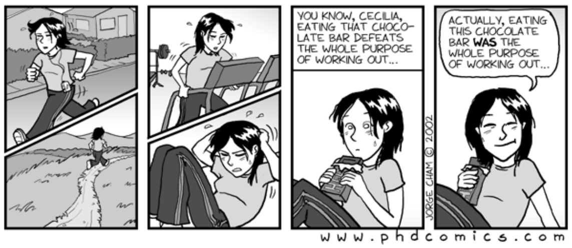
10 - basics constitutive equations - volume growth - growing tumors

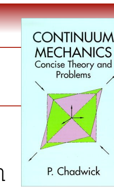


day	date		topic
tue	jan	07	motivation - everything grows!
thu	jan	09	basics maths - notation and tensors
tue	jan	14	project example - growing skin
thu	jan	16	kinematics - growing brains
tue	jan	21	basic kinematics - large deformation and growth
thu	jan	23	kinematics - growing hearts
tue	jan	28	kinematics - growing leaflets
thu	jan	30	basic balance equations - closed and open systems
tue	feb	04	basic constitutive equations - growing muscle
thu	feb	06	basic constitutive equations - growing tumors
tue	feb	11	volume growth - finite elements for growth - theory
thu	feb	13	volume growth - finite elements for growth - matlab
tue	feb	18	basic constitutive equations - growing bones
thu	feb	20	density growth - finite elements for growth
tue	feb	25	density growth - growing bones
thu	feb	27	everything grows! - midterm summary
tue	mar	04	midterm
thu	mar	06	remodeling - remodeling arteries and tendons
tue	mar	11	class project - discussion, presentation, evaluation
thu	mar	13	class project - discussion, presentation, evaluation
thu	mar	14	written part of final projects due

balance equations

balance equations [ˈbæl.əns ɪˈkweɪ.ʒəns] of mass, momentum, angular momentum and energy, supplemented with an entropy inequality constitute the set of conservation laws.
the law of **conservation of mass**/matter states that the **mass of a closed system** of substances will remain **constant**, regardless of the processes acting inside the system.
the principle of conservation of momentum states that the total momentum of a closed system of objects is constant.

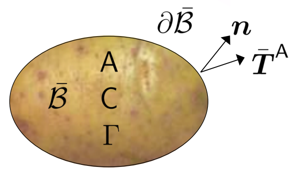
balance equations



balance equations [ˈbæl.əns ɪˈkweɪ.ʒəns] of mass, linear momentum, angular momentum and energy **apply to all material bodies**. each one gives rise to a field equation, holing on the configurations of a body in a sufficiently smooth motion and a jump condition on surfaces of discontinuity. like position, time and body, the concepts of mass, force, heating and internal energy which enter into the formulation of the balance equations are regarded as having primitive status in continuum mechanics.

chadwick "continuum mechanics" [1976]

generic balance equation

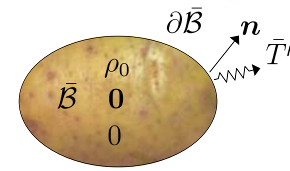


general format

A ... balance quantity
B ... flux $\quad \mathbf{B} \cdot \boldsymbol{n} = \bar{\boldsymbol{T}}^{\mathsf{A}}$
C ... source
Γ ... production

$$\mathrm{D}_t \mathsf{A} = \mathrm{Div}(\mathbf{B}) + \mathsf{C} + \Gamma$$

balance of mass

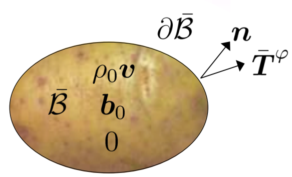


balance of mass

ρ_0 ... density
$\mathbf{0}$ ... no mass flux $\quad \bar{T}^{\rho} = 0$
0 ... no mass source
0 ... no mass production

continuity equation $\quad \mathrm{D}_t \rho_0 = 0$

balance of (linear) momentum

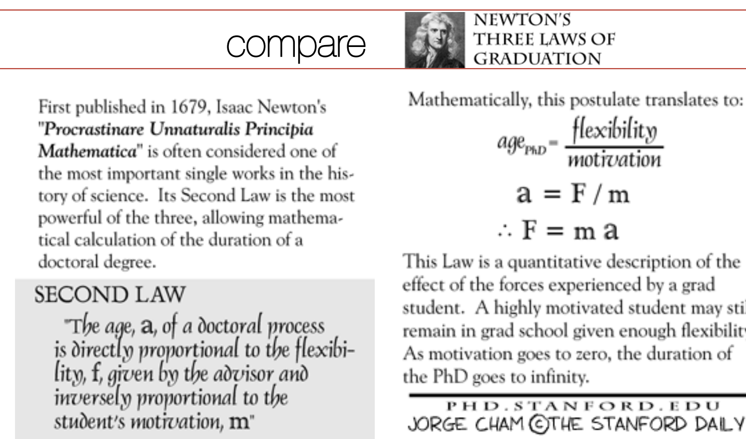


balance of momentum

$\rho_0 \boldsymbol{v}$ ... linear momentum density
$\boldsymbol{P}$ ... momentum flux - stress $\quad \boldsymbol{P} \cdot \boldsymbol{n} = \bar{\boldsymbol{T}}^{\varphi}$
$\boldsymbol{b}_0$ ... momentum source - force
0 ... no momentum production

equilibrium equation $\mathrm{D}_t(\rho_0 \boldsymbol{v}) = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0$

compare



NEWTON'S THREE LAWS OF GRADUATION

First published in 1679, Isaac Newton's "*Procrastinare Unnaturalis Principia Mathematica*" is often considered one of the most important single works in the history of science. Its Second Law is the most powerful of the three, allowing mathematical calculation of the duration of a doctoral degree.

SECOND LAW

"The age, a, of a doctoral process is directly proportional to the flexibility, f, given by the advisor and inversely proportional to the student's motivation, m"

Mathematically, this postulate translates to:

$$age_{PhD} = \frac{flexibility}{motivation}$$

$$a = F / m$$

$$\therefore F = m\, a$$

This Law is a quantitative description of the effect of the forces experienced by a grad student. A highly motivated student may still remain in grad school given enough flexibility. As motivation goes to zero, the duration of the PhD goes to infinity.

