15 - density growth

functional adaptation of proxima femur

\[ D_t \rho = R_0 \]

\[ R_0 = \frac{2}{\nu} \sigma_i \psi_0 - \psi_i^* \]

fig. 4. the density distribution resulting from a bone remodeling simulation carried out using the traditional element-based algorithm. this type of behavior is clearly nonbiological in nature and motivates the question: are the current strain-energy-based continuum formulations incapable of predicting the expected continuous results near bone ends or is this difficulty technical in nature to be overcome with appropriate numerical implementation?

JACOBS, LEVASTON, BEEUJEN, SIMO, CARTER [1995]

finite elements - integration point based

from integral equation...

- integral equations cannot be evaluated analytically

\[ R_{ij}^0 = \int_{\Omega} \int_{\Omega} \nabla N^0_i (\xi, \eta, \zeta) \cdot P_{n+1}(\xi, \eta, \zeta) \det(J(\xi, \eta, \zeta)) \, d\xi d\eta d\zeta \]

\[ K_{ij}^0 = \int_{\Omega} \int_{\Omega} \nabla N^0_i (\xi, \eta, \zeta) \cdot D_{ij} P(\xi, \eta, \zeta) \cdot \nabla N^0_j (\xi, \eta, \zeta) \det(J(\xi, \eta, \zeta)) \, d\xi d\eta d\zeta \]

- idea - numerical integration / quadrature

\[ R_{ij}^0 \approx \sum_{i=0}^n \nabla N^0_i (\xi_i, \eta_i, \zeta_i) \cdot P_{n+1}(\xi_i, \eta_i, \zeta_i) \det(J(\xi_i, \eta_i, \zeta_i)) \, w_i \]

\[ K_{ij}^0 \approx \sum_{i=0}^n \nabla N^0_i (\xi_i, \eta_i, \zeta_i) \cdot D_{ij} P(\xi_i, \eta_i, \zeta_i) \cdot \nabla N^0_j (\xi_i, \eta_i, \zeta_i) \det(J(\xi_i, \eta_i, \zeta_i)) \, w_i \]

... to discrete sum

finite elements - integration point based

finite elements - integration point based

numerical integration

- integral equations are approximated by discrete sums

\[ \int_a^b f(\xi) \, d\xi \approx \sum_{i=0}^n f(\xi_i) \, w_i \]

\[ \xi_i \ldots \text{quadrature point coordinates} \]

\[ w_i \ldots \text{quadrature point weights} \]
@ integration point level

- constitutive equations - given \( F = \nabla \phi \) calculate \( P \)
- update density for current stress state from \( \rho_{0n} \) and \( D_t \rho_0 = \left[ \frac{\rho_0}{\rho_0} \right] \psi_0(F) - \psi_0^* \) calculate \( \rho_{0n+1} \)
- calculate first piola kirchhoff stress of solid material \( P_{\text{neo}}(F) = \mu_0 F + [\lambda_0 \ln(\det(F))] - \mu_0 |F| \)
- calculate first piola kirchhoff stress of porous material \( P(F) = \left[ \frac{\rho_0}{\rho_0} \right] P_{\text{neo}} \)

stress for righthand side vector
recipe for finite element modeling

from continuous problem...

\[ D_t \rho_0 = \text{Div}(\mathbf{R}) + \mathcal{R}_0 \]
\[ \rho_0 D_t \mathbf{v} = \text{Div}(\mathbf{P}) + b_0 \]

- temporal discretization: implicit euler backward
- spatial discretization: finite element method
- staggered/simultaneous: newton raphson iteration
- linearization: gateaux derivative

... to linearized discrete initial boundary value problem

finite elements - node point based

residual equations...

\[ R^\rho = D_t \rho_0 - \text{Div}(\mathbf{R}) + \mathcal{R}_0 = 0 \quad \text{in } B_0 \]
\[ \partial B_0 = \partial B_0^e \cup \partial B_0^{te} \]
\[ R^{\rho e} = \rho_0 D_t \mathbf{v} - \text{Div}(\mathbf{P}) - b_0 = 0 \quad \text{in } B_0 \]
\[ \partial B_0 = \partial B_0^e \cup \partial B_0^{te} \]

- dirichlet / essential boundary conditions
  \[ \rho_0 - \bar{\rho}_0 = 0 \quad \text{on } \partial B_0^e \]
  \[ \Phi - \Phi = 0 \quad \text{on } \partial B_0^{te} \]

