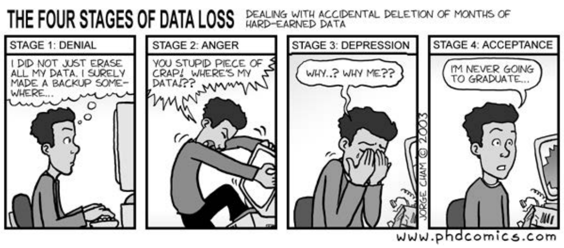
12 - finite element method - volume growth - implementation

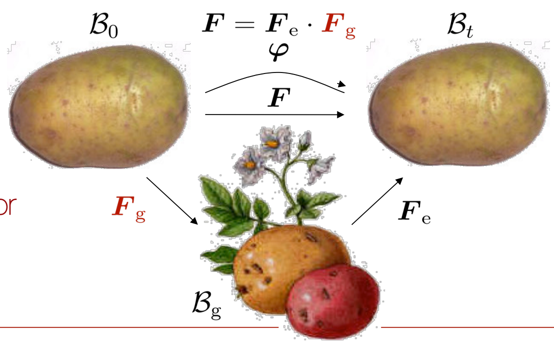


day	date		topic
tue	jan	07	motivation - everything grows!
thu	jan	09	basics maths - notation and tensors
tue	jan	14	project example - growing skin
thu	jan	16	kinematics - growing brains
tue	jan	21	basic kinematics - large deformation and growth
thu	jan	23	kinematics - growing hearts
tue	jan	28	kinematics - growing leaflets
thu	jan	30	basic balance equations - closed and open systems
tue	feb	04	basic constitutive equations - growing muscle
thu	feb	06	basic constitutive equations - growing tumors
tue	feb	11	volume growth - finite elements for growth - theory
thu	feb	13	volume growth - finite elements for growth - matlab
tue	feb	18	basic constitutive equations - growing bones
thu	feb	20	density growth - finite elements for growth
tue	feb	25	density growth - growing bones
thu	feb	27	everything grows! - midterm summary
tue	mar	04	midterm
thu	mar	06	remodeling - remodeling arteries and tendons
tue	mar	11	class project - discussion, presentation, evaluation
thu	mar	13	class project - discussion, presentation, evaluation
thu	mar	14	written part of final projects due

where are we???

the potato equations - kinematics

growth tensor



multiplicative decomposition

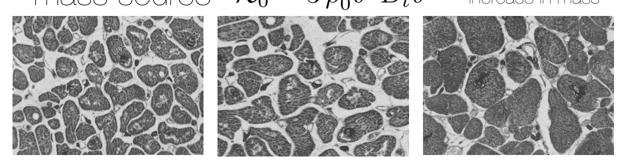
lee [1969], simo [1992], rodriguez, hoger & mc culloch [1994], epstein & maugin [2000], humphrey [2002], ambrosi & mollica [2002], himpel, kuhl, menzel & steinmann [2005]

volume growth at constant density

- free energy $\psi_0 = \psi_0^{\text{neo}}(\boldsymbol{F}_e)$
- stress $\boldsymbol{P}_e = \boldsymbol{P}_e^{\text{neo}}(\boldsymbol{F}_e)$
- growth tensor $\boldsymbol{F}_g = \vartheta\, \boldsymbol{I}$ $\quad \mathrm{D}_t\vartheta = k_\vartheta(\vartheta)\ \mathrm{tr}(\boldsymbol{C}_e \cdot \boldsymbol{S}_e)$

  growth function pressure
- mass source $\mathcal{R}_0 = 3\,\rho_0^*\vartheta^2 \mathrm{D}_t\vartheta$

  increase in mass

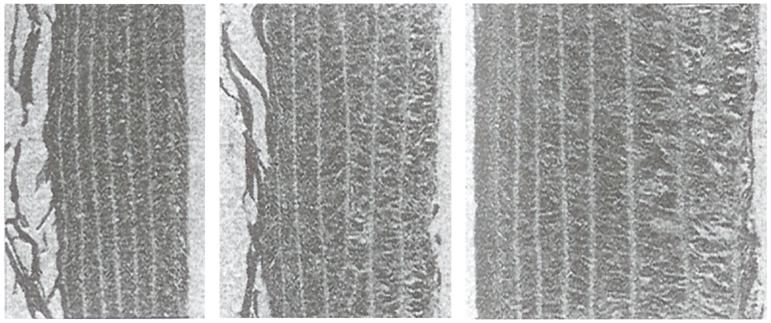


kinematic coupling of growth and deformation

rodriguez, hoger & mc culloch [1994], epstein & maugin [2000], humphrey [2002]

volume growth of the aortic wall

normosensitive hypersensitive severeley hypersensitive



wall thickening - thickening of musculoelastic fascicles

matsumoto & hayashi [1996], humphrey [2002]

compensatory wall thickening during atherosclerosis

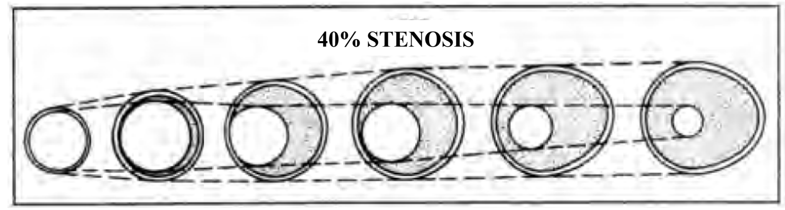
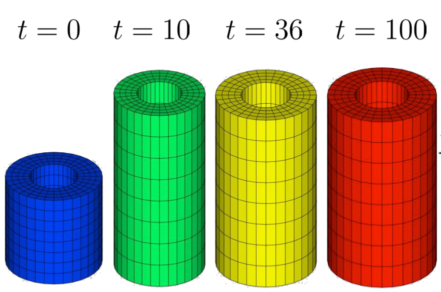
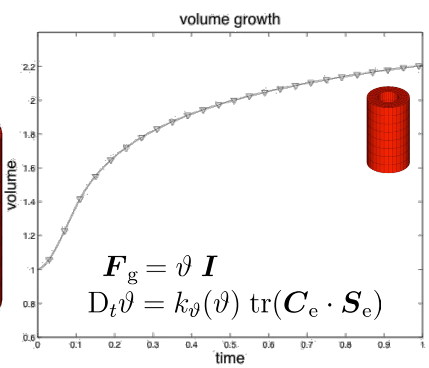


Figure 5. Diagrammic representation of a possible sequence of changes in atherosclerotic arteries leading eventually to lumen narrowing and consistent with the findings of this study. The artery enlarges initially (left to right in diagram) in association with the plaque accumulation to maintain an adequate, if not normal, lumen area. Early stages of lesion development may be associated with overcompensation. at more than 40% stenosis, however, the plaque area continues to increase to involve the entire circumference of the vessel, and the artery no longer enlarges at a rate suffcient to prevent the narrowing of the lumen.

glagov, weisenberg, zarins, stankunavicius, kolettis [1987]

volume growth in cylindrical tube





stress-induced volume growth

himpel, kuhl, menzel & steinmann [2005]

recipe for finite element modeling

from continuous problem...

$$\rho_0 \, \mathrm{D}_t \boldsymbol{v} \quad = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0$$

- temporal discretization — implicit euler backward
- spatial discretization — finite element method
- staggered/simultaneous — newton raphson iteration
- linearization — gateaux derivative

... to linearized discrete initial boundary value problem

key transformation - from strong form to weak form (1d)

- strong / differential form

$$\sum f = f^{\text{int}} + f^{\text{ext}} \doteq 0 \qquad f^{\text{int}} = P'(\varphi)$$

- strong form / residual format

$$\mathsf{R}(\varphi) = P'(\varphi) + f^{\text{ext}} \doteq 0$$

- weak / integral form - nonsymmetric $\forall \delta\varphi$

$$\mathsf{G}(\delta\varphi;\varphi) = \int \delta\varphi \cdot [\, P'(\varphi) + f^{\text{ext}}\,] \mathrm{d}x \doteq 0$$

- integration by parts

$$\int \delta\varphi \cdot P' \mathrm{d}x = \int [\, \delta\varphi \cdot P\,]' \mathrm{d}x - \int \delta\varphi' \cdot P \mathrm{d}x$$

- integral theorem & neumann bc 's

$$\int [\, \delta\varphi \cdot P\,]' \mathrm{d}x = \delta\varphi \cdot P|_{x=0}^{x=l}$$

- weak form / integral form - symmetric $\forall \delta\varphi$

$$\int \delta\varphi' \cdot P \mathrm{d}x - \delta\varphi \cdot P|_{x=0}^{x=l} - \int \delta\varphi \cdot f^{\text{ext}} \doteq 0$$

from equilibrium equation...

