05 - kinematic equations - large deformations and growth

day	date		topic
tue	jan	07	motivation - everything grows!
thu	jan	09	basics maths - notation and tensors
tue	jan	14	project example - growing skin
thu	jan	16	kinematics - growing brains
tue	jan	21	basic kinematics - large deformation and growth
thu	jan	23	kinematics - growing hearts
tue	jan	28	kinematics - growing leaflets
thu	jan	30	basic balance equations - closed and open systems
tue	feb	04	basic constitutive equations - growing muscle
thu	feb	06	basic constitutive equations - growing tumors
tue	feb	11	volume growth - finite elements for growth - theory
thu	feb	13	volume growth - finite elements for growth - matlab
tue	feb	18	basic constitutive equations - growing bones
thu	feb	20	density growth - finite elements for growth
tue	feb	25	density growth - growing bones
thu	feb	27	everything grows! - midterm summary
tue	mar	04	midterm
thu	mar	06	remodeling - remodeling arteries and tendons
tue	mar	11	class project - discussion, presentation, evaluation
thu	mar	13	class project - discussion, presentation, evaluation
thu	mar	14	written part of final projects due

growth, remodeling, and morphogenesis

growth [groʊθ] which is defined as added mass, can occur through cell division (hyperplasia), cell enlargement (hypertrophy), secretion of extracellular matrix, or accretion @external or internal surfaces. negative growth (atrophy) can occur through cell death, cell shrinkage, or resorption. in most cases, hyperplasia and hypertrophy are mutually exclusive processes. depending on the age of the organism and the type of tissue, one of these two growth processes dominates.

taber 'biomechanics of growth, remodeling and morphogenesis' [1995]

growth, remodeling, and morphogenesis

remodeling [riˈmɑd. l.ɪŋ] involves changes in material properties. these changes, which often are adaptive, may be brought about by alterations in modulus, internal structure, strength, or density. for example, bones, and heart muscle may change their internal structures through reorientation of trabeculae and muscle fibers, respectively.

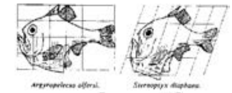
taber 'biomechanics of growth, remodeling and morphogenesis' [1995]

growth, remodeling, and morphogenesis

morphogenesis [mɔːr.foˈdʒen.ə.sɪs] is the generation of animal form. usually, the term refers to embryonic development, but wound healing and organ regeneration are also morphogenetic events. morphogenesis contains a complex series of stages, each of which depends on the previous stage. during these stages, genetric and environmental factors guide the spatial-temporal motions and differentiation (specification) of cells. a flaw in any one stage may lead to structural defects.

taber 'biomechanics of growth, remodeling and morphogenesis' [1995]

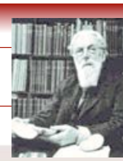
growth, remodeling, and morphogenesis

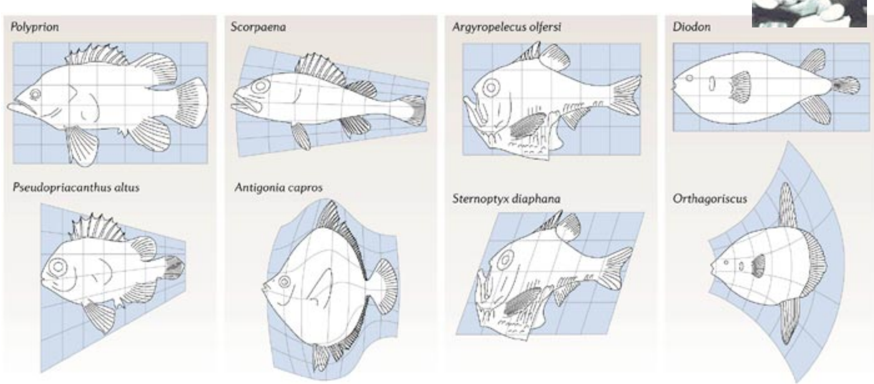


mathematical descriptions of growth [groʊθ] in plants and animals have been published since the 1940s. most of these analyses are purely **kinematic** and many borrow from the methods of continuum mechanics to describe growth rates and velocity fields. during the last quarter century, **mechanical** theories of growth have been formulated.

taber 'biomechanics of growth, remodeling and morphogenesis' [1995]

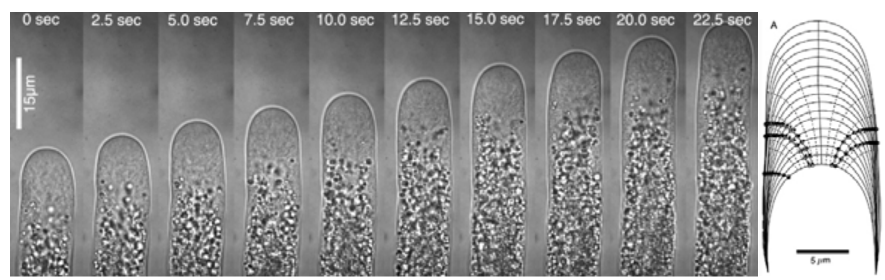
scaling growth





sir d'arcy thompson 'on growth and form' [1917]