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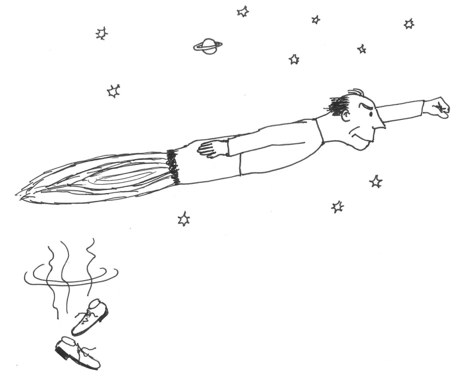
$\mathrm{D}_t(\rho_0 \boldsymbol{v}) = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0 \quad$ mass point $\quad m\,\mathrm{D}_t \boldsymbol{v} = m\,\boldsymbol{a} = \boldsymbol{F}$

thermodynamic systems

open system [ˈoʊ.pən ˈsɪs.təm] thermodynamic system which is allowed to exchange mechanical work, heat and mass, typically $\boldsymbol{P} = \boldsymbol{P}(\nabla\varphi, ...)$, $\boldsymbol{Q} = \boldsymbol{Q}(\nabla\theta, ...)$ and $\boldsymbol{R} = \boldsymbol{R}(\nabla\rho, ...)$ with its environment. enclosed by a deformable, diathermal, permeable membrane. characterized through its state of deformation φ, temperature θ and density ρ.

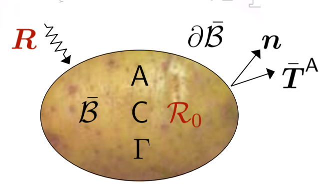
open system thermodynamics



"...thermodynamics recognizes no special role of the biological..."

bridgman, 'the nature of thermodynamics', [1941]

balance equations of growth



general format

A ... balance quantity

B ... flux $\quad \mathbf{B} \cdot \boldsymbol{n} = \bar{\boldsymbol{T}}^{\mathsf{A}}$

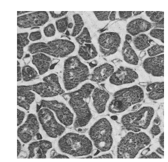
C ... source

Γ ... production

$$\mathrm{D}_t(\rho_0 \mathsf{A}) = \mathrm{Div}(\mathbf{B} + \mathsf{A} \otimes \boldsymbol{R}) + [\mathsf{C} + \mathsf{A}\mathcal{R}_0 - \nabla_X \mathsf{A} \cdot \boldsymbol{R} + \Gamma]$$

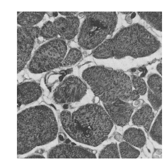
balance equations of growth

$$\mathrm{D}_t \rho_0 = \mathrm{Div}(\boldsymbol{R}) + \mathcal{R}_0$$



mass flux $\boldsymbol{R}$

- cell movement (migration)



mass source $\mathcal{R}_0$

- cell growth (proliferation)
- cell division (hyperplasia)
- cell enlargement (hypertrophy)

biological equilibrium

cowin & hegedus [1976], beaupré, orr & carter [1990], harrigan & hamilton [1992], jacobs, levenston, beaupré, simo & carter [1995], huiskes [2000], carter & beaupré [2001]

balance of (linear) momentum

- volume specific version

$$\mathrm{D}_t(\rho_0 \boldsymbol{v}) = \mathrm{Div}(\boldsymbol{P} + \boldsymbol{v} \otimes \boldsymbol{R}) + [\boldsymbol{b}_0 + \boldsymbol{v}\mathcal{R}_0 - \nabla_X \boldsymbol{v} \cdot \boldsymbol{R}]$$

- subract weighted balance of mass

$$\boldsymbol{v}\,\mathrm{D}_t\rho_0 = \mathrm{Div}(\boldsymbol{v} \otimes \boldsymbol{R}) + \boldsymbol{v}\mathcal{R}_0 - \nabla_X \boldsymbol{v} \cdot \boldsymbol{R}$$

- mass specific version

$$\rho_0\,\mathrm{D}_t\boldsymbol{v} = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0$$

mechanical equilibrium

example of open systems - rocket propulsion

balance of mass

$$\mathrm{D}_t\mathrm{m} = \mathcal{R} \quad \text{with} \quad \mathcal{R} \leq 0 \quad \text{ejection}$$

balance of momentum - volume specific

$$\mathrm{D}_t[\mathrm{m}\boldsymbol{v}] = \mathrm{m}\,\mathrm{D}_t\boldsymbol{v} + \mathrm{D}_t\mathrm{m}\,\boldsymbol{v} = \boldsymbol{f} + \mathcal{R}\boldsymbol{v}$$

balance of momentum - mass specific

$$\mathrm{m}\,\mathrm{D}_t\boldsymbol{v} = \boldsymbol{f} \quad \text{with} \quad \boldsymbol{f} = \boldsymbol{f}^{\text{closed}} + \boldsymbol{f}^{\text{open}}$$

balance of momentum - rocket head-ejection

$$\mathrm{D}_t[\mathrm{m}\boldsymbol{v}] - \mathcal{R}\bar{\boldsymbol{v}} = \boldsymbol{f}^{\text{closed}}$$

propulsive force

$$\boldsymbol{f}^{\text{open}} = [\bar{\boldsymbol{v}} - \boldsymbol{v}]\mathcal{R} \quad \text{velocity of ejection } \bar{\boldsymbol{v}}$$

example of open systems - rocket propulsion

a saturn v rocket like the one that took men to the moon has a **mass of 2.500.000 kg** at liftoff. it goes straight up vertically and burns fuel at a uniform **rate of 16.000 kg/s** for a duration of **2 minutes**. the exhaust speed of gas from the saturn **v is 3.0 km/s**

what is the speed of the rocket immediately after the combustion ceases? you should include the effect of gravity near the surface of the earth, but you can neglect air resistance.

plot the **burnout velocity as a function of time** over the range of 0 to 120 seconds to see the increase in speed of the rocket with time.

example of open systems - rocket propulsion

$$\mathrm{m}\mathrm{D}_t v = f^{\text{closed}} + f^{\text{open}}$$

$$\text{with} \quad f^{\text{closed}} = -\mathrm{m}\,g \quad \text{gravity}$$

$$f^{\text{open}} = [\bar{v} - v]\mathcal{R} = w\,\mathrm{D}_t\mathrm{m}$$

$$\mathrm{m}\mathrm{D}_t v = -\mathrm{m}\,g - \mathrm{D}_t m\,w \qquad || \;:\mathrm{m}$$

$$\mathrm{D}_t v = -g - \tfrac{1}{\mathrm{m}}\mathrm{D}_t\mathrm{m}\,w$$

integration

$$v(t) = -g\,t - w\int_{\mathrm{m}(0)}^{\mathrm{m}(t)} \tfrac{1}{\mathrm{m}}\mathrm{dm}$$