- start with nonlinear mechanical equilibrium equation

$$\rho_0 \overset{\approx \mathbf{0} \text{ quasi-static}}{\cancel{\mathrm{D}_t \boldsymbol{v}}} = \mathrm{Div}(\boldsymbol{P}) + \overset{\approx \mathbf{0} \text{ no gravity}}{\cancel{\boldsymbol{b}_0}}$$

- cast it into its residual format

$$\mathbf{R}^{\varphi}(\boldsymbol{\varphi}) = \mathbf{0} \qquad \text{in } \mathcal{B}_0$$

- with residual

$$\mathbf{R}^{\varphi} = \rho_0 \mathrm{D}_t \boldsymbol{v} \; - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0$$

... to residual format

residual equation...

- strong / differential form

$$\mathbf{R}^{\varphi} = \rho_0 \mathrm{D}_t \boldsymbol{v} \; - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 \; = \mathbf{0} \quad \text{in } \mathcal{B}_0$$

- dirichlet / essential boundary conditions (displacements)

$$\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}} \; = \mathbf{0} \quad \text{on } \partial\mathcal{B}_0^{\varphi} \quad \text{with } \partial\mathcal{B}_0^{\varphi} \cup \partial\mathcal{B}_0^{T^{\varphi}} = \partial\mathcal{B}_0$$

- neumann / natural boundary conditions (tractions)

$$\boldsymbol{P} \cdot \boldsymbol{N} - \bar{\boldsymbol{T}}^{\varphi} \; = \mathbf{0} \quad \text{on } \partial\mathcal{B}_0^{T^{\varphi}} \quad \text{and } \partial\mathcal{B}_0^{\varphi} \cap \partial\mathcal{B}_0^{T^{\varphi}} = \emptyset$$

... and boundary conditions

from strong form...

- strong / differential form

$$\mathbf{R}^{\varphi} = \rho_0 \mathrm{D}_t \boldsymbol{v} \; - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 \; = \mathbf{0} \qquad \text{in } \mathcal{B}_0$$

- mulitplication with test function & integration

$$\mathsf{G}^{\varphi}(\delta\boldsymbol{\varphi};\boldsymbol{\varphi}) = \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \mathbf{R}^{\varphi} \mathrm{d}V = 0 \qquad \forall \delta\boldsymbol{\varphi} \; \text{in } \mathcal{H}_1^0(\mathcal{B}_0)$$

- weak form / nonsymmetric

$$\mathsf{G}^{\varphi} = \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \rho_0 \mathrm{D}_t \boldsymbol{v} \,\mathrm{d}V - \int_{\mathcal{B}_0} \overset{\text{no derivative}}{\delta\boldsymbol{\varphi}} \cdot \mathrm{Div}(\overset{\text{second derivative}}{\boldsymbol{P}}) \mathrm{d}V - \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \boldsymbol{b}_0 \mathrm{d}V$$

... to nonsymmetric weak form

from non-symmetric weak form...

- integration by parts

$$\int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\mathrm{Div}(\boldsymbol{P})\mathrm{d}V = \int_{\mathcal{B}_0}\mathrm{Div}(\delta\boldsymbol{\varphi}\cdot\boldsymbol{P})\mathrm{d}V - \int_{\mathcal{B}_0}\nabla\delta\boldsymbol{\varphi}:\boldsymbol{P}\mathrm{d}V$$

- gauss theorem & boundary conditions

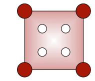
$$\int_{\mathcal{B}_0}\mathrm{Div}(\delta\boldsymbol{\varphi}\cdot\boldsymbol{P})\mathrm{d}V = \int_{\partial\mathcal{B}_0^{T\varphi}}\boldsymbol{\varphi}\cdot\boldsymbol{P}\cdot\boldsymbol{N}\mathrm{d}A = \int_{\partial\mathcal{B}_0^{T\varphi}}\delta\boldsymbol{\varphi}\cdot\bar{\boldsymbol{T}}^{\varphi}\mathrm{d}A$$

- weak form / symmetric — first derivative, first derivative

$$\mathsf{G}^{\varphi} = \int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\rho_0\,\mathrm{D}_t\boldsymbol{v}\,\mathrm{d}V + \int_{\mathcal{B}_0}\nabla\delta\boldsymbol{\varphi}:\boldsymbol{P}\mathrm{d}V - \int_{\partial\mathcal{B}_0^{T\varphi}}\delta\boldsymbol{\varphi}\cdot\bar{\boldsymbol{T}}^{\varphi}\mathrm{d}A - \int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\boldsymbol{b}_0\mathrm{d}V$$

... to symmetric weak form

spatial discretization



- discretization $\quad \mathcal{B}_0 = \bigcup_{e=1}^{n_{\mathrm{el}}}\mathcal{B}_0^e$

- interpolation of test functions

$$\delta\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \sum_{j=1}^{n_{\mathrm{en}}} N_{\varphi}^j\,\delta\boldsymbol{\varphi}_j \in \mathcal{H}_1^0(\mathcal{B}_0) \qquad \nabla\delta\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \sum_{j=1}^{n_{\mathrm{en}}} \delta\boldsymbol{\varphi}_j \otimes \nabla N_{\varphi}^j$$

- interpolation of trial functions

$$\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \sum_{l=1}^{n_{\mathrm{en}}} N_{\varphi}^l\,\boldsymbol{\varphi}_l \in \mathcal{H}_1(\mathcal{B}_0) \qquad \nabla\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \sum_{l=1}^{n_{\mathrm{en}}} \boldsymbol{\varphi}_l \otimes \nabla N_{\varphi}^l$$

... to discrete weak form

from discrete weak form...

- discrete weak form

$$\mathsf{G}^{\varphi} = \delta\boldsymbol{\varphi}_J\cdot\mathbf{R}_J^{\varphi}(\boldsymbol{\varphi}_{n+1}^h) = 0 \qquad \forall\delta\boldsymbol{\varphi}_J$$

- discrete residual format

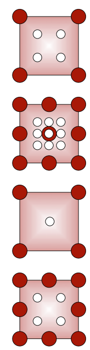
$$\mathbf{R}_J^{\varphi}(\boldsymbol{\varphi}_{n+1}^h) = \mathbf{0} \qquad \forall\ J = 1,\dots,n_{\mathrm{np}}$$

- discrete residual

$$\mathbf{R}_J^{\varphi} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}}\int_{\mathcal{B}_0^e}N_{\varphi}^j\mathrm{D}_t\boldsymbol{v}_{n+1}\mathrm{d}V + \int_{\mathcal{B}_0^e}\nabla N_{\varphi}^j\cdot\boldsymbol{P}_{n+1}\mathrm{d}V - \int_{\partial\mathcal{B}_0^e}N_{\varphi}^j\,\bar{\boldsymbol{T}}_{n+1}^{\varphi}\mathrm{d}A - \int_{\mathcal{B}_0^e}N_{\varphi}^j\boldsymbol{b}_{0\,n+1}\mathrm{d}V$$