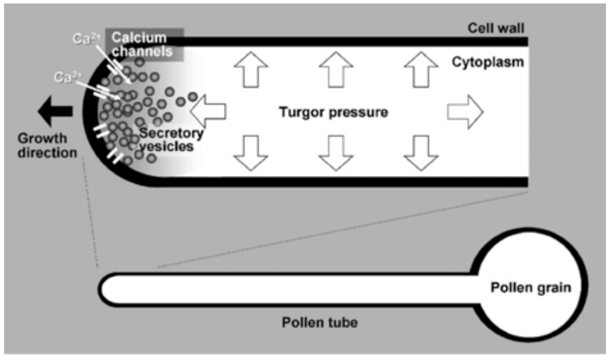
tip growth



time lapse sequence of a growing lily pollen tube. note that the morphology of the tube is drawn by the expanding tip and does not change behind it. tip growth is a common mode of cell morphogenesis observed in root hairs, fungal hyphae, pollen tubes, and many unicellular algae. these organisms have cell walls with distinct polymer compositions and structures.

dumais, long, shaw [2004]

tip growth



unlike diffusely growing cells that expand over their entire surface or large portions of it, cell wall expansion in pollen tubes is confined to the apex of the cell. this highly polarized mechanism is called tip growth. pollen tubes have the function to rapidly grow and deliver the sperm cells from the pollen grain to the ovule. kroeger, geitmann, grant [2008]

tip growth

scanning electron microscope of growing lily pollen grains germinated in vitro. the spherical objects are the pollen grains, the cylindrical objects are the pollen tubes, or cellular protuberances growing from the grains (left). brightfield microscopy of the apical region of a lily pollen tube. the outermost end of the tube is filled mainly with delivery vesicles. kroeger & geitmann [2012]

surface growth

skalak, farrow, hoger [1997]

surface growth

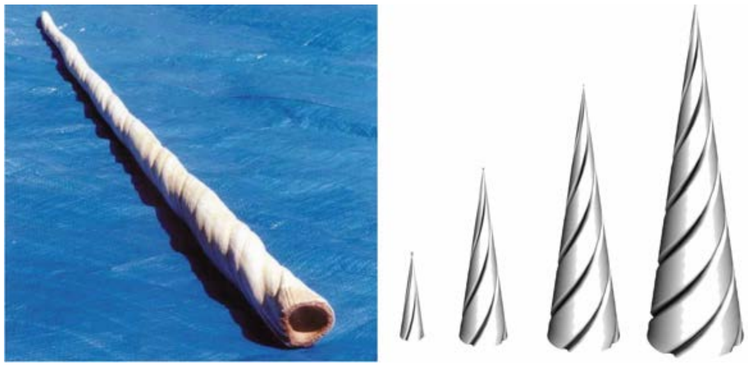


figure 1. surface growth of a narwhal tusk. photograph of a narwhal tusk, left, demonstrates the characteristic helical growth pattern. computational simulation of surface growth, right, with an outward pointing velocity of the growth surface, here characterized through the bottom ring, and a helically upward pointing velocity of material grown at the surface.. menzel & kuhl [2012]

```
%%% initial configuration %%%%%%%%%%%%%%%%%%%%%

for i=1:imax+1  % loop over all points in space
  if ((mod(i,imax/4))<=1); col=0.9; else; col=1.0; end;

  X1(i) = cos((i-1)*2*pi/imax)*rad*col;
  X2(i) = sin((i-1)*2*pi/imax)*rad*col;
  X3(i) = 0.0d0;
end          % loop over all points in space
%plot3(X1,X2,X3,'LineWidth',2,'Color',[0 1 0])

x1(:,1) = X1(:);
x2(:,1) = X2(:);
x3(:,1) = X3(:);

%%% initial configuration %%%%%%%%%%%%%%%%%%%%%
```

kinematics of finite growth

```
%%% deformed configuration %%%%%%%%%%%%%%%%%

for t=1:tmax          % loop over all points in time
for i=1:imax+1        % loop over all points in space

if (x1(i,t) >= 0) && (x2(i,t) >= 0)
    phi(i,t)  =+atan(x2(i,t)/x1(i,t)) + 0.d0 * pi;
elseif (x1(i,t) < 0)
    phi(i,t)  = atan(x2(i,t)/x1(i,t)) + 1.d0 * pi;
else
    phi(i,t)  = atan(x2(i,t)/x1(i,t)) + 2.d0 * pi;
end
r(i,t) = sqrt(x1(i,t)^2+x2(i,t)^2);      z(i,t) = x3(i,t);

phi(i,t+1) =+phi(i,t) + dphi;
r(i,t+1) = r(i,t) - dr;                   z(i,t+1)  = z(i,t) + dz;

x1(i,t+1)  = r(i,t+1) * cos(phi(i,t+1));
x2(i,t+1)  = r(i,t+1) * sin(phi(i,t+1));
x3(i,t+1)  = z(i,t+1);

end                   % loop over all points in space
end                   % loop over all points in time

%%% deformed configuration %%%%%%%%%%%%%%%%%
```

kinematics of finite growth

surface growth

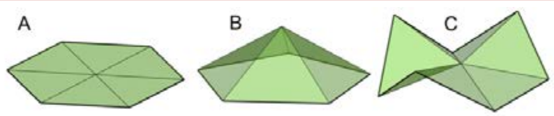


Fig. 1. Illustration of Gauss's Theorema Egregium. Change of metric in a regular hexagon (A), induced by the removal of a triangle, produces a cup-like shape (positive Gaussian curvature) (B). Conversely, insertion of a triangle produces a saddle shape (negative curvature) (C).

