15 - finite element method - density growth - alternative formulation

functional adaptation of proxima femur

$$\mathrm{D}_t\rho_0 = \mathcal{R}_0 \qquad \mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m}\psi_0 - \psi_0^*$$

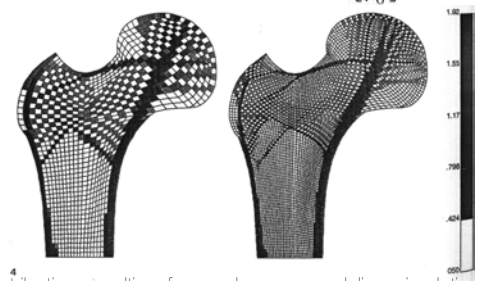


fig. 4. the density distribution resulting from a bone remodeling simulation carried out using the traditional element-based algorithm. this type of behavior is clearly nonbiological in nature and motivates the question: are the current strain-energy-based continuum formulations incapable of predicting the expected continuous results near bone ends or is this difficulty technical in nature to be overcome with appropriate numerical implementation?

jacobs, levenston, beaupre, simo, carter [1995]

from integral equation…

- integral equations cannot be evaluated analytically

$$\mathbf{R}^{\mathrm{e}}_j = \int_\zeta\int_\eta\int_\xi \nabla N^j_\varphi(\xi,\eta,\zeta)\cdot \boldsymbol{P}_{n+1}(\xi,\eta,\zeta)\,\det(\boldsymbol{J}(\xi,\eta,\zeta))\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

$$\mathbf{K}^{\mathrm{e}}_{jl} = \int_\zeta\int_\eta\int_\xi \nabla N^j_\varphi(\xi,\eta,\zeta)\cdot \mathrm{D}_F\boldsymbol{P}(\xi,\eta,\zeta)\cdot\nabla N^l_\varphi(\xi,\eta,\zeta)\,\det(\boldsymbol{J}(\xi,\eta,\zeta))\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

- idea - numerical interation / quadrature

$$\mathbf{R}^{\mathrm{e}}_j \approx \sum_{i=0}^{n} \nabla N^j_\varphi(\xi_i,\eta_i,\zeta_i)\cdot \boldsymbol{P}_{n+1}(\xi_i,\eta_i,\zeta_i)\,\det(\boldsymbol{J}(\xi_i,\eta_i,\zeta_i)\,w_i$$

$$\mathbf{K}^{\mathrm{e}}_{jl} \approx \sum_{i=0}^{n} \nabla N^j_\varphi(\xi_i,\eta_i,\zeta_i)\cdot \mathrm{D}_F\boldsymbol{P}(\xi_i,\eta_i,\zeta_i)\cdot\nabla N^l_\varphi(\xi_i,\eta_i,\zeta_i)\,\det(\boldsymbol{J}(\xi_i,\eta_i,\zeta_i)\,w_i$$

… to discrete sum

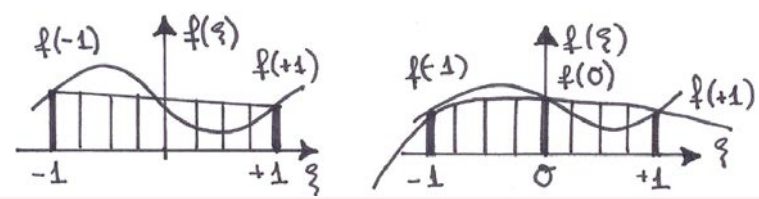
numerical integration

- integral equations are approximated by discrete sums

$$\int_a^b f(\xi)\mathrm{d}\xi \approx [\,b-a\,]\sum_{i=0}^{n} f(\xi_i)\,w_i$$

ξ_i … quadrature point coordinates

w_i… quadrature point weights



newton cotes quadrature - accuracy [n-1]

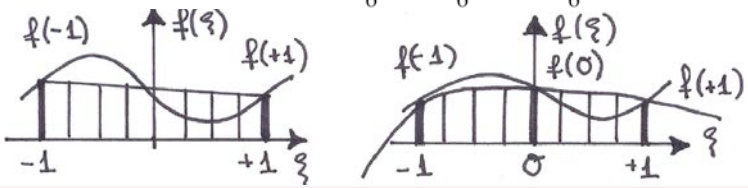
equidistant quadrature points @ $\xi_i = -1 + 2\frac{i}{n}$

n=2 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(\xi_0)\,w_0 + f(\xi_1)\,w_1\,]$