... to discrete residual

integration point based solution of growth equation



loop over all time steps	`nlin_fem`
global newton iteration	`nlin_fem`
loop over all elements	`brick_3d`
loop over all quadrature points	`cnst_vol`
local newton iteration ϑ_{n+1}	`upd_vol`
determine element residual & tangent	`cnst_vol`
determine global residual and tangent	`brick_3d`
determine $\boldsymbol{\varphi}_{n+1}$	`nlin_fem`
determine state of biological equilibrium	`nlin_fem`

growth multiplier ϑ as internal variable

nlin_fem.m

```
%%% loop over all load steps %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for is = (nsteps+1):(nsteps+inpstep);
   iter = 0;  residuum = 1;
%%% global newton-raphson iteration %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
   while residuum > tol
      iter=iter+1;
      R = zeros(ndof,1); K = sparse(ndof,ndof);
      e_spa = extr_dof(edof,dof);
%%%%% loop over all elements %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
      for ie = 1:nel
         [Ke,Re,Ie] = element1(e_mat(ie,:),e_spa(ie,:),i_var(ie,:),mat);
         [K, R, I ] = assm_sys(edof(ie,:),K,Ke,R,Re,I,Ie);
      end
%%%%% loop over all elements %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
      u_inc(:,2)=dt*u_pre(:,2);  R = R - time*F_pre;   dofold = dof;
      [dof,F] = solve_nr(K,R,dof,iter,u_inc);
      residuum= res_norm((dof-dofold),u_inc);
    end
%%% global newton-raphson iteration %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    time = time + dt;   i_var = I;   plot_int(e_spa,i_var,nel,is);
end
%%% loop over all load steps %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```


discrete residual

- discrete residual

$$\mathbf{R}^{\varphi}_J = \mathbf{A}_{e=1}^{n_{\text{el}}} \int_{\mathcal{B}_0^e} \nabla N^j_{\varphi} \cdot \boldsymbol{P}_{n+1} \mathrm{d}V = \mathbf{0} \qquad \forall\ J = 1, ..., n_{\text{np}}$$

righthand side vector for global system of equations

@ the element level

- determine global residual

check in matlab!

$$\mathbf{R}^{\varphi}_J = \mathbf{A}_{e=1}^{n_{\text{el}}} \int_{\mathcal{B}_0^e} \nabla N^j_{\varphi} \cdot \boldsymbol{P}_{n+1} \mathrm{d}V$$

- residual of mechanical equilibrium/balance of momentum

righthand side vector for global system of equations

discrete residual

check in matlab!

$$\mathbf{R}^{\varphi}_J = \mathbf{A}_{e=1}^{n_{\text{el}}} \int_{\mathcal{B}_0^e} \nabla N^j_{\varphi} \cdot \boldsymbol{P}_{n+1} \mathrm{d}V$$

quads_2d.m/brick_3d.m

```
for i=1:nod
  en=(i-1)*2;
  Re(en+ 1) = Re(en+ 1) +(P(1,1)*dNx(1,i)' ...
                 + P(1,2)*dNx(2,i)') * detJ * wp(ip);
  Re(en+ 2) = Re(en+ 2) +(P(2,1)*dNx(1,i)' ...
                 + P(2,2)*dNx(2,i)') * detJ * wp(ip);
end
```

righthand side vector for global system of equations

from discrete residual …

- linearization / newton raphson scheme
$$\mathbf{R}^{\varphi\,k+1}_{J\,n+1} = \mathbf{R}^{\varphi\,k}_{J\,n+1} + \mathrm{d}\mathbf{R}^{\varphi}_{J} \doteq 0 \qquad \forall\; J = 1, ..., n_{\mathrm{np}}$$
- incremental residual
$$\mathrm{d}\mathbf{R}^{\varphi}_{J} = \textstyle\sum_{L=1}^{n_{\mathrm{en}}} \mathbf{K}^{\varphi\varphi}_{JL} \cdot \mathrm{d}\boldsymbol{\varphi}_L \qquad\qquad \mathbf{K}^{\varphi\varphi}_{JL} = \frac{\mathrm{d}\mathbf{R}^{\varphi}_{J}}{\mathrm{d}\boldsymbol{\varphi}_L}$$
- system of equations
$$\mathbf{K}^{\varphi\varphi}_{JL}\;\mathrm{d}\boldsymbol{\varphi}_L = -\mathbf{R}^{\varphi\,k}_{J\,n+1}$$
- incremental iterative update
$$\Delta\boldsymbol{\varphi}_L = \Delta\boldsymbol{\varphi}_L + \mathrm{d}\boldsymbol{\varphi}_L \qquad\qquad \forall\; L = 1, ..., n_{\mathrm{np}}$$

… to linearized residual

linearized residual

- stiffness matrix / iteration matrix

$$\mathbf{K}^{\varphi\varphi}_{JL} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} N^j_{\varphi} \rho \mathrm{D}_{\varphi}(\mathrm{D}_t \boldsymbol{v}) N^l_{\varphi} \mathrm{d}V + \int_{\mathcal{B}_0^e} \nabla N^j_{\varphi} \cdot \mathrm{D}_F \boldsymbol{P} \cdot \nabla N^l_{\varphi} \mathrm{d}V$$

4th order tensor - derivatives of 2nd order tensors wrt 2nd order tensor

- linearization of residual wrt nodal dofs

iteration matrix for global system of equations

@ the element level

- stiffness matrix / iteration matrix

check in matlab!

$$\mathbf{K}^{\varphi\varphi}_{JL} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} \nabla N^j_{\varphi} \cdot \mathrm{D}_F \boldsymbol{P} \cdot \nabla N^l_{\varphi} \mathrm{d}V$$

- linearization of residual wrt nodal dofs

iteration matrix for global system of equations

linearized residual

check in matlab!

$$\mathbf{K}^{\varphi\varphi}_{JL} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} \nabla N^j_{\varphi} \cdot \mathrm{D}_F \boldsymbol{P} \cdot \nabla N^l_{\varphi} \mathrm{d}V$$

quads_2d.m/brick_3d.m