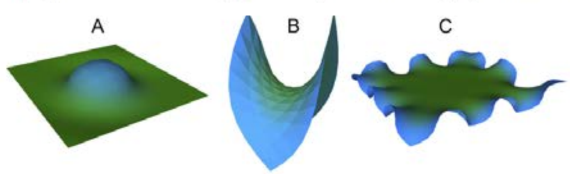


Fig. 3. Snapshots from an interactive program illustrating relations between growth, metric, and form (Matthews, 2002). The simulation begins with a relaxed square shape. Deposition of a growth-inducing morphogen (blue) in the central parts of the surface causes the formation of a cup-like shape (A). Deposition of the morphogen at the margin, with the concentrations slowly decreasing towards the centre, induces a saddle shape (B). Deposition of the morphogen along the margin, with the concentration quickly decreasing towards the centre, results in a wavy border (C).

prusinkiewicz & de reuille 'constraints of space in plant development' [2010]

kinematics of growth

surface growth

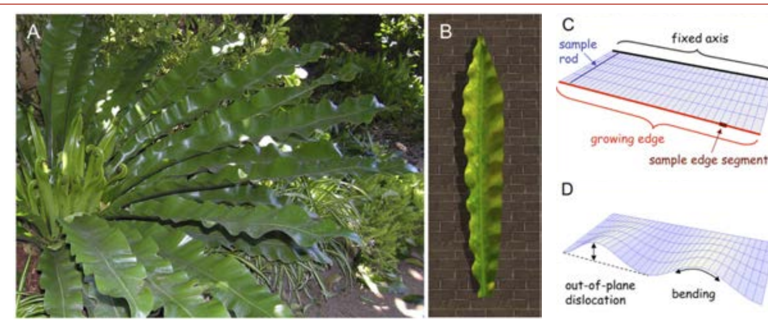


Fig. 4. Simulation study of wavy leaves. A photograph (A) and a simulation model (B) of *Asplenium australasicum* leaves showing simple waves along the margin. The model was constructed by joining surface models representing the left and right parts of the blade along the midrib. Each surface was represented as a sequence of rods spanning the area between a fixed axis and a growing edge (C). An increase in the growing edge length causes buckling, which is controlled by the relative strength of springs that counter out-of-plane dislocation and springs that counter bending of the growing edge (D). Simulations show that increasing the strength of the former type of springs compared to the latter type decreases the wavelength and amplitude of the waves

prusinkiewicz & de reuille 'constraints of space in plant development' [2010]

kinematics of growth

surface growth

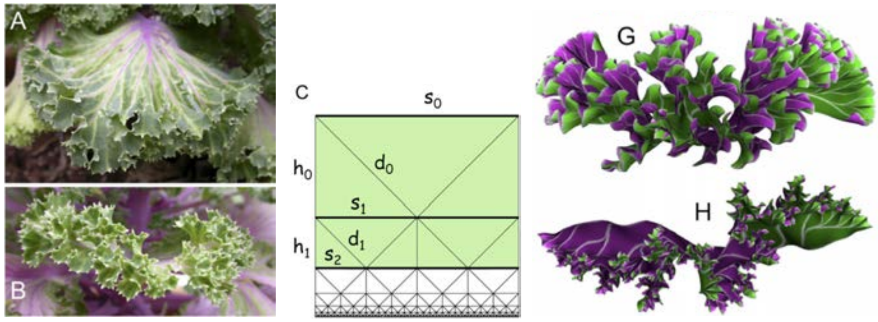


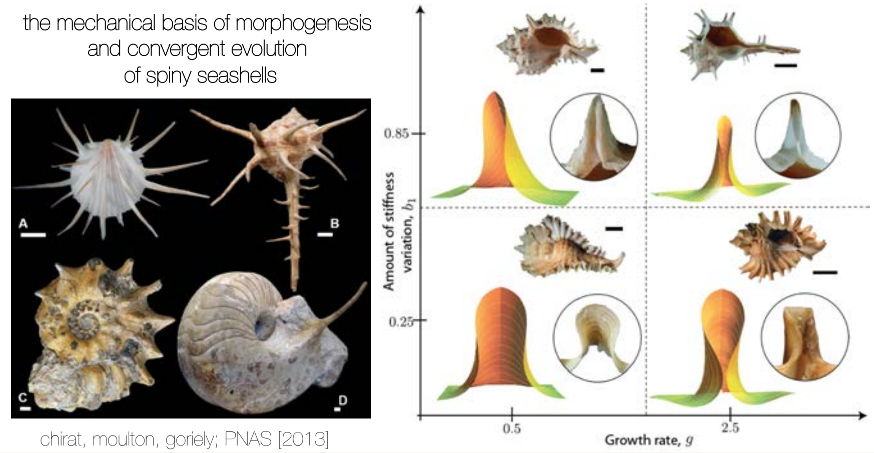
Fig. 5. Simulation study of surfaces with a fractal cascade of waves at the margin. (A) A kale (a variety of *Brassica oleracea*) leaf showing a superposition of waves with a decreasing amplitude and wavelength towards the leaf margin. (B) The fractal character of the leaf margin. (C) A computational representation of a leaf. The surface is divided into rows of geometrically similar rectangles, each row with twice the number of rectangles as its predecessor. The first two rows are highlighted in green. Each rectangle is further subdivided into three triangles. Proportions are controlled by the scaling ratio r, initially set to $\frac{1}{2}$, such that $s_{i+1}=h_i=rs_i$ and $d_i=r\sqrt{2}s_i$ for $i=0,1,2,\ldots$.

