1 Tensor calculus

1.1 Tensor algebra

1.1.1 Vector algebra

1.1.1.1 Notation

- Einstein’s summation convention

\[ u_i = \sum_{j=1}^{3} A_{ij} x_j + b_i = A_{ij} x_j + b_i \]  

(1.1.1)

summation over indices that appear twice in a term or symbol, with silent (dummy) index \( j \) and free index \( i \), and thus

\[ u_1 = A_{11} x_1 + A_{12} x_2 + A_{13} x_3 + b_1 \]
\[ u_2 = A_{21} x_1 + A_{22} x_2 + A_{23} x_3 + b_2 \]
\[ u_3 = A_{31} x_1 + A_{32} x_2 + A_{33} x_3 + b_3 \]  

(1.1.2)

- Kronecker symbol \( \delta_{ij} \)

\[ \delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \]  

(1.1.3)

multiplication with Kronecker symbol corresponds to exchange of silent index with free index of Kronecker symbol

\[ u_i = \delta_{ij} u_j \]  

(1.1.4)
1 Tensor calculus

• permutation symbol $^3\epsilon_{ijk}$

\[
^3\epsilon_{ijk} = \begin{cases} 
1 & \text{for } \{i, j, k\} \ldots \text{even permutation} \\
-1 & \text{for } \{i, j, k\} \ldots \text{odd permutation} \\
0 & \text{else}
\end{cases} \quad (1.1.5)
\]

1.1.1.2 Euclidian vector space

• consider linear vector space $\mathcal{V}^3$ characterized through addition of its elements $u, v$ and multiplication with real scalars $\alpha, \beta$

$\alpha, \beta \in \mathcal{R} \quad \mathcal{R} \ldots \text{real numbers}$

$u, v \in \mathcal{V}^3 \quad \mathcal{V}^3 \ldots \text{linear vector space}$

definition of linear vector space $\mathcal{V}^3$ through the following axioms

\[
\alpha(u + v) = \alpha u + \alpha v \\
(\alpha + \beta)u = \alpha u + \beta u \\
(\alpha \beta)u = \alpha (\beta u)
\]

zero element and identity

\[
0u = 0 \quad 1u = u \quad (1.1.7)
\]

linear independence of elements $e_1, e_2, e_3 \in \mathcal{V}^3$ if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is the only (trivial) solution to

\[
\alpha_i e_i = 0 \quad (1.1.8)
\]

• consider Euclidian vector space $\mathcal{E}^3$ equipped with the Euclidian norm

\[
n(u) = \sqrt{u \cdot u} = |u| = |u_1^2 + u_2^2 + u_3^2|^{1/2} \quad (1.1.11)
\]

mapping elements of the Euclidian vector space $\mathcal{E}^3$ to the space of real numbers $\mathcal{R}$

\[
n : \mathcal{E}^3 \to \mathcal{R} \quad \text{Euclidian norm} \quad (1.1.12)
\]

representation of three–dimensional vector $a \in \mathcal{E}^3$

\[
a = a_1 e_1 + a_2 e_2 + a_3 e_3 \quad (1.1.13)
\]

with $a_1, a_2, a_3$ coordinates (components) of $a$ relative to the basis $e_1, e_2, e_3$

\[
a = [a_1, a_2, a_3]^t \quad (1.1.14)
\]
### 1.1.1.3 Scalar product

Euclidian norm enables the definition of scalar (inner) product between two vectors \( u, v \) and introduces a scalar \( \alpha \in \mathbb{R} \):

\[
 u \cdot v = \alpha \tag{1.1.15}
\]

geometric interpretation with \( 0 \leq \theta \leq \pi \) being the angle enclosed by the vectors \( u \) and \( v \), then

\[
 |u| \cos \theta
\]

can be interpreted as the projection of \( u \) onto the direction of \( v \) and

\[
 u \cdot v = |u||v| \cos \theta
\]

corresponds to the grey area in the picture with the above interpretation with \( 0 \leq \theta \leq \pi \), obviously

\[
 ||u \cdot v|| \leq ||u|| ||v|| \tag{1.1.16}
\]

properties of scalar product

\[
 u \cdot v = v \cdot u
\]

\[
 (\alpha u + \beta v) \cdot w = \alpha (u \cdot w) + \beta (v \cdot w) \tag{1.1.17}
\]

\[
 w \cdot (\alpha u + \beta v) = \alpha (w \cdot u) + \beta (w \cdot v)
\]

positive definiteness of scalar product

\[
 u \cdot u \geq 0, \quad u \cdot u = 0 \iff u = 0 \tag{1.1.18}
\]

orthogonal vectors \( u \) and \( v \)

\[
 u \cdot v = 0 \iff u \perp v \tag{1.1.19}
\]

### 1.1.1.4 Vector product

vector product of two vectors \( u, v \) defines a new vector \( w \in \mathbb{E}^3 \):

\[
 u \times v = w \tag{1.1.20}
\]

geometric interpretation with \( 0 \leq \theta \leq \pi \) being the angle enclosed by the vectors \( u \) and \( v \), then

\[
 |v| \sin \theta
\]

can be interpreted as the height of the grey polygon and

\[
 u \times v = |u||v| \sin \theta n
\]

introduces the vector \( w \) orthogonal to \( u \) and \( v \) whereby its length corresponds to the grey area with the above interpretation, obviously \( u \parallel v \) if

\[
 u \times v = 0 \iff u \parallel v \tag{1.1.21}
\]

index representation of \( w = u \times v \)

\[
 \begin{bmatrix}
 w_1 \\
 w_2 \\
 w_3 
\end{bmatrix}
 =
 \begin{bmatrix}
 u_2 v_3 - u_3 v_2 \\
 u_3 v_1 - u_1 v_3 \\
 u_1 v_2 - u_2 v_1 
\end{bmatrix} \tag{1.1.22}
\]

properties of vector product

\[
 u \times v = -v \times u
\]

\[
 (\alpha u + \beta v) \times w = \alpha (u \times w) + \beta (v \times w) \tag{1.1.23}
\]

\[
 u \cdot (u \times v) = 0
\]

\[
 (u \times v) \cdot (u \times v) = (u \cdot u)(v \cdot v) - (u \cdot v)^2
\]
1.1.1.5 Scalar triple vector product

scalar triple vector product of three vectors \( u, v, w \) introduces a scalar \( \alpha \in \mathbb{R} \)
\[
[u, v, w] = u \cdot (v \times w) = \alpha \tag{1.1.24}
\]

geometric interpretation
with vector product
\[
v \times w = ||v|| ||w|| \sin \theta n
\]
defining area of ground surface
\[
[u, v, w] = u \cdot (v \times w)
\]
defines volume of parallelepiped

obviously
\[
\alpha = u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v) \tag{1.1.25}
\]

index representation of \( \alpha = [u, v, w] \)
\[
\alpha = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) \tag{1.1.26}
\]

properties of scalar triple product
\[
[u, v, w] = [v, w, u] = [w, u, v] = -[u, w, v] = -[v, u, w] = -[w, v, u] \tag{1.1.27}
\]
\[
[\alpha u + \beta v, w, d] = \alpha [u, w, d] + \beta [v, w, d]
\]

three vectors \( u, v, w \) are linearly independent if
\[
[u, v, w] \neq 0 \tag{1.1.28}
\]