- neumann / natural boundary conditions
  \[ \mathbf{R} \cdot \mathbf{N} - T^e = 0 \quad \text{on } \partial B_0^{te} \]
  \[ \mathbf{P} \cdot \mathbf{N} - T^{\rho e} = 0 \quad \text{on } \partial B_0^{te} \]

... and boundary conditions

from biological and mechanical equilibrium...

\[ D_t \rho_0 = \text{Div}(\mathbf{R}) + \mathcal{R}_0 \]
\[ \rho_0 D_t \mathbf{v} = \text{Div}(\mathbf{P}) + b_0 \]

- strong form / residual format
  \[ R^\rho (\rho_0, \Phi) = 0 \quad \text{in } B_0 \]
  \[ R^{\rho e} (\rho_0, \Phi) = 0 \quad \text{in } B_0 \]

- residuals
  \[ R^\rho = D_t \rho_0 - \text{Div}(\mathbf{R}) - \mathcal{R}_0 \]
  \[ R^{\rho e} = \rho_0 D_t \mathbf{v} - \text{Div}(\mathbf{P}) - b_0 \]

... to residual format

finite elements - node point based

from strong form...

\[ R^\rho = D_t \rho_0 - \text{Div}(\mathbf{R}) - \mathcal{R}_0 = 0 \]
\[ R^{\rho e} = \rho_0 D_t \mathbf{v} - \text{Div}(\mathbf{P}) - b_0 = 0 \]

- weak form
  \[ G^\rho (\delta \rho; \rho_0, \Phi) = 0 \quad \forall \delta \rho \quad \text{in } \mathcal{H}_1^0 (B_0) \]
  \[ G^{\rho e} (\delta \Phi; \rho_0, \Phi) = 0 \quad \forall \delta \Phi \quad \text{in } \mathcal{H}_1^0 (B_0) \]

- weak form expressions

\[ G^\rho = \int_{B_0} \delta \rho \quad D_t \rho_0 dV - \int_{B_0} \delta \rho \text{Div}(\mathbf{R}) dV - \int_{B_0} \delta \rho \mathcal{R}_0 dV \]
\[ G^{\rho e} = \int_{B_0} \delta \Phi \quad \rho_0 D_t \mathbf{v} dV - \int_{B_0} \delta \Phi \text{Div}(\mathbf{P}) dV - \int_{B_0} \delta \Phi b_0 dV \]

... to nonsymmetric weak form

finite elements - node point based
from nonsymmetric weak form...

- integration by parts
  \[ \int_{\Omega} \delta \rho \text{Div}(\mathbf{R}) \, dV = \int_{\Omega} \text{Div}(\delta \rho \mathbf{R}) \, dV - \int_{\partial \Omega} \delta \rho \nabla \cdot \mathbf{R} \cdot \mathbf{n} \, dA \]
- Gauss theorem & boundary conditions
  \[ \int_{\Omega} \delta \phi \text{Div}(\mathbf{P}) \, dV = \int_{\Omega} \text{Div}(\delta \phi \cdot \mathbf{P}) \, dV - \int_{\partial \Omega} \delta \phi \nabla \phi \cdot \mathbf{n} \, dA \]

- weak form

\[ G^p = \int_{\Omega} \delta \rho \mathbf{R} \cdot \nabla \phi \, dV - \int_{\partial \Omega} \delta \rho \mathbf{T} \cdot \mathbf{n} \, dA - \int_{\partial \Omega} \delta \rho \mathbf{R} \cdot \mathbf{n} \, dA \]

\[ G^\phi = \int_{\Omega} \delta \phi \rho_0 \mathbf{dv} + \int_{\partial \Omega} \delta \phi \mathbf{n} \cdot \mathbf{N} \, dA - \int_{\partial \Omega} \delta \phi \mathbf{b}_0 \, dA \]

... to symmetric weak form

finite elements - node point based

spatial discretization

- discretization
  \[ B_0 = \bigcup_{e=1}^{n_{\text{el}}} B_0^e \]
- interpolation of test functions
  \[ \delta \rho^h|_{B_0} = \sum_{i=1}^{n_{\text{en}}} N_i^e \delta \rho_i \in \mathcal{H}_1^0(B_0) \]
  \[ \nabla \delta \rho^h|_{B_0} = \sum_{i=1}^{n_{\text{en}}} \rho_i \nabla N_i^e \]
- interpolation of trial functions
  \[ \rho^h|_{B_0} = \sum_{k=1}^{n_{\text{en}}} N_k^e \rho_k \in \mathcal{H}_1(B_0) \]
  \[ \nabla \rho^h|_{B_0} = \sum_{k=1}^{n_{\text{en}}} \rho_k \nabla N_k^e \]

... to discrete weak form

finite elements - node point based

from discrete weak form...