prusinkiewicz & de reuille 'constraints of space in plant development' [2010]

kinematics of growth

surface growth

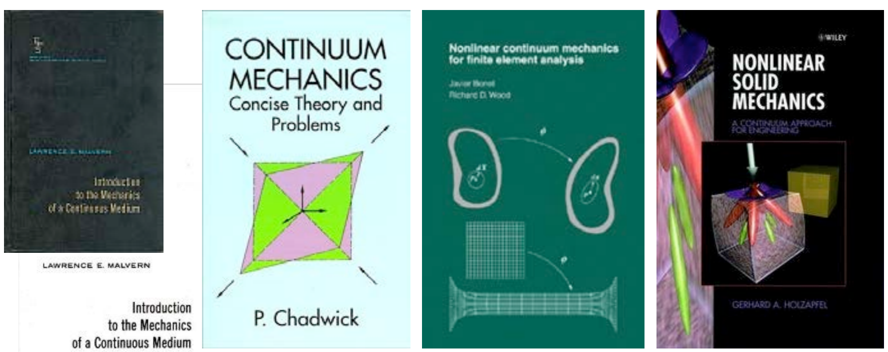
the mechanical basis of morphogenesis
and convergent evolution
of spiny seashells



chirat, moulton, goriely; PNAS [2013]

kinematics of finite growth

suggested reading



malvern le: introduction to the mechanics of a continuous medium, prentice hall, 1969
chadwick p: continuum mechanics - concise theory and problems, dover reprint, 1976
bonet j, wood rd: nonlinear continuum mechanics for fe analysis, cambridge university press, 1997
holzapfel ga: nonlinear solid mechanics, a continuum approach for engineering, john wiley & sons, 2000

introduction to continuum mechanics

continuum mechanics

continuum mechanics [kənˈtɪn.ju.əm məˈkæn.ɪks] is a branch of physics (specifically mechanics) that deals with continuous matter. the fact that matter is made of atoms and that it commonly has some sort of heterogeneous microstructure is ignored in the simplifying approximation that physical quantities, such as energy and momentum, can be handled in the infinitesimal limit. differential equations can thus be employed in solving problems in continuum mechanics.

WIKIPEDIA
The Free Encyclopedia

introduction to continuum mechanics

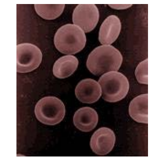
continuum mechanics

continuum mechanics [kənˈtɪn.ju.əm məˈkæn.ɪks] is the branch of mechanics concerned with the stress in solids, liquids and gases and the deformation or flow of these materials. the adjective continuous refers to the simplifying concept underlying the analysis: we disregard the molecular struc-ture of matter and picture it as being without gaps or empty spaces. we suppose that all the mathematical functions entering the theory are continuous functions.this hypothetical continuous material we call a continuum.

malvern 'introduction to the mechanics of a continuous medium' [1969]

continuum mechanics

continuum hypothesis [kənˈtɪn.ju.əm haɪˈpɑː.θə.sɪs] we assume that the characteristic length scale of the microstructure is much smaller than the characteristic length scale of the overall problem, such that the properties at each point can be understood as averages over a characteristic length scale



$$l^{\text{micro}} \ll l^{\text{averg}} \ll l^{\text{conti}}$$

example: biomechanics

$$l^{\text{micro}} = l^{\text{cells}} \approx 10\mu\text{m}$$
$$l^{\text{conti}} = l^{\text{tissue}} \approx 10\text{cm}$$

the continuum hypothesis can be applied when analyzing tissues

the potato  equations

- kinematic equations – what's strain? $\epsilon = \frac{\Delta l}{l}$
  general equations that characterize the deformation of a physical body without studying its physical cause
- balance equations – what's stress? $\sigma = \frac{F}{A}$
  general equations that characterize the cause of motion of any body
- constitutive equations - how are they related? $\sigma = E\,\epsilon$
  material specific equations that complement the set of governing equations

the potato  equations

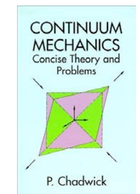
- kinematic equations - why not $\epsilon = \frac{\Delta l}{l}$?
  inhomogeneous deformation » non-constant
  finite deformation » non-linear $\boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$
  inelastic deformation » growth tensor $\boldsymbol{F} = \boldsymbol{F}_{\text{e}} \cdot \boldsymbol{F}_{\text{g}}$
- balance equations - why not $\sigma = \frac{F}{A}$? $\text{Div}(\boldsymbol{P}) + \rho\, \boldsymbol{b}_0 = \boldsymbol{0}$
  equilibrium in deformed configuration » multiple stress measures
- constitutive equations - why not $\sigma = E\,\epsilon$?
  finite deformation » non-linear $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F})$
  inelastic deformation » internal variables $\boldsymbol{P} = \boldsymbol{P}(\rho, \boldsymbol{F}, \boldsymbol{F}_{\text{g}})$

kinematics

kinematic equations [kɪnəˈmætɪk ɪˈkweɪ.ʒəns] describe the motion of objects without the consideration of the masses or forces that bring about the motion. the basis of kinematics is the choice of coordinates. the 1st and 2nd time derivatives of the position coordinates give the velocities and accelerations. the difference in placement between the beginning and the final state of two points in a body expresses the numerical value of strain. strain expresses itself as a change in size or shape.