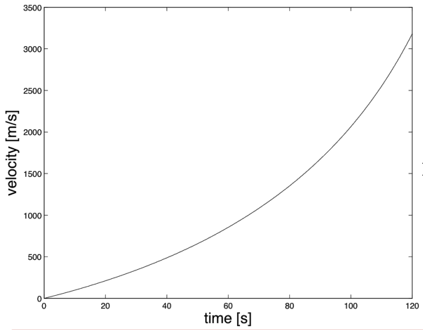
$$\mathrm{m}(t) = \mathrm{m}(0) + \mathcal{R}\,t$$

velocity

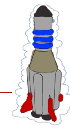
$$v(t) = -g\,t - w\,\ln\frac{\mathrm{m}(0) + \mathcal{R}\,t}{\mathrm{m}(0)}$$

example of open systems - rocket propulsion

velocity $v(t) = -g\,t - w\,\ln\frac{\mathrm{m}(0)+\mathcal{R}\,t}{\mathrm{m}(0)}$

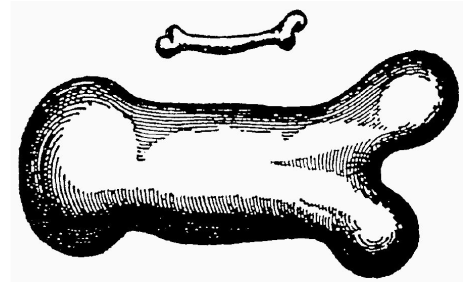


$$g = 9.8\ \mathrm{m/s^2}$$
$$w = 3000\mathrm{m/s}$$
$$\mathrm{m}(0) = 2.5 \cdot 10^6\ \mathrm{kg}$$
$$\mathcal{R}_0 = -16000\mathrm{kg/s}$$



velocity $v(t = 120\mathrm{s}) = 3207.1\ \mathrm{m/s}$

example of open systems - the galileo giant



„...dal che e manifesto, che chi volesse mantener in un vastissimo gigante le proporzioni, che hanno le membra in un huomo ordinario, bisognerebbe o trouar materia molto piu dura, e resistente per formarne l'ossa o vero ammettere, che la robustezza sua fusse a proporzione assai piu fiacca, che negli huomini de statura mediocre; altrimente crescendogli a smisurata altezza si vedrebbono dal proprio peso opprimere, e cadere…"

galileo,"discorsi e dimostrazioni matematiche", [1638]

example of open systems - the galileo giant

the tallest man in medical history is robert pershing wadlow. he was born at alton, illinois, on february 22, 1918. he was **2.72m** / 8ft 11.1", tall.

his weight was **222.71kg**. his shoe size was 47cm / 18.5", and his hands measured 32.4cm / 12.75". his arm span was 2.88m / 9 ft 5.75", and his peak daily food consumption was 8000 calories.

guiness world records [2010]

example of open systems - the galileo giant

consider and compare the two cases:

closed system. calculale the vertical displacement and the energy for a giant with a **constant bone mineral density along the height**

open system. calculate the vertical displacement and the bone mineral density for a giant with a **constant energy along the height**

guiness world records [2010]

example of closed systems - the galileo giant

- balance equation closed systems $\rho_0 \, \mathrm{D}_t v = \mathrm{Div}(P) + b_0$
- neo hookean free energy $\psi_0 = \frac{1}{4} \, E_0 [F^2 - 1 - 2 \, \ln(F)]$
- stress from dissipation inequality $P = \mathrm{D}_F \psi_0 = \frac{1}{4} E_0 [\, 2F - 2 \frac{1}{F} \,]$
- quasi-static case $\rho_0 \, \mathrm{D}_t v = 0$
- constant gravity load $b_0 = \text{const along the height } h$
- from balance eqn linear stress $P = [X - h] \, b_0$
  ...linear along the height h
- closed system, constant density $\rho = \rho_0^* \qquad [\rho - \rho_0^*] / \rho_0^* = 0$

example of open systems - the galileo giant

density $[\rho - \rho_0^*] / \rho_0^* = 0$

displacement $u = \varphi - X$

energy $[\rho_0 / \rho_0^*]^{-m} \psi_0 = \psi_0$

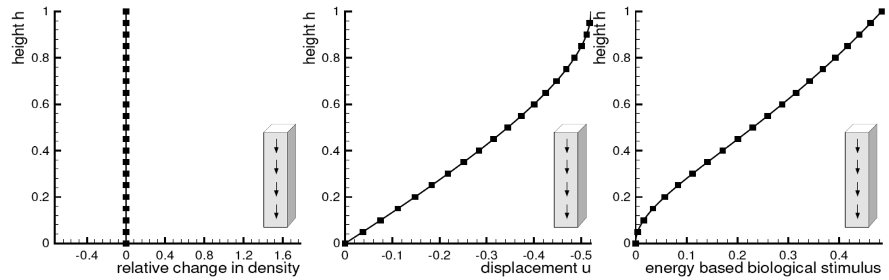


Figure 7.1: One–dimensional model problem - Closed system - Homogeneous density

example of open systems - the galileo giant

- balance equations open systems
$$\begin{aligned} \mathrm{D}_t \rho_0 &= \mathrm{Div}(R) + \mathcal{R}_0 \\ \rho_0 \, \mathrm{D}_t v &= \mathrm{Div}(P) + b_0 \end{aligned}$$
- free energy $\psi_0 = [\rho_0 / \rho_0^*]^n \, \psi_0^{\mathrm{neo}}$ $\quad \psi_0^{\mathrm{neo}} = \frac{1}{4} \, E_0 [F^2 - 1 - 2 \, \ln(F)]$
- stress from dissipation inequality $P = \mathrm{D}_F \psi_0 = \frac{1}{4} E_0 [\, 2F - 2 \frac{1}{F} \,]$
- quasi-static case $\rho_0 \, \mathrm{D}_t v = 0$
- constant gravity load $b_0 = \text{const along the height } h$
- from balance eqn linear stress $P = [X - h] \, b_0$
  ...linear along the height h
- open system, varying density $\mathrm{D}_t \rho_0 = \mathcal{R}_0 = 0$
  $\mathcal{R}_0 = [\rho_0 / \rho_0^*]^{-m} \rho_0 - \rho_0^*$