```
for i=1:nod; for j=1:nod
  eni=(i-1)*2; enj=(j-1)*2;
  Ke(eni+1,enj+1)=Ke(eni+1,enj+1)+(dNx(1,i)*A(1,1,1,1)*dNx(1,j) ...
                  +dNx(1,i)*A(1,1,1,2)*dNx(2,j) ...
                  +dNx(2,i)*A(1,2,1,1)*dNx(1,j) ...
                  +dNx(2,i)*A(1,2,1,2)*dNx(2,j))*detJ*wp(ip);
end; end
```

iteration matrix for global system of equations

from integral equations...

- integral equations cannot be evaluated analytically

$$\mathbf{R}_j^e = \int_\zeta \int_\eta \int_\xi \nabla N_\varphi^j(\xi,\eta,\zeta) \cdot \boldsymbol{P}_{n+1}(\xi,\eta,\zeta)\, \det(\boldsymbol{J}(\xi,\eta,\zeta))\, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$$

$$\mathbf{K}_{jl}^e = \int_\zeta \int_\eta \int_\xi \nabla N_\varphi^j(\xi,\eta,\zeta) \cdot \mathrm{D}_F \boldsymbol{P}(\xi,\eta,\zeta) \cdot \nabla N_\varphi^l(\xi,\eta,\zeta)\, \det(\boldsymbol{J}(\xi,\eta,\zeta))\, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$$

- idea - numerical interation / quadrature

$$\mathbf{R}_j^e \approx \sum_{i=0}^{n} \nabla N_\varphi^j(\xi_i,\eta_i,\zeta_i) \cdot \boldsymbol{P}_{n+1}(\xi_i,\eta_i,\zeta_i)\, \det(\boldsymbol{J}(\xi_i,\eta_i,\zeta_i)\, w_i$$

$$\mathbf{K}_{jl}^e \approx \sum_{i=0}^{n} \nabla N_\varphi^j(\xi_i,\eta_i,\zeta_i) \cdot \mathrm{D}_F \boldsymbol{P}(\xi_i,\eta_i,\zeta_i) \cdot \nabla N_\varphi^l(\xi_i,\eta_i,\zeta_i)\, \det(\boldsymbol{J}(\xi_i,\eta_i,\zeta_i)\, w_i$$

... to discrete sums

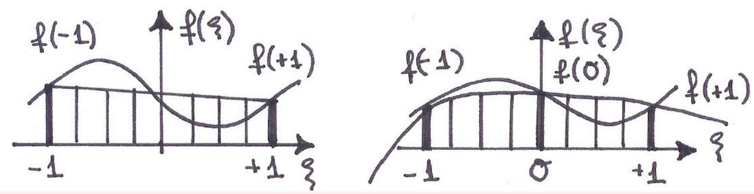
numerical integration

- integral equations are approximated by discrete sums

$$\int_a^b f(\xi)\mathrm{d}\xi \approx [\, b - a\,] \sum_{i=0}^{n} f(\xi_i)\, w_i$$

ξ_i ... quadrature point coordinates

w_i... quadrature point weights



newton cotes quadrature - accuracy [n-1]

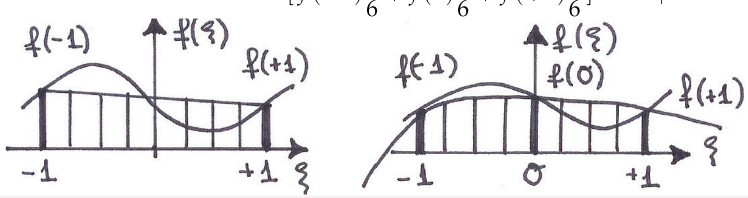
equidistant quadrature points @ $\xi_i = -1 + 2\,\frac{i}{n}$

n=2 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \;\approx\; 2\,[\, f(\xi_0)\, w_0 + f(\xi_1)\, w_1\,]$

$= f(-1) + f(+1)$ trapezoidal rule

n=3 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \;\approx\; 2\,[\, f(\xi_0)\, w_0 + f(\xi_1)\, w_1 + f(\xi_2)\, w_2\,]$

$= 2\,[\, f(-1)\frac{1}{6} + f(0)\frac{4}{6} + f(+1)\frac{1}{6}\,]$ simpson rule



gauss legendre quadrature - accuracy [2n-1]

optimized quadrature points

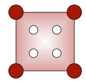
n=1 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \;\approx\; 2\,[\, f(0)\, 1\,]$

n=2 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \;\approx\; 2\,[\, f(-\frac{1}{\sqrt{3}})\,\frac{1}{2} + f(+\frac{1}{\sqrt{3}})\,\frac{1}{2}\,]$

n=3 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \;\approx\; 2\,[\, f(-\frac{3}{\sqrt{5}})\,\frac{5}{18} + f(0)\,\frac{8}{18} + f(+\frac{3}{\sqrt{5}})\,\frac{5}{18}\,]$

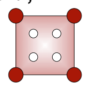
most fe programs prefer gauss over newton!

quads_2d.m



```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [Ke,Re,Ie]=element1(e_mat,e_spa,i_var,mat)
%%% element stiffness matrix Ke, residual Re, internal variables Ie %%%%
Ie = i_var;
Re = zeros(8,1);
Ke = zeros(8,8);
nod=4;            delta = eye(2);
indx=[1;3;5;7];  ex_mat=e_mat(indx);
indy=[2;4;6;8];  ey_mat=e_mat(indy);
%%% integration points %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
g1=0.577350269189626; w1=1;
gp(:,1)=[-g1; g1;-g1; g1];   w(:,1)=[ w1; w1; w1; w1];
gp(:,2)=[-g1;-g1; g1; g1];   w(:,2)=[ w1; w1; w1; w1];
wp=w(:,1).*w(:,2);      xsi=gp(:,1);      eta=gp(:,2);
%%% shape functions and derivatives in isoparametric space %%%%%%%%%%%%%
N(:,1)=(1-xsi).*(1-eta)/4;   N(:,2)=(1+xsi).*(1-eta)/4;
N(:,3)=(1+xsi).*(1+eta)/4;   N(:,4)=(1-xsi).*(1+eta)/4;
dNr(1:2:8  ,1)=-(1-eta)/4;   dNr(1:2:8  ,2)= (1-eta)/4;
dNr(1:2:8  ,3)= (1+eta)/4;   dNr(1:2:8  ,4)=-(1+eta)/4;
dNr(2:2:8+1,1)=-(1-xsi)/4;   dNr(2:2:8+1,2)=-(1+xsi)/4;
dNr(2:2:8+1,3)= (1+xsi)/4;   dNr(2:2:8+1,4)= (1-xsi)/4;
JT=dNr*[ex_mat;ey_mat]';
%%% element stiffness matrix Ke, residual Re, internal variables Ie %%%%
```

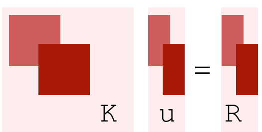

quads_2d.m



```
%%% loop over all integration points %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for ip=1:4
indx=[2*ip-1; 2*ip];  detJ=det(JT(indx,:));
if detJ<10*eps;  disp('Jacobi determinant less than zero!');  end;
JTinv=inv(JT(indx,:));  dNx=JTinv*dNr(indx,:);
F=zeros(2,2);
for j=1:4
   jndx=[2*j-1; 2*j];
   F=F+e_spa(jndx)'*dNx(:,j)';
end
var = i_var(ip);
[A,P,var]=cnst_law(F,var,mat);
Ie(ip) = var;
for i=1:nod
   en=(i-1)*2;
   Re(en+ 1) =  Re(en+ 1) +(P(1,1)*dNx(1,i)' ...
                          + P(1,2)*dNx(2,i)') * detJ * wp(ip);
   Re(en+ 2) =  Re(en+ 2) +(P(2,1)*dNx(1,i)' ...
                          + P(2,2)*dNx(2,i)') * detJ * wp(ip);
end
%%% loop over all integration points %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% element stiffness matrix Ke, residual Re, internal variables Ie %%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```


assm_sys.m

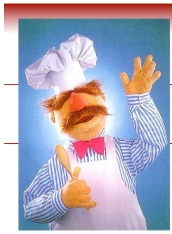
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [K,R,I]=assm_sys(edof,K,Ke,R,Re,I,Ie)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% assemble local element contributions to global tangent & residual %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% input:  edof  = [ elem X1 Y1 X2 Y2 ]     ... incidence matrix
%%%         Ke    = [ nedof x nedof ]        ... element tangent Ke
%%%         Re    = [ fx_1 fy_1 fx_2 fy_2 ] ... element residual Re
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% output: K     = [ ndof x ndof ]          ... global tangent K
%%%         R     = [ ndof x 1 ]             ... global residual R
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[nie,n]=size(edof);
I(edof(:,1),:)=Ie(:);
t=edof(:,2:n);
for i = 1:nie
   K(t(i,:),t(i,:)) = K(t(i,:),t(i,:))+Ke;
   R(t(i,:))        =R(t(i,:))        +Re;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```



@ integration point level

- constitutive equations - given $\boldsymbol{F} = \nabla\varphi$ calculate $\boldsymbol{P}$
- update growth multiplier for current stress state from ϑ_{n} and $\mathrm{D}_t\vartheta = k_\vartheta(\vartheta)\,\mathrm{tr}(\boldsymbol{C}_{\mathrm{e}} \cdot \boldsymbol{S}_{\mathrm{e}})$ calculate $\vartheta_{\mathrm{n+1}}$
- update growth tensor $\boldsymbol{F}^{\mathrm{g}} = \vartheta\,\boldsymbol{I}$ and $\boldsymbol{F}^{\mathrm{g}\,-1} = \dfrac{1}{\vartheta}\,\boldsymbol{I}$
- calculate elastic tensor $\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}} \cdot \boldsymbol{F}^{\mathrm{g}} \qquad \boldsymbol{F}^{\mathrm{e}} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{g}\,-1} = \boldsymbol{F}/\vartheta$
- calculate stress $\boldsymbol{P}(\boldsymbol{F}^{\mathrm{e}}) = \mu\,\boldsymbol{F}^{\mathrm{e}} + [\,\lambda\,\ln(\det(\boldsymbol{F}^{\mathrm{e}})) - \mu\,]\,\boldsymbol{F}^{\mathrm{e}\,-\mathrm{t}}$

stress for righthand side vector



recipe for temporal discretization

explicit euler forward

- evolution of growth multiplier

$$D_t\vartheta = \frac{1}{\Delta t}[\,\vartheta_{n+1} - \vartheta_n\,] = \dot{\vartheta} \qquad \text{finite difference approximation}$$

$$D_t\vartheta = k(\vartheta_n)\,\mathrm{tr}(\boldsymbol{C}^e \cdot \boldsymbol{S}^e) \qquad \text{euler forward}$$