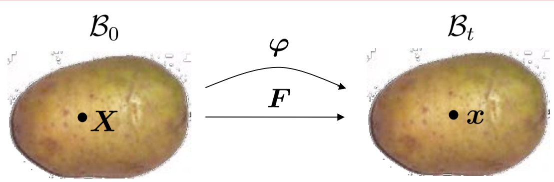
kinematics

kinematics [kɪnəˈmætɪks] is the study of motion per se, regardless of the forces causing it. the primitive concepts concerned are position, time and body, the latter abstracting into mathematical terms intuitive ideas about aggregations of matter capable of motion and deformation.



chadwick 'continuum mechanics' [1976]

kinematics



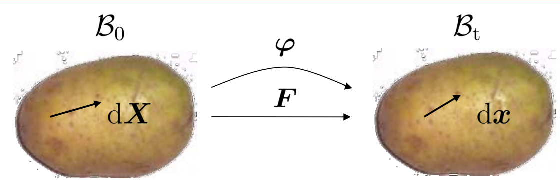
- nonlinear deformation map φ

$$\boldsymbol{x} = \varphi(\boldsymbol{X}, t) \quad \text{with} \quad \varphi : \mathcal{B}_0 \times \mathbb{R} \to \mathcal{B}_t$$

- spatial derivative of φ - deformation gradient

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} \quad \text{with} \quad \boldsymbol{F} : T\mathcal{B}_0 \to T\mathcal{B}_t \qquad \boldsymbol{F} = \left.\frac{\partial \varphi}{\partial \boldsymbol{X}}\right|_{t \text{ fixed}}$$

kinematics



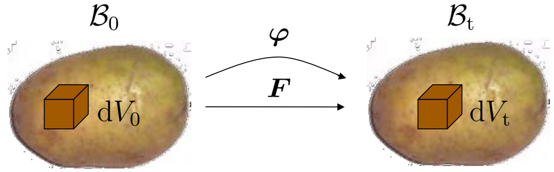
- transformation of line elements - deformation gradient F_{ij}

$$\mathrm{d}x_i = F_{ij}\, \mathrm{d}X_j \quad \text{with} \quad F_{ij} : T\mathcal{B}_0 \to T\mathcal{B}_t \qquad F_{ij} = \left.\frac{\partial \varphi_i}{\partial X_j}\right|_{t \text{ fixed}}$$

- uniaxial tension (incompressible), simple shear, rotation

$$F_{ij}^{\text{uni}} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha^{-\frac{1}{2}} \end{bmatrix} \quad F_{ij}^{\text{shr}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F_{ij}^{\text{rot}} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

kinematics



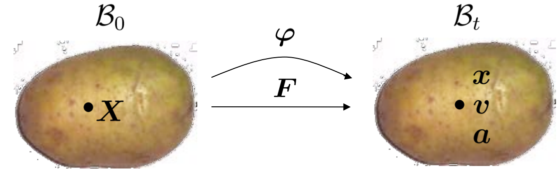
- transformation of volume elements - determinant of $\boldsymbol{F}$

$$\mathrm{d}V_0 = \mathrm{d}\boldsymbol{X}_1 \cdot [\mathrm{d}\boldsymbol{X}_2 \times \mathrm{d}\boldsymbol{X}_3] \qquad \mathrm{d}V_t = \mathrm{d}\boldsymbol{x}_1 \cdot [\mathrm{d}\boldsymbol{x}_2 \times \mathrm{d}\boldsymbol{x}_3]$$
$$= \det([\mathrm{d}\boldsymbol{x}_1, \mathrm{d}\boldsymbol{x}_2, \mathrm{d}\boldsymbol{x}_3])$$
$$= \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3]) \qquad = \det(\boldsymbol{F}) \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3])$$

- changes in volume - determinant of deformation tensor J

$$\mathrm{d}V_t = J\,\mathrm{d}V_0 \qquad J = \det(\boldsymbol{F})$$

kinematics



- temporal derivative of $\boldsymbol{\varphi}$ - velocity (material time derivative)

$$\boldsymbol{v} = \mathrm{D}_t\boldsymbol{\varphi} = \frac{\partial \boldsymbol{\varphi}}{\partial t}\bigg|_{X\ \text{fixed}} \qquad \text{with} \quad \boldsymbol{v} : \mathcal{B}_0 \times \mathbb{R} \to \mathbb{R}^3$$

- temporal derivative of $\boldsymbol{v}$ - acceleration

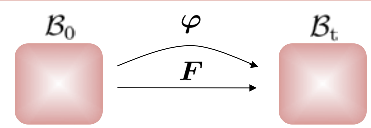
$$\boldsymbol{a} = \mathrm{D}_t\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t}\bigg|_{X\ \text{fixed}} = \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}\bigg|_{X\ \text{fixed}} \qquad \text{with} \quad \boldsymbol{a} : \mathcal{B}_0 \times \mathbb{R} \to \mathbb{R}^3$$

volume growth

volume growth ['val.ju:m groʊθ] is conceptually comparable to thermal expansion. in linear elastic problems, growth stresses (such as thermal stresses) can be superposed on the mechanical stress field. in the nonlinear problems considered here, another approach must be used. the fundamental idea is to refer the strain measures in the constitutive equations of each material element to its current zero-stress configuration, which changes as the element grows.

