to the reference curve by the non-linear deformation map \( \chi_l(\Theta) = \varphi_l(\chi_0(\Theta)) \). Tangents of the curves belonging to the respective tangent spaces are defined as

\[
T := \frac{d\chi_0(\Theta)}{d\Theta} \in T_{\mathcal{B}} \quad \text{and} \quad t := \frac{d\chi_l(\Theta)}{d\Theta} \in T_{\mathcal{S}} \quad (2.3.1)
\]
Through the chain rule, we obtain
\[ t = \frac{d\chi_t(\Theta)}{d\Theta} = \nabla_X \varphi_t(X) \cdot \frac{d\chi_0(\Theta)}{d\Theta} = F \cdot T, \quad (2.3.2) \]
where we introduced the deformation gradient \( F := \nabla_X \varphi_t(X) \) as the Fréchet derivative of the deformation.

The deformation gradient is also called the tangent map
\[
F(X, t) : \begin{cases} \quad T_X B \rightarrow T_x S, \\ T \mapsto t = F \cdot T, \end{cases} \quad (2.3.3)
\]
between the tangent spaces \( T_X B \) and \( T_x S \) of the configurations (manifolds) \( B \) and \( S \), respectively. Note that the deformation gradient is a two-point tensor that involves points from both configurations; that is, in index notation
\[ F_{aA} := \frac{\partial \varphi_t}{\partial X_A} \quad \text{and} \quad t_a = F_{aA} T_A. \]

The determinant of the deformation gradient \( \det(F) \) is restricted to non-zero (\( \det(F) \neq 0 \)), positive (\( \det(F) > 0 \)) values due to the facts that \( \varphi_t(X) \) is one-to-one and we rule out interpenetration of the material in the course of a motion.

**Examples:**

**Pure shear (Homogeneous)**

\[
\begin{align*}
x_1 &= \frac{\lambda+\lambda^{-1}}{2} X_1 + \frac{\lambda-\lambda^{-1}}{2} X_2 \\
x_2 &= \frac{\lambda-\lambda^{-1}}{2} X_1 + \frac{\lambda+\lambda^{-1}}{2} X_2 \\
x_3 &= X_3
\end{align*}
\]

\[
F = \frac{1}{2} \begin{bmatrix}
\lambda+\lambda^{-1} & \lambda-\lambda^{-1} & 0 \\
\lambda-\lambda^{-1} & \lambda+\lambda^{-1} & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

\[
T = e_2, \quad t = F \cdot T = \frac{1}{2} [(\lambda-\lambda^{-1})e_1 + (\lambda+\lambda^{-1})e_2]
\]

**Periodic deformation (Inhomogeneous)**

\[
\begin{align*}
x_1 &= X_1 + a(t) \sin(2\pi X_2) + a \\
x_2 &= X_2 - a(t) \sin(2\pi X_1) \\
x_3 &= X_3
\end{align*}
\]

\[
F = \begin{bmatrix}
1 & 2\pi a \cos(2\pi X_1) & 0 \\
-2\pi a \cos(2\pi X_1) & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}_{X=(1,0,5,0)}
\]

\[
F = \begin{bmatrix}
1 & -2\pi a & 0 \\
-2\pi a & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
T = e_2, \quad t = F \cdot T = -2\pi a e_1 + e_2
\]
2.3.2 Volume Transformation. Jacobi Map

Let $dV$ and $dv$ denote the infinitesimal volumes of parallelepipeds

$$dV := dX_1 \cdot (dX_2 \times dX_3) \quad \text{and} \quad dv := dx_1 \cdot (dx_2 \times dx_3)$$

(2.3.4)

defined as the scalar triple product of vectors $dX_{i=1,2,3} \in T_x B$ and $dx_{i=1,2,3} \in T_x S$, respectively. Each spatial tangent vector $dx_i$ is defined as a tangential map of its material counterpart, i.e. $dx_i := F \cdot dX_i$ for $i = 1, 2, 3$. We define the volume map

$$dv = (F \cdot dX_1) \cdot [(F \cdot dX_2) \times (F \cdot dX_3)]$$

$$= \det(F) \cdot (dX_2 \times dX_3) = \det(F) dV =: J dV$$

(2.3.5)

through the conventional coordinate-free definition of the determinant of a second order tensor. Recall that the value of the Jacobian $J$ is restricted to positive real numbers $\mathcal{R}_+$ in order to ensure the one-to-one relation between $x$ and $X$ and the impenetrability of a material. Then, we say that the volume map, $\det(F)$, maps the reference volume elements onto their spatial counterparts

$$J = \det(F) := \begin{cases} 
\mathcal{R}_+ & \to \mathcal{R}_+, \\
\mathcal{R}_- & \to \mathcal{R}_- 
\end{cases},$$

(2.3.6)

Example:

Assume that vectors $u, v, w$ are defined as

$$u := F \cdot e_1, \quad v := F \cdot e_2 \quad \text{and} \quad w := F \cdot e_3.$$

Components of the deformation gradient are then

$$u_i = F_{i1}, \quad v_i = F_{i2}, \quad w_i = F_{i3},$$

$$F = \begin{bmatrix} u_1 & v_1 & w_1 \\
u_2 & v_2 & w_2 \\
u_3 & v_3 & w_3 \end{bmatrix}$$

Volume of the deformed parallelepiped is identical to $\det(F)$:

$$dv := w \cdot (u \times v) = (F \cdot e_1) \cdot [(F \cdot e_2) \times (F \cdot e_3)]$$

$$= \det(F) e_1 \cdot [e_2 \times e_3]$$

$$= \det(F)$$
2.3.3 Adjoint Transformation. Normal Map

The reference and spatial area normals are defined as
\[ N\,dA := dX_2 \times dX_3 \quad \text{and} \quad nda := dx_2 \times dx_3. \]

With these definitions, we can recast (2.3.5) into the following from
\[ dx_1 \cdot nda = |dX_1| \cdot N\,dA. \quad (2.3.7) \]

If we incorporate the identity \( dx_1 = FdX_1 \) in (2.3.7)
\[ (F \cdot dX_1) \cdot nda = dX_1 \cdot (F^T \cdot nda) = |dX_1| \cdot NdA. \]

and solve this equality for \( nda \) for an arbitrary tangent vector \( dX_1 \), we end up with the area (normal) map
\[ nda = \text{cof}(F) \cdot NdA \quad \text{with} \quad \text{cof}(F) := |F^{-T}|. \quad (2.3.8) \]

It transforms the normals of material surfaces onto the normal vectors of spatial surfaces. Furthermore, we observe that the tensorial quantity carrying out the mapping operation in (2.3.8) is none other than \( F^{-T} \). Thus, we consider \( F^{-T} \) as the normal map transforming the reference normals \( N \) onto the spatial normals (co-vectors) \( n \) belonging to the respective co-tangent spaces \( T^*_xB \) and \( T^*_xS \). The normal map is then defined as
\[ F^{-T} : \begin{cases} T^*_xB \rightarrow T^*_xS, \\ N \mapsto n = F^{-T} \cdot N. \end{cases} \quad (2.3.9) \]

The co-factor of the deformation gradient \( \text{cof}(F) \) is also defined as the derivative of the volume map \( J := \det F \) with respect to the deformation gradient \( F \)
\[ \text{cof} F := \partial_F \det F = \det(F)F^{-T}. \quad (2.3.10) \]

Example:

Simple shear

\[ F = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F^{-T} = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = e_1, \]

\[ n = F^{-T} \cdot N = e_1 - \gamma e_2. \]

Observe that the area change of the side surface is given by
\[ ||n|| = \sqrt{1 + \gamma^2}. \]