example of open systems - the galileo giant

density $[\rho - \rho_0^*] / \rho_0^*$

displacement $u = \varphi - X$

energy $[\rho_0 / \rho_0^*]^{-m} \psi_0$

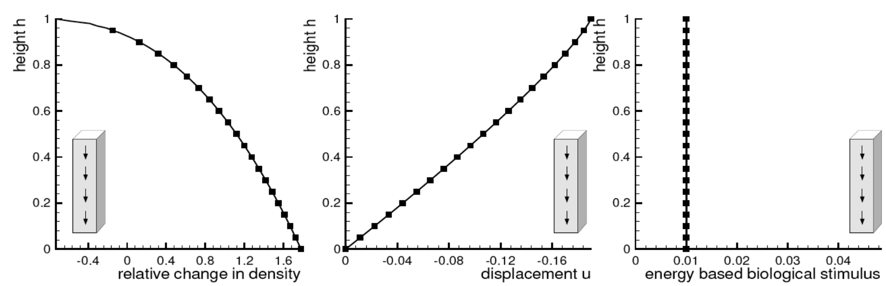


Figure 7.2: One–dimensional model problem - Open system - Homogeneous stimulus

constitutive equations

constitutive equations [kənˈstɪ.tu.tɪv ɪˈkweɪ.ʒəns] in structural analysis, constitutive relations **connect applied stresses** or forces **to strains** or deformations. the constitutive relations for linear materials are linear. more generally, in physics, a constitutive equation is a relation between two physical quantities (often tensors) that is specific to a material, and does not follow directly from physical law. some constitutive equations are **simply phenomenological**; others are **derived from first principles**.

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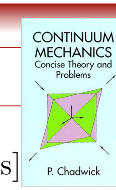
constitutive equations



constitutive equations [kənˈstɪ.tu.tɪv ɪˈkweɪ.ʒəns] or equations of state bring in the **characterization of particular materials** within continuum mechanics. mathematically, the purpose of these relations is to supply connections between kinematic, mechanical and thermal fields. physically, constitutive equations represent the various forms of **idealized material response** which serve as **models** of the behavior of actual substances.

chadwick 'continuum mechanics' [1976]

neo hooke'ian elasticity

- free energy
$$\psi_0^{\mathrm{neo}} = \tfrac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F})) + \tfrac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\mathrm{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\mathrm{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$$
- definition of stress
$$\begin{aligned}\boldsymbol{P}^{\mathrm{neo}} &= \mathrm{D}_{\boldsymbol{F}}\psi_0^{\mathrm{neo}}\\ &= \tfrac{1}{2}\,\lambda_0\,2\,\ln(\det\boldsymbol{F})\,\boldsymbol{F}^{\text{-t}} + \tfrac{1}{2}\mu_0\,2\,\boldsymbol{F} - \mu_0\boldsymbol{F}^{\text{-t}}\\ &= \mu_0\,\boldsymbol{F} + [\,\lambda_0\ln(\det(\boldsymbol{F})) - \mu_0\,]\,\boldsymbol{F}^{\text{-t}}\end{aligned}$$
- definition of tangent operator
$$\begin{aligned}\mathsf{A}^{\mathrm{neo}} &= \mathrm{D}_{\boldsymbol{F}\boldsymbol{F}}\psi_0^{\mathrm{neo}} = \mathrm{D}_{\boldsymbol{F}}\boldsymbol{P}^{\mathrm{neo}}\\ &= \lambda_0\boldsymbol{F}^{\text{-t}}\otimes\boldsymbol{F}^{\text{-t}} + \mu_0\boldsymbol{I}\,\overline{\otimes}\,\boldsymbol{I}\\ &\quad + [\mu_0 - \lambda_0\ln(\det(\boldsymbol{F}))]\,\boldsymbol{F}^{\text{-t}}\,\underline{\otimes}\,\boldsymbol{F}^{\text{-1}}\end{aligned}$$

tensor analysis - these derivatives might be handy

$$\{\bullet\otimes\circ\}_{ijkl} = \{\bullet\}_{ij}\,\{\circ\}_{kl}$$
$$\{\bullet\,\overline{\otimes}\,\circ\}_{ijkl} = \{\bullet\}_{ik}\,\{\circ\}_{jl}$$
$$\{\bullet\,\underline{\otimes}\,\circ\}_{ijkl} = \{\bullet\}_{il}\,\{\circ\}_{jk}$$

$$\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{F}} = \boldsymbol{I}\,\overline{\otimes}\,\boldsymbol{I} \qquad \frac{\partial F_{ij}}{\partial F_{kl}} = \delta_{ik}\,\delta_{jl}$$
$$\frac{\partial\boldsymbol{F}^{\text{-1}}}{\partial\boldsymbol{F}} = -\boldsymbol{F}^{\text{-1}}\,\overline{\otimes}\,\boldsymbol{F}^{\text{-t}} \qquad \frac{\partial F^{\text{-1}}_{ij}}{\partial F_{kl}} = -F^{\text{-1}}_{ik}\,F^{\text{-1}}_{lj}$$
$$\frac{\partial\boldsymbol{F}^{\mathrm{t}}}{\partial\boldsymbol{F}} = \boldsymbol{I}\,\underline{\otimes}\,\boldsymbol{I} \qquad \frac{\partial F_{ji}}{\partial F_{kl}} = \delta_{il}\,\delta_{jk}$$
$$\frac{\partial\boldsymbol{F}^{\text{-t}}}{\partial\boldsymbol{F}} = -\boldsymbol{F}^{\text{-t}}\,\underline{\otimes}\,\boldsymbol{F}^{\text{-1}} \qquad \frac{\partial F^{\text{-1}}_{ji}}{\partial F_{kl}} = -F^{\text{-1}}_{li}\,F^{\text{-1}}_{jk}$$

tensor analysis - these derivatives might be handy

$$\frac{\partial \det(\boldsymbol{F})}{\partial \boldsymbol{F}} = \det(\boldsymbol{F})\, \boldsymbol{F}^{\text{-t}} \qquad \frac{\partial \ln(\det(\boldsymbol{F}))}{\partial \boldsymbol{F}} = \boldsymbol{F}^{\text{-t}}$$