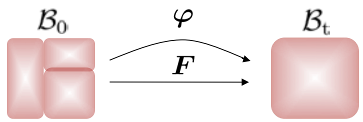
taber 'biomechanics of growth, remodeling and morphogenesis' [1995]

kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

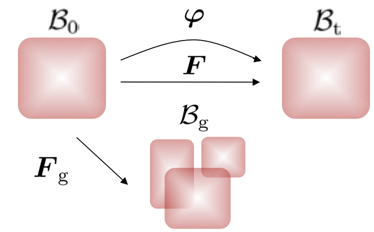
kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

kinematics of finite growth

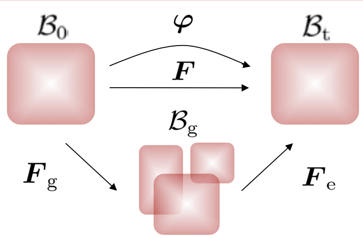


[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

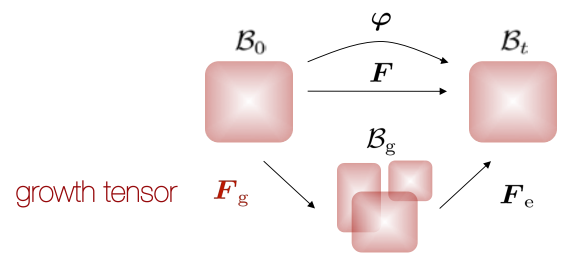
[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[4] loading generates compatible current configuration $\mathcal{B}_t$

kinematics of finite growth

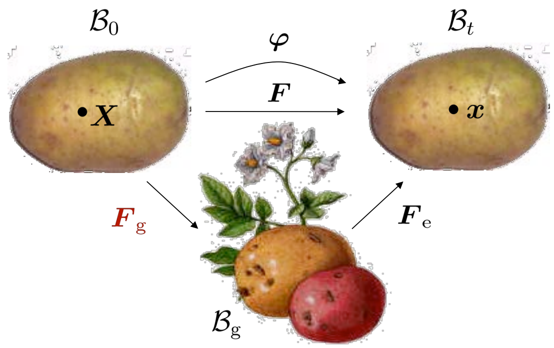
$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_g$$



multiplicative decomposition

lee [1969], simo [1992], rodriguez, hoger & mc culloch [1994], epstein & maugin [2000], humphrey [2002], ambrosi & mollica [2002], himpel, kuhl, menzel & steinmann [2005]

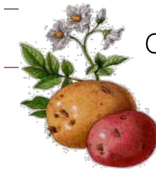
kinematics of finite growth



- incompatible growth configuration $\mathcal{B}_{\mathrm{g}}$ & growth tensor $\boldsymbol{F}_{\mathrm{g}}$

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{g}}$$

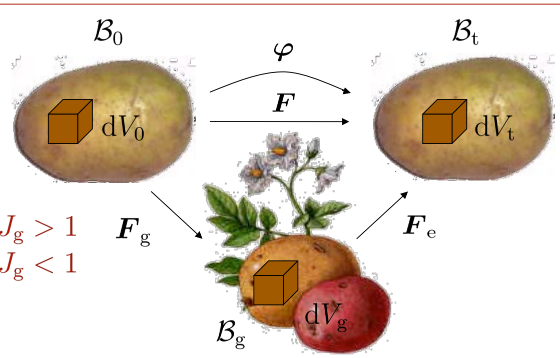
rodriguez, hoger & mc culloch [1994]



concept of incompatible growth configuration

biologically, the notion of **incompatibility** implies that subelements of the grown configuration may overlap or have gaps. the implication of incompatibility is the existence of residual stresses necessary to `squeeze` these grown subelements back together. mathematically, the notion of **incompatibility** implies that unlike the deformation gradient, $\boldsymbol{F} = \left.\frac{\partial \varphi}{\partial \boldsymbol{X}}\right|_{t\text{ fixed}}$ the growth tensor cannot be derived as a gradient of a vector field. incompatible configurations are useful in finite strain inelasticity such as viscoelasticity, thermoelasticity, elastoplasticity and growth.

kinematics of finite growth



growth $J_{\mathrm{g}} > 1$
resorption $J_{\mathrm{g}} < 1$

- changes in volume - determinant of growth tensor J_{g}

$$\mathrm{d}V_{\mathrm{g}} = J_{\mathrm{g}}\, \mathrm{d}V_0 \qquad J_{\mathrm{g}} = \det(\boldsymbol{F}_{\mathrm{g}})$$