- direct update of growth multiplier

$$\vartheta_{n+1} = \vartheta_n + k(\vartheta_n)\,\mathrm{tr}(\boldsymbol{C}^e \cdot \boldsymbol{S}^e)\Delta t \doteq 0$$

$$k(\vartheta_n) = k_\vartheta^+ \left[\frac{\vartheta^{max} - \vartheta_n}{\vartheta^{max} - 1}\right] \quad \text{if} \quad \mathrm{tr}(\boldsymbol{C}^e \cdot \boldsymbol{S}^e) > 0$$

$$k(\vartheta_n) = k_\vartheta^+ \left[\frac{\vartheta_n - \vartheta^{min}}{\vartheta^{min} - 1}\right] \quad \text{if} \quad \mathrm{tr}(\boldsymbol{C}^e \cdot \boldsymbol{S}^e) < 0$$

conditionally stable - limited to small time steps

recipe for temporal discretization

implicit euler backward

- evolution of growth multiplier

$$D_t\vartheta = \frac{1}{\Delta t}[\,\vartheta_{n+1} - \vartheta_n\,] = \dot{\vartheta} \qquad \text{finite difference approximation}$$

$$D_t\vartheta = k(\vartheta_{n+1})\,\mathrm{tr}(\boldsymbol{C}^e \cdot \boldsymbol{S}^e) \qquad \text{euler backward}$$

- discrete residual

$$R^\vartheta_{n+1} = \vartheta_{n+1} - \vartheta_n - k(\vartheta_{n+1})\,\mathrm{tr}(\boldsymbol{C}^e \cdot \boldsymbol{S}^e)\Delta t \doteq 0$$

- local newton iteration

$$R^{\vartheta k+1}_{n+1} = R^{\vartheta k}_{n+1} + dR^\vartheta \doteq 0 \qquad dR^\vartheta = \frac{dR^\vartheta}{d\vartheta}\,d\vartheta$$

$$\vartheta_{n+1} \leftarrow \vartheta_n + d\vartheta \qquad d\vartheta = \left[\frac{dR^\vartheta}{d\vartheta}\right]^{-1} R^{\vartheta k}_{n+1} \qquad \text{iterative update}$$

unconditionally stable - larger time steps

@ the integration point level

- constitutive equations - given $\boldsymbol{F}$ calculate $\boldsymbol{P}$

check in matlab!

$$\boldsymbol{P}(\boldsymbol{F}^e) = \mu\,\boldsymbol{F}^e + [\,\lambda\,\ln(\det(\boldsymbol{F}^e)) - \mu\,]\,\boldsymbol{F}^{e\,-t}$$

- stress calculation @ integration point level

stress for righthand side vector

@ the integration point level

check in matlab!

$$\boldsymbol{P}(\boldsymbol{F}^e) = \mu\,\boldsymbol{F}^e + [\,\lambda\,\ln(\det(\boldsymbol{F}^e)) - \mu\,]\,\boldsymbol{F}^{e\,-t}$$

const_vol.m

```
Fe     = F / theta;
Fe_inv = inv(Fe);
Je     = det(Fe);
delta  = eye(ndim);

P = xmu * Fe + (xlm * log(Je) - xmu) * Fe_inv';
```

stress for righthand side vector

@ the integration point level

- tangent operator / constitutive moduli

check in matlab!

$$\mathsf{A} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\boldsymbol{F}} = \left.\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}}\right|_{\boldsymbol{F}^{\mathrm{g}}} + \left.\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}^{\mathrm{g}}} : \frac{\partial \boldsymbol{F}^{\mathrm{g}}}{\partial \vartheta} \otimes \frac{\partial \vartheta}{\partial \boldsymbol{F}}\right|_{\boldsymbol{F}}$$

- linearization of stress wrt deformation gradient

tangents for iteration matrix

@ the integration point level

check in matlab!

$$\mathsf{A} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\boldsymbol{F}} = \left.\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}}\right|_{\boldsymbol{F}^{\mathrm{g}}} + \left.\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}^{\mathrm{g}}} : \frac{\partial \boldsymbol{F}^{\mathrm{g}}}{\partial \vartheta} \otimes \frac{\vartheta}{\boldsymbol{F}}\right|_{\boldsymbol{F}}$$

const_vol.m

```
for i=1:ndim; for j=1:ndim; for k=1:ndim; for l=1:ndim
 A(i,j,k,l) = xlm * Fe_inv(j,i)*Fe_inv(l,k) ...
           - (xlm * log(Je) - xmu) * Fe_inv(l,i)*Fe_inv(j,k) ...
           + xmu * delta(i,k)* delta(j,l) ...
           + ten1(i,j)* ten2(k,l);
end, end, end, end
A = A / theta;
```

tangent for iteration matrix

@ the integration point level

- discrete update of growth multiplier

check in matlab!

$$\mathsf{R}^{\vartheta}_{\mathrm{n+1}} = \vartheta_{\mathrm{n+1}} - \vartheta_{\mathrm{n}} - k\,\mathrm{tr}(\boldsymbol{M}^{\mathrm{e}})\,\Delta t$$

- residual of biological equilibrium

local newton iteration

@ the integration point level

check in matlab!

$$\mathsf{R}^{\vartheta}_{\mathrm{n+1}} = \vartheta_{\mathrm{n+1}} - \vartheta_{\mathrm{n}} - k\,\mathrm{tr}(\boldsymbol{M}^{\mathrm{e}})\,\Delta t$$

updt_vol.m

```
while abs(res) > tol
   res  = k * tr_Me * dt - the_k1 + the_k0;
   dres =(dk_dthe * tr_Me + k * dtrM_dthe)*dt -1;
end
```

local newton iteration

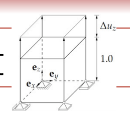
cnst_vol.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function[A,P,var]=cnst_vol(F,var,mat,ndim)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
determine tangent, stress and internal variable
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
emod = mat(1);   nue = mat(2);  kt = mat(3);  kc = mat(4);
mt   = mat(5);   mc  = mat(6);  tt = mat(7);  tc = mat(8); dt=mat(9);
xmu = emod / 2 / (1+nue);       xlm= emod * nue / (1+nue) /(1-2*nue);
%%% update internal variable%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[var,ten1,ten2]=updt_vol(F,var,mat,ndim);
theta =var(1)+1; Fe=F/theta;Fe_inv=inv(Fe);Je=det(Fe);delta=eye(ndim);
%%% first piola kirchhoff stress %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
P = xmu * Fe + (xlm * log(Je) - xmu) * Fe_inv';
%%% tangent %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for i=1:ndim; for j=1:ndim; for k=1:ndim; for l=1:ndim
    A(i,j,k,l) =  xlm * Fe_inv(j,i)*Fe_inv(l,k) ...
               - (xlm * log(Je) - xmu) * Fe_inv(l,i)*Fe_inv(j,k) ...
               +                  xmu  *  delta(i,k)* delta(j,l) ...
               +                           ten1(i,j)*  ten2(k,l);
end, end, end, end;
A = A / theta;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```