- discrete residual format
  \[ R_I = \mathbf{a}^h_{\rho} \int_{\Omega} N_i^1 \rho_0 n_{\text{en}} + \rho n_{\text{en}} \cdot \mathbf{n} \cdot \mathbf{N} \cdot dV + \int_{\partial \Omega} \nabla N_i^1 \cdot \mathbf{R} \cdot n_{\text{en}} + \nabla \mathbf{P} \cdot dA \]
- discrete residuals
  \[ R_I = \mathbf{a}^h_{\rho} \int_{\Omega} N_i^1 \rho_0 n_{\text{en}} + \rho n_{\text{en}} \cdot \mathbf{n} \cdot \mathbf{N} \cdot dV + \int_{\partial \Omega} \nabla N_i^1 \cdot \mathbf{R} \cdot n_{\text{en}} + \nabla \mathbf{P} \cdot dA \]

... to discrete residuals

finite elements - node point based

temporal discretization

- discretization
  \[ T = \bigcup_{n=0}^{n_{\text{step}}-1} [t_n, t_{n+1}] \]
  \[ \Delta t = t_{n+1} - t_n \]
- time discrete weak form
  \[ G^p(\delta \rho; \rho_{0_n-1}, \varphi_{n+1}) = 0 \quad \forall \delta \rho \text{ in } \mathcal{H}_0^1(B_0) \]
  \[ G^\phi(\delta \varphi; \rho_{0_n-1}, \varphi_{n+1}) = 0 \quad \forall \delta \varphi \text{ in } \mathcal{H}_0^1(B_0) \]
- interpolation of material time derivatives
  \[ D_t \rho_0 = \frac{1}{\Delta t} (\rho_{0_{n+1}} - \rho_{0_n}) \]
  \[ D_t \mathbf{v} = \frac{1}{\Delta t} (\mathbf{v}_{n+1} - \mathbf{v}_n) \]

... to semidiscrete weak form

finite elements - node point based
finite elements - node point based

loop over all time steps
  global newton iteration
    loop over all elements
      loop over all quadrature points
        evaluate balance of mass and momentum
determine element residuals & partial derivatives
determine global residuals and iteration matrices
determine $\rho_{n+1}$ and $\varphi_{n+1}$
determine state of biological equilibrium

density $\rho_0$ as nodal degree of freedom

finite elements - integration point based

loop over all time steps
  global newton iteration
    loop over all elements
      loop over all quadrature points
        determine element residual & partial derivative
determine global residual and iteration matrix
determine $\rho_{n+1}$
determine state of biological equilibrium

density $\rho_0$ as internal variable
**finite element method**

**discontinuous model problem**

**integration vs node point based**

**exact solution**

**approximation**

**continuous model problem**

**integration vs node point based**

**finite element method**

**different load cases**

1. Midstance phase of gait
   - $2317 \text{ N}$
   - $24^\circ$
   - $703 \text{ N}$
   - $28^\circ$

2. Extreme range of abduction
   - $1158 \text{ N}$
   - $15^\circ$
   - $351 \text{ N}$
   - $-8^\circ$

3. Extreme range of adduction
   - $1548 \text{ N}$
   - $56^\circ$
   - $468 \text{ N}$
   - $35^\circ$

**example - adaptation in bone**

Example - adaptation in bone


Example - adaptation in bone

Node point based - h-refinement

$\rho_0 - \rho_0$

Kuhl, Menzel, Steinmann [2003]
node point based - p-refinement

\[ n_{el} = 658 \]
\[ n_{dof} = 2175 \]
\[ n_{ip} = 2632 \]

integration point based - p-refinement

\[ n_{el} = 658 \]
\[ n_{dof} = 1450 \]
\[ n_{ip} = 2632 \]

example - adaptation in bone

example - adaptation in bone

parameter sensitivity - instabilities

\[ \psi_0 = \left[ \frac{\rho_0}{\rho^*_0} \right]^n \psi_0^{geo}(F) \]
\[ \mathcal{R}_0 = \left[ \frac{\rho_0}{\rho^*_0} \right]^m (\psi_0(F) - \psi_0^*) \]
\[ n = 2 \]
\[ m = 0 \]