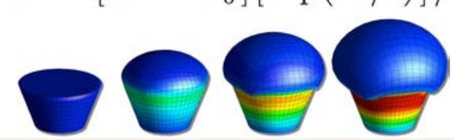
growth in constrained geometries

- isotropic volume growth

$$\boldsymbol{F}^{\mathrm{g}} = \vartheta \boldsymbol{I}$$

- morphogenetic growth

$$\dot{\vartheta}^{\mathrm{g}} = [\,\vartheta^{\max} - \vartheta_0\,]\,[\,\exp\,(-t/\tau)\,]\,/\,\tau$$



constrained growth during development

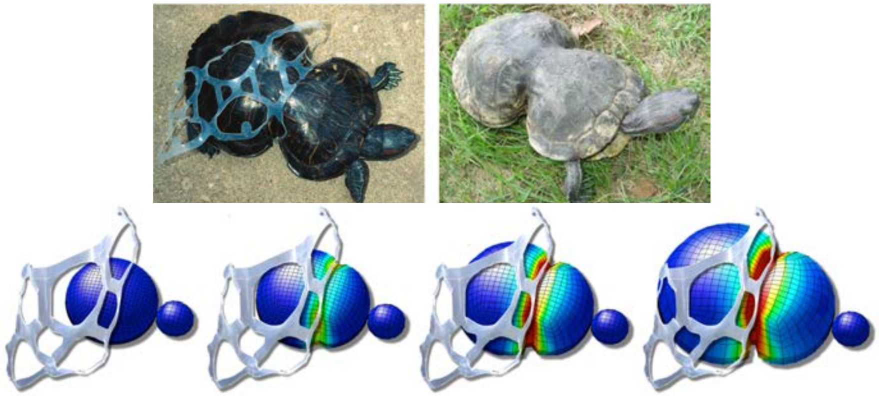


figure. growth-induced microenvironmental changes in a nine-year old female red-eared slider turtle trapped in a plastic six-pack ring. the turtle had worn the ring for five years. during this time, the ring had constrained the growth of the outer shell and created growth-induced stresses on the inner organs.

example - growing turtles

finite growth and tissue tension in rhubarb

Exercise 21.

86. To demonstrate the tissue tension.—Take a portion of the petiole of a caladium, or of celery, or other plant, about 15*cm* long. Cut the ends off squarely. With a knife strip off a layer from the outside about 2–3*mm* in thickness, and the full length of the piece. Now attempt to replace it, comparing the length of each part. Remove another strip lying next this one, and so on until all the outer portion has been removed. Describe what takes place as the successive strips are removed. When all are removed, compare an outside strip with the central portion. What has happened? Is there now a greater difference in length between the outside strip and the central portion? What is the cause of this? Describe the tensions in the outside and inner portion of the petiole.

Cut a section of the petiole about 8*cm* long, remove strips on two opposite sides and split the remainder down the middle, securing two pieces with the center and outside portion attached. Place one of these in fresh water and the other in a 5 per cent salt solution and note the result. If convenient treat celery petioles in the same way. The flower stems of dandelions split into quarters are excellent objects to compare when placed in water, and in a 5 per cent salt solution.

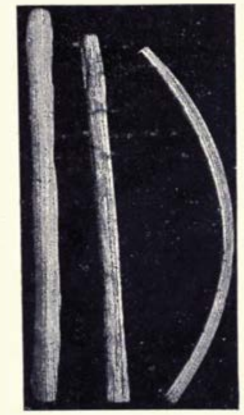


Fig. 37. Centre of petiole. Fig. 38. Outside strip. Fig. 39. Outside strip attached to centre.

george francis atkinson 'lessons in botany' [1900]

example - growing plants

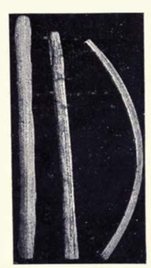
finite growth and tissue tension in rhubarb

GROWTH.

174. Meaning of growth.—By growth is usually meant an increase in the bulk of the plant accompanied generally by an increase in plant substance.

Exercise 21.

86. To demonstrate the tissue tension.—Take a portion of the petiole of a caladium, or of celery, or other plant, about 15*cm* long. Cut the ends off squarely. With a knife strip off a layer from the outside about 2–3*mm* in thickness, and the full length of the piece. Now attempt to replace it, comparing the length of each part. Remove another strip lying next this one, and so on.



george francis atkinson "lessons in botany" [1900]

example - growing plants

finite growth and tissue tension in rhubarb

118. Differential Growth.—Not all the tissues of a stem or other part grow at the same rate.[1] On this account, and since adjacent tissues are closely united, those which elongate or grow more slowly are stretched by those which grow more rapidly. As a result either a state of tension exists, or the organ is distorted, or both.