$$\frac{\partial \det(\boldsymbol{F}^{\text{-1}})}{\partial \boldsymbol{F}} = -\frac{1}{\det(\boldsymbol{F})}\, \boldsymbol{F}^{\text{-t}} \qquad \frac{\partial \ln(\det(\boldsymbol{F})^{\text{-1}})}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{\text{-t}}$$

$$\frac{\partial \ln(\det(\boldsymbol{F}))}{\partial(\det(\boldsymbol{F}))} = \frac{1}{\det(\boldsymbol{F})} \qquad \frac{\partial \ln^2(\det(\boldsymbol{F}))}{\partial \boldsymbol{F}} = 2\, \ln(\det(\boldsymbol{F}))\boldsymbol{F}^{\text{-t}}$$

neo hooke'ian elasticity

- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(F_{ij})) + \frac{1}{2}\,\mu_0[\,F_{ij}F_{ij} - n^{\text{dim}} - 2\,\ln(\det(F_{ij}))\,]$
- definition of stress

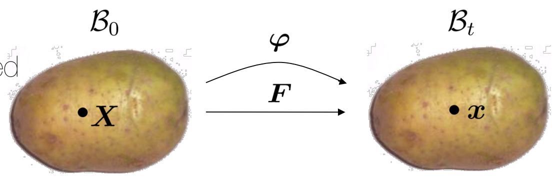
$$\begin{aligned} P_{ij}^{\text{neo}} &= \mathrm{D}_{F_{ij}}\psi_0^{\text{neo}} \\ &= \tfrac{1}{2}\,\lambda_0\, 2\,\ln(\det F_{ij})F^{\text{-1}}_{ji} + \tfrac{1}{2}\mu_0\, 2\, F_{ij} - \mu_0 F^{\text{-1}}_{ji} \\ &= \mu_0\, F_{ij} + [\,\lambda_0 \ln(\det(F_{ij})) - \mu_0\,]F^{\text{-1}}_{ji} \end{aligned}$$

- definition of tangent operator

$$\begin{aligned} \mathsf{A}_{ijkl}^{\text{neo}} &= \mathrm{D}_{F_{ij}F_{kl}}\psi_0^{\text{neo}} = \mathrm{D}_{F_{kl}}P_{ij}^{\text{neo}} \\ &= \lambda_0 F^{\text{-1}}_{ji}\; F^{\text{-1}}_{lk} + \mu_0 I_{ik}\; I_{jl} \\ &\quad + [\mu_0 - \lambda_0 \ln(\det(F_{ij}))]\; F^{\text{-1}}_{li}\; F^{\text{-1}}_{jk} \end{aligned}$$

neo hooke'ian elasticity

undeformed potato



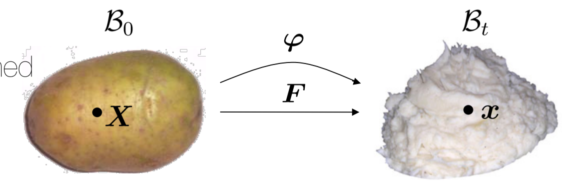
deformed potato

- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F})) + \frac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$
- definition of stress

$$\begin{aligned} \boldsymbol{P}^{\text{neo}} &= \mathrm{D}_F\psi_0^{\text{neo}} \\ &= \mu_0\,\boldsymbol{F} + [\,\lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{\text{-t}} \end{aligned}$$

neo hooke'ian elasticity

undeformed potato



mashed potatoes

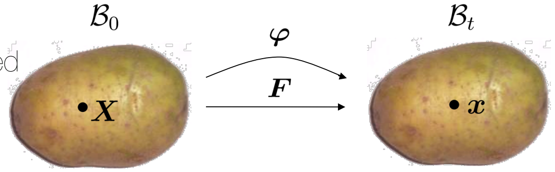
- free energy $\psi^{\text{neo}} = \frac{1}{2}\,\lambda \ln^2(\det(\boldsymbol{F})) + \frac{1}{2}\,\mu[\,\boldsymbol{F}^{\text{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$
- definition of stress

$$\begin{aligned} \boldsymbol{P}^{\text{neo}} &= \rho_0\mathrm{D}_F\psi \\ &= \mu\,\boldsymbol{F} + [\,\lambda \ln(\det(\boldsymbol{F})) - \mu\,]\boldsymbol{F}^{\text{-t}} \end{aligned}$$

- remember! mashing potatoes is not an elastic process!

neo hooke'ian elasticity

undeformed potato



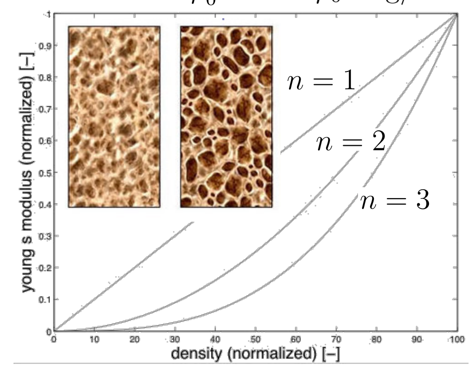
deformed potato

- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F}))$
  $+ \frac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$
- large strain - lamé parameters and bulk modulus
  $\lambda = \frac{E\,\nu}{[1+\nu][1-2\,\nu]} \qquad \mu = \frac{E}{2\,[1+\nu]} \qquad \kappa = \frac{E}{3\,[1-2\,\nu]}$
- small strain – young's modulus and poisson's ratio
  $E = 3\,\kappa\,[1-2\,\nu] \qquad \nu = \frac{3\,\kappa - 2\,\mu}{2\,[3\,\kappa + \mu]}$

neo hooke'ian elasticity - cellular tissues

free energy $\psi_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^n \psi_0^{\text{neo}}(\boldsymbol{F})$