updt_vol.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% local newton-raphson iteration %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
while abs(res) > tol
  iter=iter+1;
  Fe = F/the_k1; Fe_inv = inv(Fe);   Ce = Fe'*Fe; Ce_inv = inv(Ce);
  Je = det(Fe);  delta  = eye(ndim);
  Se = xmu * delta + (xlm * log(Je) - xmu) * Ce_inv;
  Me = Ce*Se;      tr_Me = trace(Me);
  CeLeCe = ndim * ndim * xlm - 2 * ndim * (xlm * log(Je) - xmu);
  dtrM_dthe = - 1/the_k1 * ( 2*tr_Me + CeLeCe );
  if tr_Me > 0
     k        = kt*((tt-the_k1)/(tt-1))^mt;
     dk_dthe = k  /(the_k1-tt)          *mt;
  else
     k        = kc*((the_k1-tc)/(1-tc))^mc;
     dk_dthe = k  /(the_k1-tc)          *mc;
  end
res  = k * tr_Me * dt - the_k1 + the_k0;
dres =(dk_dthe * tr_Me + k * dtrM_dthe)*dt -1;
the_k1 =  the_k1 -res/dres;
if(iter>20); disp(['*** NO LOCAL CONVERGENCE ***']); return; end;
%%% local newton-raphson iteration %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```


probing the material @the integration point



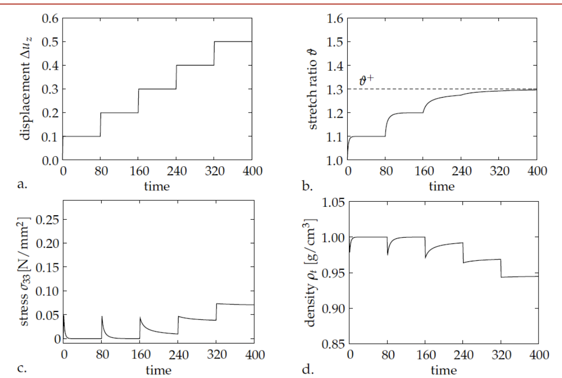
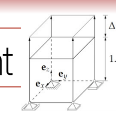


Figure 4.2: Isotropic simple tension test on a growing cube. (a) An incrementally increasing stretch is applied. (b) The stretch ratio converges time-dependently to the biological equilibrium. (c) The stresses vanish in the biological equilibrium state as long as $\vartheta < \vartheta^+$. (d) The density in the biological equilibrium state does not change as long as $\vartheta < \vartheta^+$.

himpel, kuhl, menzel & steinmann [2005]

probing the material @the integration point



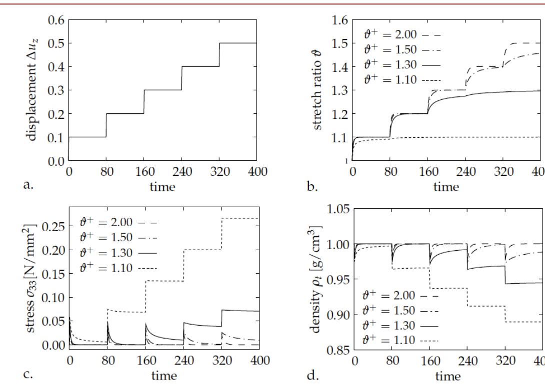
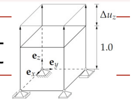


Figure 4.3: Variation of the limiting stretch ratio ϑ^+ in the simple tension test. The stretch ratio increases until the limiting value is reached. If the limiting value of the stretch ratio is reached the material behavior is purely elastic.

himpel, kuhl, menzel & steinmann [2005]

probing the material @the integration point



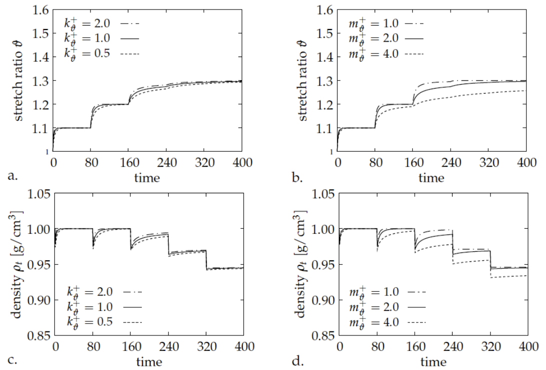
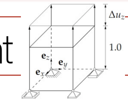


Figure 4.4: Variation of the material parameters k_ϑ^+ and m_ϑ^+ in the simple tension test. They influence the relaxation time, but not the final state at biological equilibrium.

himpel, kuhl, menzel & steinmann [2005]

probing the material @the integration point



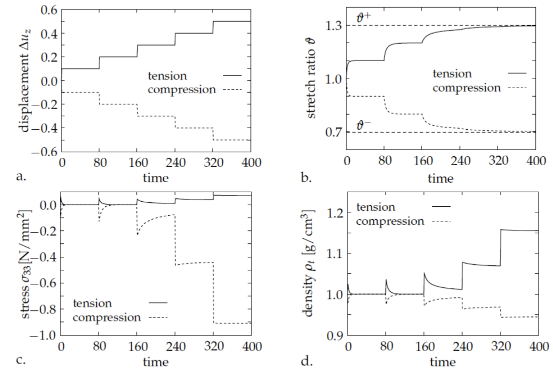
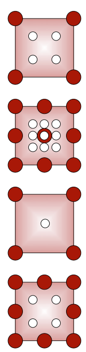


Figure 4.5: The material distinguishes between tension and compression. In case of tension the material grows, and in case of compression the material decreases.

himpel, kuhl, menzel & steinmann [2005]

integration point based solution of growth equation



- loop over all time steps — nlin_fem
  - global newton iteration — nlin_fem
    - loop over all elements — brick_3d
      - loop over all quadrature points — cnst_vol
        - local newton iteration ϑ_{n+1} — upd_vol
      - determine element residual & tangent — cnst_vol
    - determine global residual and tangent — brick_3d
  - determine φ_{n+1} — nlin_fem
- determine state of biological equilibrium — nlin_fem

growth multiplier ϑ as internal variable

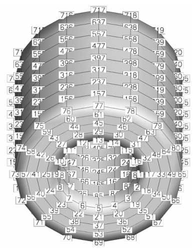
ex_tube1.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [q0,edof,emat,bc,F_ext,mat,ndim,nel,node,ndof,nip,nlod] = ex_tube1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% material parameters for volume growth %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
emod = 3.0; nue = 0.3;
kt   = 0.5; kc   = 0.25; mt = 4.0; mc = 5.0; tt = 1.5; tc = 0.5;dt=1.0;
mat=[emod,nue,kt,kc,mt,mc,tt,tc,dt];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
l   =    2.0;      % length
ra  =    1.0;      % inner radius
ri  =    0.5;      % outer radius
nez =      8;      % number of elements in longitudinal direction
ner =      4;      % number of elements in radial direction
nep =     16;      % number of elements in circumferential direction
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tol  = 1e-8;
ndim = 3;
nip  = 8;
nel = nez * ner * nep;
node= (nez+1)*(ner+1)*nep;
ndof = ndim*node;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

ex_tube1.m

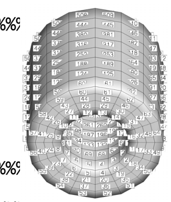
```
%%% coordinates %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
q0 = zeros(ndim*node,1);
nn = 0;

delta_z =       l / nez;
delta_r = (ra-ri) / ner;
delta_t =    2*pi / nep;
for iz = 0:nez
  z = iz * delta_z;
  for ir = 0:ner
    r = ri + ir * delta_r;
    for ip = 0:(nep-1)
      p = ip * delta_t;
      nn = nn +ndim;
      q0(nn-2,1) = r*cos(p);
      q0(nn-1,1) = r*sin(p);
      q0(nn  ,1) = z;
    end
  end
end
%%% coordinates %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```



ex_tube1.m