\[ \rho^*_0 = +0.20 \]
\[ \rho^*_0 = +0.00 \]
\[ \rho^*_0 = -0.20 \]
\[ \rho^*_0 = -0.40 \]
\[ \rho^*_0 = -0.60 \]
\[ \rho^*_0 = -0.80 \]

certain parameters induce checkerboard modes

harrigon & hamilton [1992], [1994]

total hip replacement vs hip resurfacing

• about 120,000 artificial hip replacements in us per year
• aseptic loosening caused by adaptive bone remodeling
• goal prediction of redistribution of bone density

example - hip replacement
**Total Hip Replacement**

Total hip replacement is a surgical procedure in which the hip joint is replaced by a prosthetic implant. A total hip replacement consists of replacing both the acetabulum and the femoral head. Hip replacement is currently the most successful and reliable orthopaedic operation. Risks and complications include aseptic loosening, dislocation, and pain. In the long term, many problems relate to bone resorption and subsequent loosening or fracture often requiring revision surgery.

**Example - Hip Replacement**

**Conventional Total Hip Replacement**

Conventional total hip replacement is a surgical procedure which has been developed as an intervention alternative to total hip replacement. The potential advantages of hip resurfacing include less bone removal, a potentially lower number of hip dislocations due to a relatively larger femoral head size, and possibly easier revision surgery for a subsequent total hip replacement device. The potential disadvantages are femoral neck fractures, aseptic loosening, and metal wear.

**Example - Hip Replacement**

**Hip Resurfacing**

Hip resurfacing is a surgical procedure which has been developed as an intervention alternative to total hip replacement. The potential advantages of hip resurfacing include less bone removal, a potentially lower number of hip dislocations due to a relatively larger femoral head size, and possibly easier revision surgery for a subsequent total hip replacement device. The potential disadvantages are femoral neck fractures, aseptic loosening, and metal wear.
new birmingham hip resurfacing

ward’s triangle • trabeculae • dense cortical shaft

example - hip replacement

computer tomography of human femur

cr of overall hip • representative cut • binary data model • solid model of proxima femur

spline models of individual cuts

patient specific medical treatment

example - hip replacement

osteoaarthritis

osteoarthritis, also known as degenerative joint disease or osteoarthrosis, is a group of mechanical abnormalities involving degradation of joints, including articular cartilage and subchondral bone. it affects about 27 million people in the united states alone. symptoms may include joint pain, tenderness, stiffness, locking, and sometimes an effusion. a variety of causes, hereditary, developmental, metabolic, and mechanical, may initiate processes leading to loss of cartilage. when bone surfaces become less well protected by cartilage, bone may be exposed and damaged. treatment generally involves a combination of exercise, lifestyle modification, or, in severe cases, surgical joint replacement.
Figure 1. Healthy proximal tibia. Isometric density distribution, left, visualized through DEXA scan, which displays the characteristic heterogeneous density distribution with a dense region in the medial plateau, a lower density region in the lateral plateau, and the lowest density in the central region. Pang et al. [2012]. Anisotropic density distribution, right, visualized through a thin section. Trabeculae are aligned with axis of maximum principal loading, Wolf [1870].

Figure 6. Geometry of the upper part of the proximal tibia. (a) Finite element mesh with applied concentrated forces $F_1$, $F_2$ and (b) location of slices I (sagittal slice), II (axial slice) and III (coronal slice) used in Figs. 9-13.

Waffenschmidt, Menzel, Kuhl [2012].
anisotropy of bone mineral density

example - osteoarthritis

anisotropy of bone mineral density

example - osteoarthritis

wound healing

• epidermal migration / spreading of existing cells $R$
• increase of mitotic activity of about 15 times in 1mm wide zone at wound edge $R_0$

example - wound healing

wound healing

…thou shouldst bind it with fresh meet the first day, and thou shouldst treat afterword with grease and honey every day until he recovers...

edwin smith surgical papyrus [13th dynasty]
the phases of cutaneous wound healing: Beanes, Dang, Soo, Ting [2003]

Increased cellular activity
Density increase at wound edge

Example - wound healing

Tension - single edge notched specimen
Kuhl, Steinmann [2004]

Increased cell activity @wound edge

Example - wound healing
example - wound healing

increased cell activity - wound healing & closure

kuhl, steinmann [2004]

example - wound healing

increased cell activity @wound edge

bending - single edge notched specimen

example - wound healing

bending - crack mouth opening displacement

increased cell activity - wound healing & closure

kuhl, steinmann [2004]