$\text{E} = 3.790\;\rho_0^3\;\text{MPa} \qquad \rho_0 \text{ in g/cm}^3$



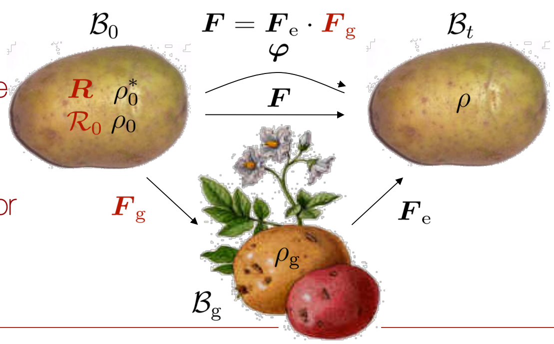
carter & hayes [1977]

additional constitutive equations for volume growth

mass source

mass flux

growth tensor

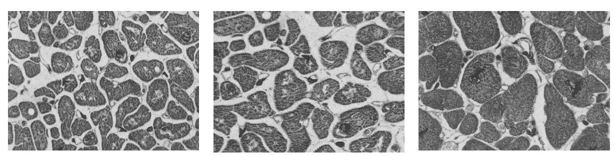


multiplicative decomposition - but what is $\boldsymbol{F}_{\text{g}}$?

lee [1969], simo [1992], rodriguez, hoger & mc culloch [1994], epstein & maugin [2000], humphrey [2002], ambrosi & mollica [2002], himpel, kuhl, menzel & steinmann [2005]

volume growth at constant density

- free energy $\psi_0 = \psi_0^{\text{neo}}(\boldsymbol{F}_{\text{e}})$
- stress $\boldsymbol{P}_{\text{e}} = \boldsymbol{P}_{\text{e}}^{\text{neo}}(\boldsymbol{F}_{\text{e}})$
- growth tensor $\boldsymbol{F}_{\text{g}} = \vartheta\,\boldsymbol{I} \qquad \text{D}_t\vartheta = k_\vartheta(\vartheta)\;\text{tr}(\boldsymbol{C}_{\text{e}} \cdot \boldsymbol{S}_{\text{e}})$
  growth function pressure
- mass source $\mathcal{R}_0 = 3\,\rho_0\vartheta^2\text{D}_t\vartheta$
  increase in mass



kinematic coupling of growth and deformation

rodriguez, hoger & mc culloch [1994], epstein & maugin [2000], humphrey [2002]

volume growth - cylindrical tumor

ductal carcinoma is a very common type of breast cancer in women. infiltrating ductal carcinoma refers to the development of invasive cancer cells within the milk ducts of the breast. it accounts for 80% of all types of breast cancer. on a mammography, it is usually visualized as a mass with fine spikes radiating from the edges, and small microcalcification may be seen as well. on physical examination, the lump usually feels hard or firm. on microscopic examination, the cancerous cells invade and replace the surrounding normal tissue inside the breast. tumors under 1 cm in diameter are unlikely to spread systemically. tumors under 4 cm in diameter are surgically removed. Additionally, the patient may be treated with chemotherapy, radiotherapy or hormonal therapy.

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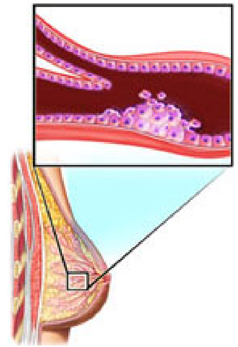
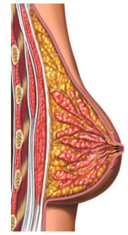
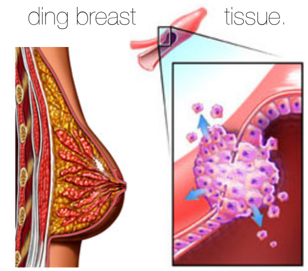
volume growth - cylindrical tumor

normal cells grow and multiply at a specific rate. cells that grow and multiply without stopping are called cancerous or malignant. however, they are not detectable when they first start growing.





invasive or infiltrating ductal carcinoma is the most common type of breast cancer. it occurs when the cells that line the milk duct become abnormal and spread into the surrounding breast tissue.

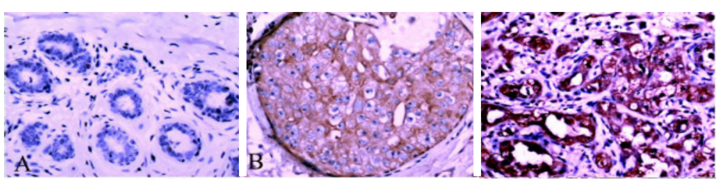


ductal carcinoma in situ is a non-invasive change in the cells that line the milk tubes that bring milk from the milk lobules to the nipple

volume growth - cylindrical tumor

model assumptions

- ductal cardinoma - tumor grows in breast duct for up to 10 cm
- model - homogeneous growth inside a rigid cylinder
- assumption - rotational symmetry
- strategy - solve for deformation that satisfies equi'm and boundary cond's



normal ductal epithelial cells (negative) | tumor cells of ductal carcinoma (positive) | invasive ductal carcinoma cells (strongly positive)

kinematic coupling of growth and deformation

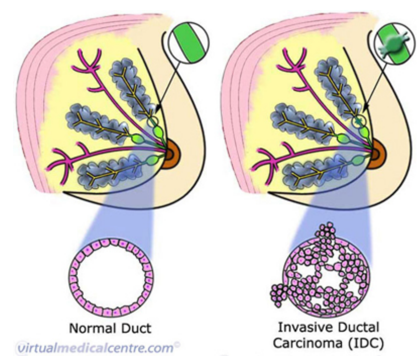
volume growth - cylindrical tumor

- homogeneous deformation inside a rigid cylinder

$$x = X \qquad y = Y \qquad z = \lambda Z$$

- deformation gradient

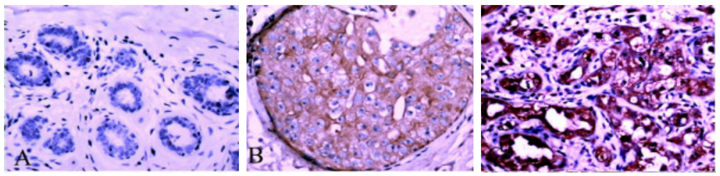
$$\boldsymbol{F} = \nabla_X \boldsymbol{\varphi} \qquad \boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$



kinematic coupling of growth and deformation

volume growth - cylindrical tumor

- free energy $\psi_0 = \psi_0^{\mathrm{bko}}(\boldsymbol{F}_\mathrm{e}) = \frac{1}{2}\mu[[I-3] - \frac{2}{q}[III^{q/2}-1]]$
- stress $\boldsymbol{P}_\mathrm{e} = \boldsymbol{P}_\mathrm{e}^{\mathrm{bko}}(\boldsymbol{F}_\mathrm{e}) = \frac{1}{J_\mathrm{e}}\mu[-J_\mathrm{e}^q \boldsymbol{F}_\mathrm{e}^{\text{-t}} + \boldsymbol{F}_\mathrm{e}] \cdot \boldsymbol{F}_\mathrm{g}^{\text{-t}}$
- growth tensor $\boldsymbol{F}_\mathrm{g} = \vartheta\, \boldsymbol{I}$ $\quad \vartheta(t) = \exp(\alpha\, t/3)$
- deformation $x = X \quad y = Y \quad z = \lambda Z$