```
%%% connectivity %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for ie = 1:nel
   edof(ie,:)=[ie, ndim*enod(ie,1)-2  ndim*enod(ie,1)-1  ndim*enod(ie,1) ...
                   ndim*enod(ie,2)-2  ndim*enod(ie,2)-1  ndim*enod(ie,2) ...
                   ndim*enod(ie,3)-2  ndim*enod(ie,3)-1  ndim*enod(ie,3) ...
                   ndim*enod(ie,4)-2  ndim*enod(ie,4)-1  ndim*enod(ie,4) ...
                   ndim*enod(ie,5)-2  ndim*enod(ie,5)-1  ndim*enod(ie,5) ...
                   ndim*enod(ie,6)-2  ndim*enod(ie,6)-1  ndim*enod(ie,6) ...
                   ndim*enod(ie,7)-2  ndim*enod(ie,7)-1  ndim*enod(ie,7) ...
                   ndim*enod(ie,8)-2  ndim*enod(ie,8)-1  ndim*enod(ie,8)];
end
%%% boundary conditions %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
du = l/2; nb = 0;
for ib = 1:(nep*(ner+1))
  if(abs(q0(ndim*(node-nep*(ner+1))+ndim*ib-2)-0.0)<tol)
    nb = nb+1;     bc(nb,:) = [ndim*ib-2  0];
  end  if(abs(q0(ndim*(node-nep*(ner+1))+ndim*ib-1)-0.0)<tol)
end
%%% loading %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
load  = 0.0;F_ext = zeros(ndof,1);  nlod  = 1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```



ex_tube1.m

Comp Meth Eng Sci. 2005;8:119-134

Computational modelling of isotropic multiplicative growth

G. Himpel, E. Kuhl, A. Menzel, P. Steinmann

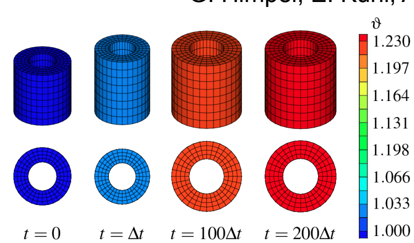


Figure 10 : Deformation of the tube and evolution of the stretch ratio for an axial stretch $u = 1.0$.

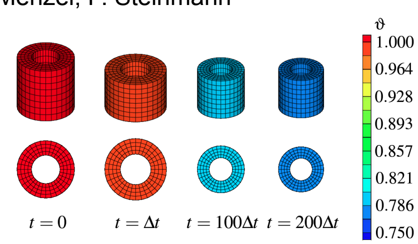
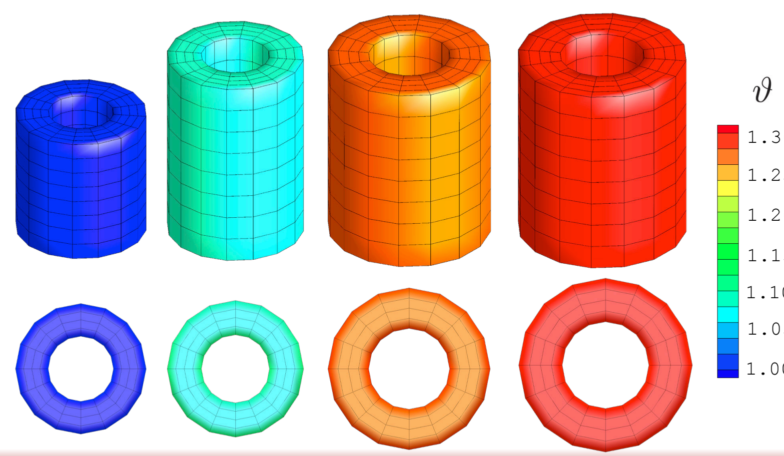


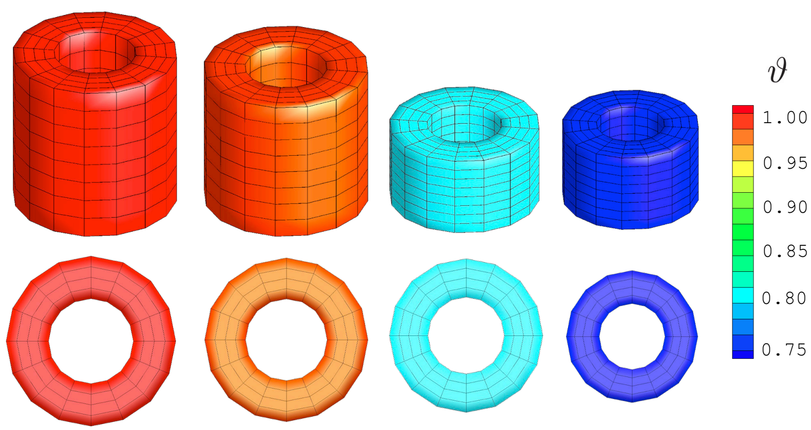
Figure 12 : Deformation of the tube and evolution of the stretch ratio for an axial compression $u = -1.0$.

himpel, kuhl, menzel & steinmann [2005]

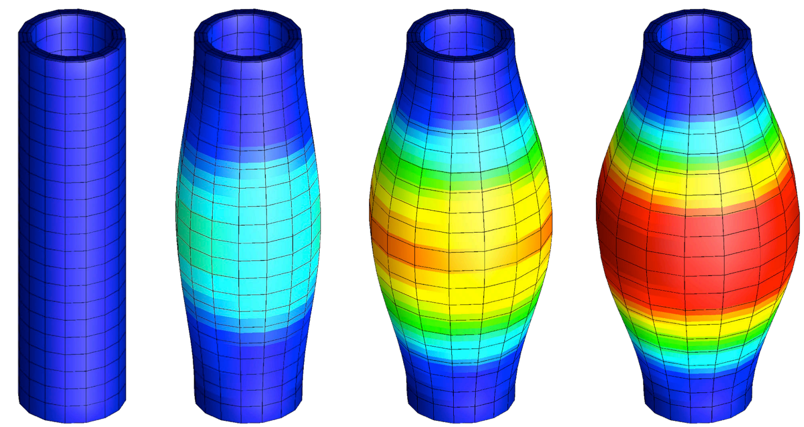
ex_tube1.m



ex_tube2.m



ex_tube3.m



atherosclerosis

atherosclerosis is a condition in which an artery wall thickens as the result of a build-up of fatty materials. the atheromatous plaques, although compensated for by artery enlargement, eventually lead to plaque rupture and clots inside the arterial lumen. the clots leave behind stenosis, a narrowing of the artery, and insufficient blood supply to the tissues and organ it feeds. if the artery enlargement is excessive, a net aneurysm results. these complications of advanced atherosclerosis are chronic, slowly progressive and cumulative.

Wikipedia The Free Encyclopedia

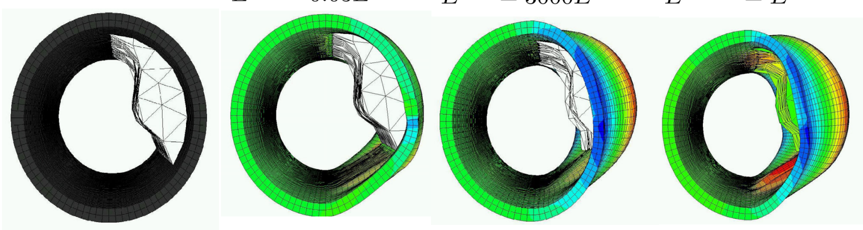
atherosclerosis



[greek] arteria = artery / sclerosis = hardening

qualitative simulation of atherosclerosis

initial conditions | adiposis $E^{\text{fat}} = 0.03 E^{\text{tissue}}$ | calcification $E^{\text{lime}} = 3000 E^{\text{tissue}}$ | growing plaque $E^{\text{plaque}} = E^{\text{tissue}}$