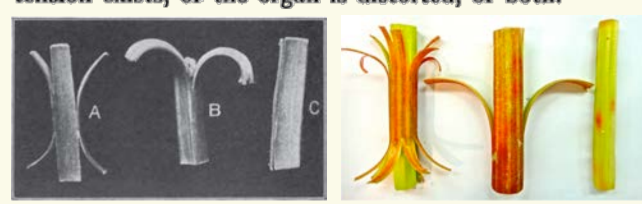
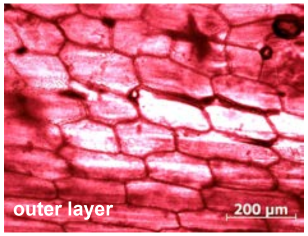


Fig. 74.—Longitudinal tissue-tension in leaf-stalk of rhubarb. In *C* the strip of outer tissues, entirely removed from the main piece, is seen to have shortened, showing that, before being removed, it was in a state of *longitudinal tissue-tension*.

charles stuart gager "fundamentals of botany" [1916], holland, kosmata, goriely, kuhl [2013]

example - growing plants

microstructure of rhubarb

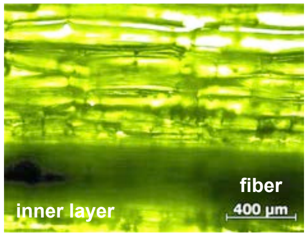


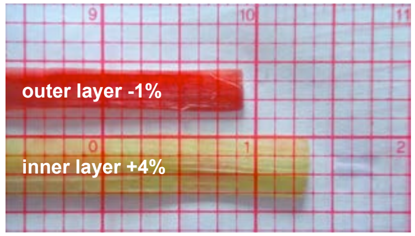
differential growth

outer layer
epidermis and collenchyma layers
tension - shortens by **-1%**

inner layer
parenchyma
compression - lengthens by **+4%**

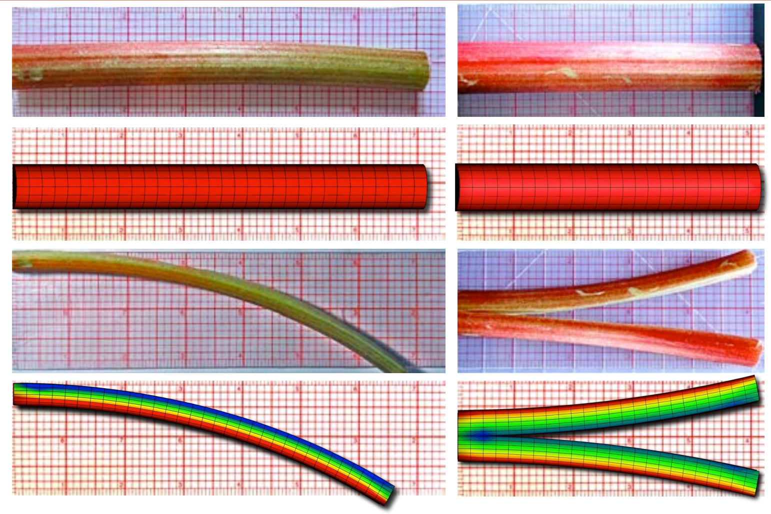
residual stresses



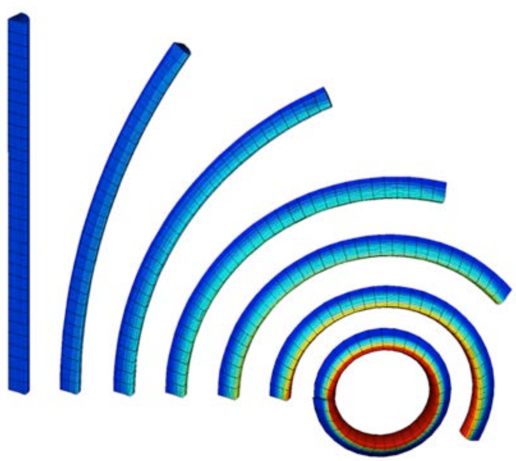


vandiver, goriely [2009], holland, kosmata, goriely, kuhl [2013]

growth of thin biological surfaces

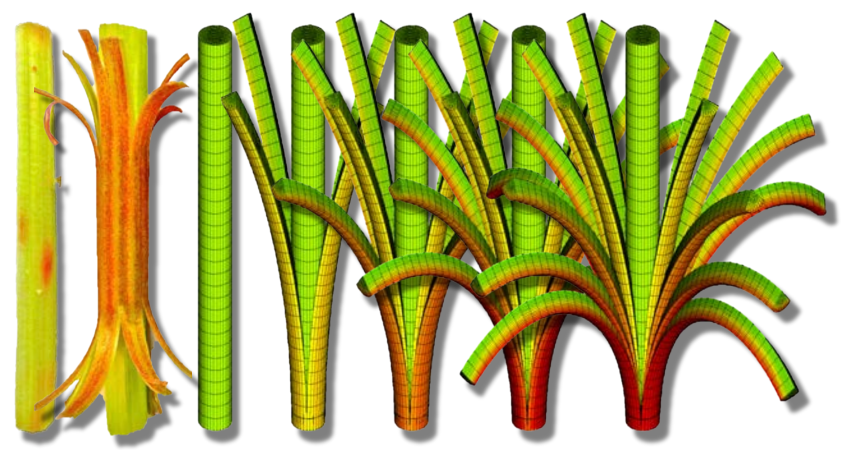


growth of thin biological surfaces



holland, kosmata, goriely, kuhl [2013]

growth of thin biological surfaces



holland, kosmata, goriely, kuhl [2013]