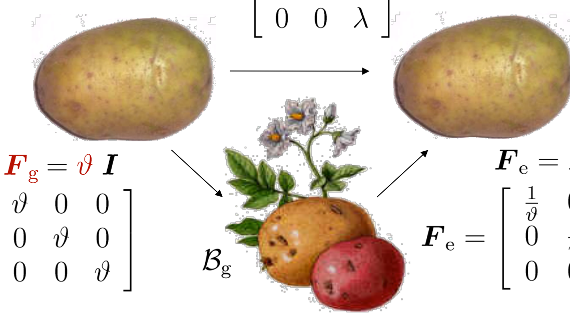
normal ductal epithelial cells (negative) | tumor cells of ductal carcinoma (positive) | invasive ductal carcinoma cells (strongly positive)

kinematic coupling of growth and deformation

volume growth - cylindrical tumor

$$\boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$$

$$\mathcal{B}_0 \qquad \boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad \mathcal{B}_t$$



$$\boldsymbol{F}_\mathrm{g} = \vartheta\, \boldsymbol{I} \qquad \boldsymbol{F}_\mathrm{g} = \begin{bmatrix} \vartheta & 0 & 0 \\ 0 & \vartheta & 0 \\ 0 & 0 & \vartheta \end{bmatrix}$$

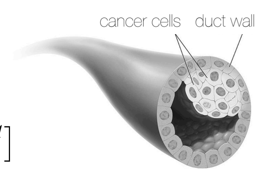
$$\mathcal{B}_\mathrm{g}$$

$$\boldsymbol{F}_\mathrm{e} = \boldsymbol{F} \cdot \boldsymbol{F}_\mathrm{g}^{-1} \qquad \boldsymbol{F}_\mathrm{e} = \begin{bmatrix} \frac{1}{\vartheta} & 0 & 0 \\ 0 & \frac{1}{\vartheta} & 0 \\ 0 & 0 & \frac{\lambda}{\vartheta} \end{bmatrix}$$

ambrosi & mollica [2002]

volume growth - cylindrical tumor

- stress $\boldsymbol{\sigma}_\mathrm{e} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$



$$\sigma_{xx} = \sigma_{yy} = \mu\, \frac{\vartheta^3}{\lambda} \left[\frac{1}{\vartheta^2} - \left[\frac{\lambda}{\vartheta^3}\right]^q\right]$$

$$\sigma_{zz} = \mu\, \frac{\vartheta^3}{\lambda} \left[\frac{\lambda^2}{\vartheta^2} - \left[\frac{\lambda}{\vartheta^3}\right]^q\right]$$

- bc's $\sigma_{zz} = 0$
- axial displacement λ as a function of growth ϑ

$$\lambda = \vartheta^{[2-3q]/[2-q]}$$

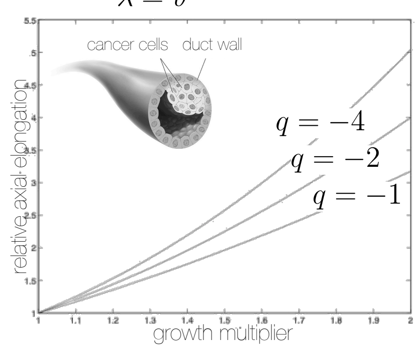
- growth induced stress σ_{xx} on tumor wall

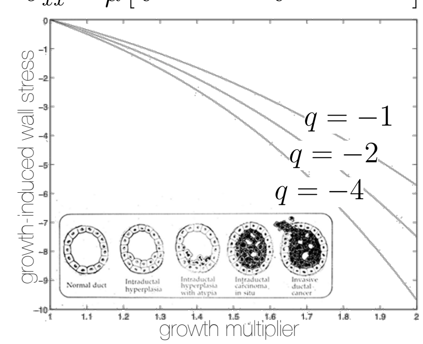
$$\sigma_{xx} = \mu\,[\,\vartheta^{2q/[2-q]} - \vartheta^{[4-4q]/[2-q]}\,]$$

ambrosi & mollica [2002]

volume growth - cylindrical tumor

$$\lambda = \vartheta^{[2-3q]/[2-q]} \qquad \sigma_{xx} = \mu\,[\,\vartheta^{2q/[2-q]} - \vartheta^{[4-4q]/[2-q]}\,]$$





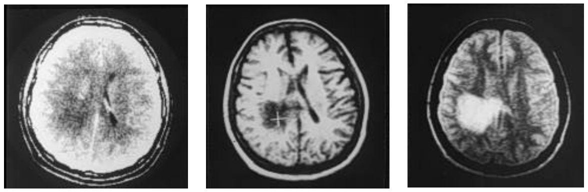
tumor pressure on duct walls increases with growth

ambrosi & mollica [2002]

volume growth - spherical tumor

model assumptions

- tumor model - inhomogeneous growth of a sphere
- inhomogeneity - residual stress even in the absense of applied loads
- assumption - spherical symmetry
- strategy - solve for deformation that satisfies equi'm and boundary cond's