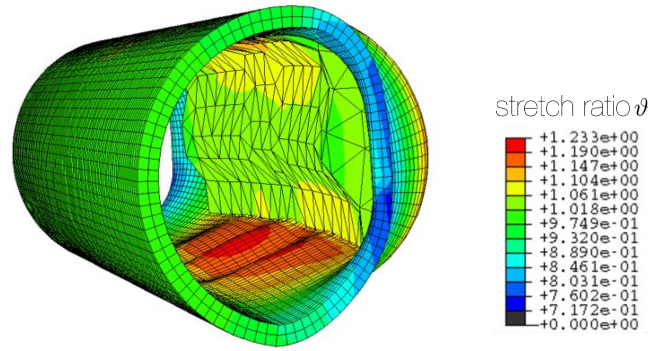
early stage soft plaque pressed into wall | later stage stiff plaque induces high stresses

overall thickening - thickening of individual fascicles

holzapfel [2001], holzapfel & ogden [2003], kuhl, maas, himpel & menzel [2007]

qualitative simulation of atherosclerosis

restenotic conditions



re-nerrowing of x-section in response to high stress

kuhl, maas, himpel & menzel [2007]

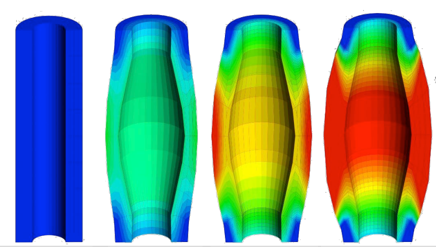
in-stent restenosis

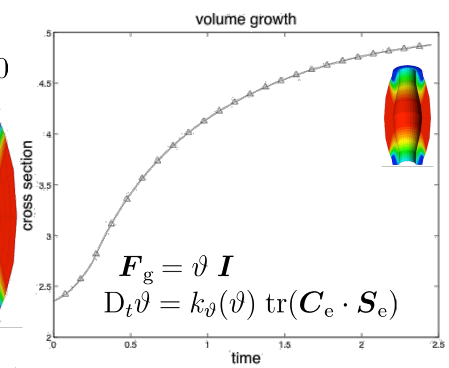
restenosis is the reoccurrence of stenosis, the narrowing of a blood vessel, leading to restricted blood flow. restenosis usually pertains to a blood vessel that has become narrowed, received treatment, and subsequently became renarrowed. in some cases, surgical procedures to widen blood vessels can cause further narrowing. during balloon angioplasty, the balloon 'smashes' the plaques against the arterial wall to widen the size of the lumen. however, this damages the wall which responds by using physiological mechanisms to repair the damage and the wall thickens.

Wikipedia The Free Encyclopedia

qualitative simulation of stent implantation

$t = 0$ $t = 10$ $t = 36$ $t = 100$



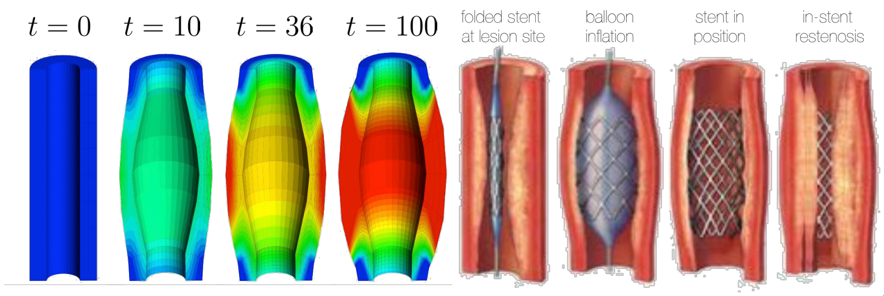


$$\boldsymbol{F}_g = \vartheta \, \boldsymbol{I}$$
$$\mathrm{D}_t \vartheta = k_\vartheta(\vartheta) \, \mathrm{tr}(\boldsymbol{C}_e \cdot \boldsymbol{S}_e)$$

stress-induced volume growth

kuhl, maas, himpel & menzel [2007]

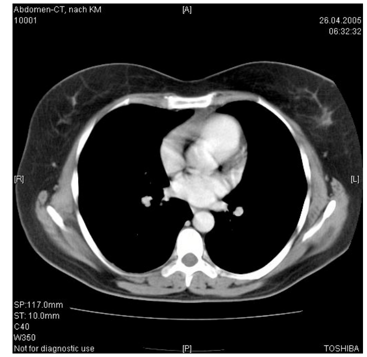
qualitative simulation of stent implantation



stress-induced volume growth

kuhl, maas, himpel & menzel [2007]

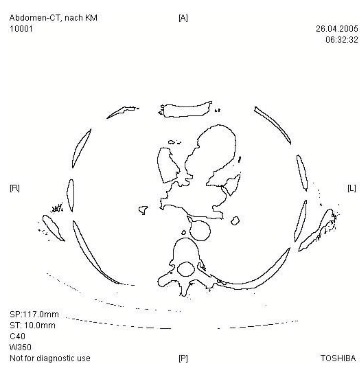
generation of patient specific model



computer tomography - typical cross section

kuhl, maas, himpel & menzel [2007]

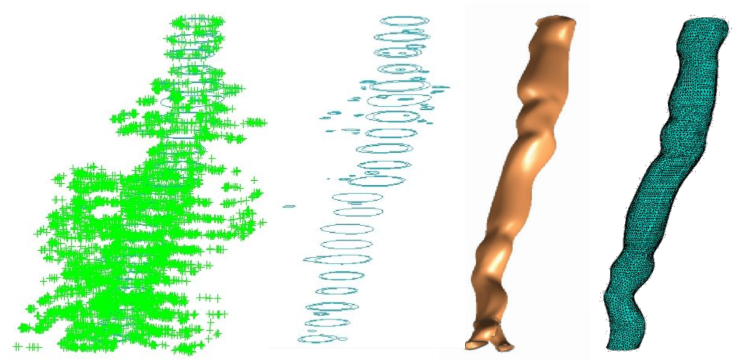
generation of patient specific model



outline of ct image - typical cross section

kuhl, maas, himpel & menzel [2007]

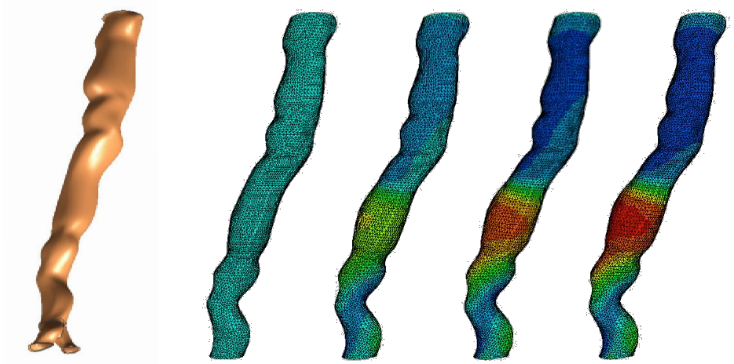
generation of patient specific model



from computer tomography to finite element model

kuhl, maas, himpel & menzel [2007]

virtual stent implantation - patient specific model

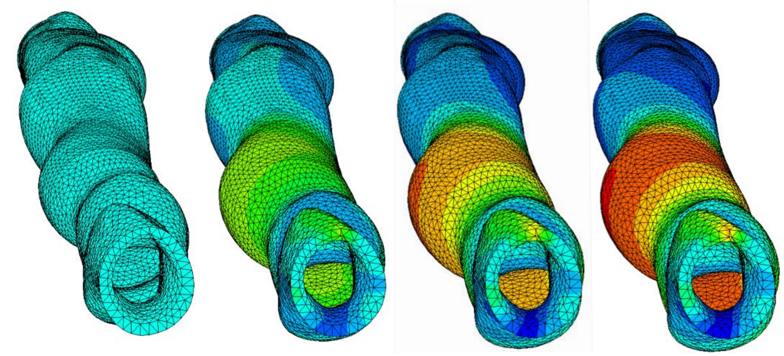


tissue growth - response to virtual stent implantation

kuhl, maas, himpel & menzel [2007]

example - stenting and restenosis

virtual stent implantation - patient specific model



tissue growth - response to virtual stent implantation

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example - stenting and restenosis