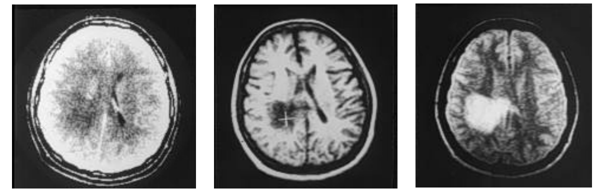


kinematic coupling of growth and deformation

ambrosi & mollica [2002]

volume growth - spherical tumor

- free energy $\psi_0 = \psi_0^{\text{bko}}(\boldsymbol{F}_\text{e}) = \frac{1}{2}\mu[[I-3] - \frac{2}{q}[III^{q/2}-1]]$
- stress $\boldsymbol{P}_\text{e} = \boldsymbol{P}_\text{e}^{\text{bko}}(\boldsymbol{F}_\text{e}) = \frac{1}{J_\text{e}}\mu[-J_\text{e}^q \boldsymbol{F}_\text{e}^{\text{-t}} + \boldsymbol{F}_\text{e}] \cdot \boldsymbol{F}_\text{g}^{\text{-t}}$
- growth tensor $\boldsymbol{F}_\text{g} = \vartheta\, \boldsymbol{I}$ $\quad \vartheta(R) = \frac{\alpha}{R}\sinh(k\,R)$
- deformation $r = \lambda(R) \quad \theta = \Theta \quad \phi = \Phi$



kinematic coupling of growth and deformation

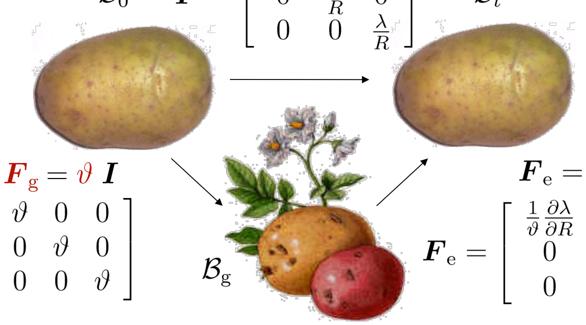
ambrosi & mollica [2002]

volume growth - spherical tumor

$$\boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$$

$$\boldsymbol{F} = \begin{bmatrix} \frac{\partial\lambda}{\partial R} & 0 & 0 \\ 0 & \frac{\lambda}{R} & 0 \\ 0 & 0 & \frac{\lambda}{R} \end{bmatrix}$$

$\mathcal{B}_0$ $\qquad$ $\mathcal{B}_t$



$\mathcal{B}_\text{g}$

$$\boldsymbol{F}_\text{g} = \vartheta\, \boldsymbol{I} \qquad \boldsymbol{F}_\text{g} = \begin{bmatrix} \vartheta & 0 & 0 \\ 0 & \vartheta & 0 \\ 0 & 0 & \vartheta \end{bmatrix}$$

$$\boldsymbol{F}_\text{e} = \boldsymbol{F} \cdot \boldsymbol{F}_\text{g}^{\text{-1}} \qquad \boldsymbol{F}_\text{e} = \begin{bmatrix} \frac{1}{\vartheta}\frac{\partial\lambda}{\partial R} & 0 & 0 \\ 0 & \frac{\lambda}{\vartheta R} & 0 \\ 0 & 0 & \frac{\lambda}{\vartheta R} \end{bmatrix}$$

ambrosi & mollica [2002]

volume growth - spherical tumor

- stress

$$\boldsymbol{P}_\text{e} = \begin{bmatrix} P_{RR} & 0 & 0 \\ 0 & P_{\Theta\Theta} & 0 \\ 0 & 0 & P_{\Phi\Phi} \end{bmatrix}$$

$$P_{RR} = \mu\,\vartheta \left[\frac{\partial\lambda}{\partial R} - J_\text{e}^q\,\vartheta^2\left[\frac{\partial\lambda}{\partial R}\right]^{-1}\right]$$

$$P_{\Theta\Theta} = P_{\Phi\Phi} = \mu\,\vartheta\left[\frac{\lambda}{R} - J_\text{e}^q\,\vartheta^2\frac{R}{\lambda}\right]$$

$$J_\text{e} = \frac{\partial\lambda}{\partial R}\frac{\lambda^2}{\vartheta^3 R^2}$$

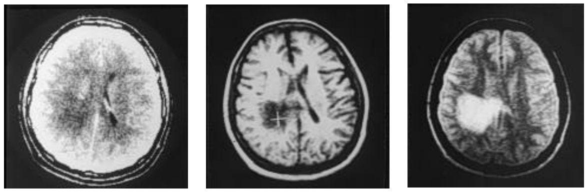
- balance of momentum $\dfrac{\text{d}P_{RR}}{\text{d}R} + \dfrac{2}{R}[P_R R - P_{\Theta\Theta}] = 0$
- bc's $\lambda(R=0) = 0 \qquad P_{RR}(R=\bar{R}) = 0$

ambrosi & mollica [2002]

volume growth - spherical tumor

ordinary differential eqn of 2nd order in λ 2B solved numerically

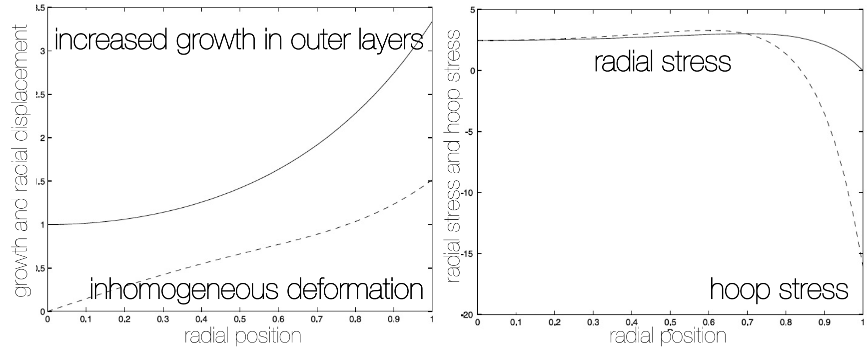
$$\frac{\partial^2 \lambda}{\partial R^2} = \frac{\frac{2}{R}\left[\frac{1}{R} + \frac{\vartheta^2 J_e^q [1-q]}{\lambda\, \partial\lambda/\partial R}\right]\left[\lambda - \frac{\partial \lambda}{\partial R} R\right] - \frac{1}{\vartheta}\left[\frac{\partial \lambda}{\partial R} - \frac{3\,\vartheta^2 J_e^q [1-q]}{\partial\lambda/\partial R}\frac{\partial \vartheta}{\partial R}\right]}{1 + \frac{\vartheta^2 J_e [1-q]}{[\partial\lambda/\partial R]^2}}$$



kinematic coupling of growth and deformation

ambrosi & mollica [2002]

volume growth - spherical tumor



hoop stress tensile inside / compressive outside

ambrosi & mollica [2002]