$= f(-1) + f(+1)$ trapezoidal rule

n=3 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(\xi_0)\,w_0 + f(\xi_1)\,w_1 + f(\xi_2)\,w_2\,]$

$= 2\,[\,f(-1)\frac{1}{6} + f(0)\frac{4}{6} + f(+1)\frac{1}{6}\,]$ simpson rule



gauss legendre quadrature - accuracy [2n-1]

optimized quadrature points

n=1 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(0)\,1\,]$

n=2 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(-\frac{1}{\sqrt{3}})\,\frac{1}{2} + f(+\frac{1}{\sqrt{3}})\,\frac{1}{2}\,]$

n=3 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(-\frac{3}{\sqrt{5}})\,\frac{5}{18} + f(0)\,\frac{8}{18} + f(+\frac{3}{\sqrt{5}})\,\frac{5}{18}\,]$

most fe programs prefer gauss over newton!

@ integration point level

- constitutive equations - given $\boldsymbol{F} = \nabla\boldsymbol{\varphi}$ calculate $\boldsymbol{P}$ 
- update density for current stress state
  from $\rho_{0\,n}$ and $\mathrm{D}_t\rho_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m}\psi_0(\boldsymbol{F}) - \psi_0^*$ calculate $\rho_{0\,n+1}$
- calculate first piola kirchhoff stress of solid material
  $\boldsymbol{P}^{\mathrm{neo}}(\boldsymbol{F}) = \mu_0\,\boldsymbol{F} + [\,\lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{-\mathrm{t}}$
- calculate first piola kirchhoff stress of porous material
  $\boldsymbol{P}\,(\boldsymbol{F}) = \left[\frac{\rho_0}{\rho_0^*}\right]^n \boldsymbol{P}^{\mathrm{neo}}$

stress for righthand side vector

staggered solution - integration point based

weinans, huiskes & grootenboer [1992], harrigan & hamilton [1992],, [1994], jacobs, levenston, beaupré,simo & carter [1995]

simultaneous solution - node point based

jacobs, levenston, beaupré,simo & carter [1995], fischer, jacobs, levenston & carter [1997], nackenhorst [1997], levenston [1997]]

sequential solution - element based

huiskes, weinans, grootenboer, dalstra, fudala & slooff [1987], carter, orr, fhyrie [1989], beaupré, orr & carter [1990], weinans, huiskes & grootenboer [1992], [1994], jacobs, levenston, beaupré, simo & carter [1995], huiskes [2000], carter & beaupré [2001]

recipe for finite element modeling

from continuous problem…

$$\begin{aligned} \mathrm{D}_t\rho_0 &= \mathrm{Div}(\boldsymbol{R}) + \mathcal{R}_0 \\ \rho_0\,\mathrm{D}_t\boldsymbol{v} &= \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0 \end{aligned}$$

- temporal discretization — implicit euler backward
- spatial discretization — finite element method
- staggered/simultaneous — newton raphson iteration
- linearization — gateaux derivative

… to linearized discrete initial boundary value problem

from biological and mechanical equilibrium…

$$\begin{aligned} \mathrm{D}_t\rho_0 &= \mathrm{Div}(\boldsymbol{R}) + \mathcal{R}_0 \\ \rho_0\,\mathrm{D}_t\boldsymbol{v} &= \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0 \end{aligned}$$

- strong form / residual format

$$\begin{aligned} \mathsf{R}^\rho(\rho_0,\boldsymbol{\varphi}) &= 0 \quad \text{in } \mathcal{B}_0 \\ \mathbf{R}^\varphi(\rho_0,\boldsymbol{\varphi}) &= \mathbf{0} \quad \text{in } \mathcal{B}_0 \end{aligned}$$

- residuals

$$\begin{aligned} \mathsf{R}^\rho &= \mathrm{D}_t\rho_0 - \mathrm{Div}(\boldsymbol{R}) - \mathcal{R}_0 \\ \mathbf{R}^\varphi &= \rho_0\,\mathrm{D}_t\boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 \end{aligned}$$

… to residual format

residual equations…

$$\begin{aligned} \mathsf{R}^\rho &= \mathrm{D}_t\rho_0 - \mathrm{Div}(\boldsymbol{R}) - \mathcal{R}_0 = 0 \quad \text{in } \mathcal{B}_0 \quad \partial\mathcal{B}_0 = \partial\mathcal{B}_0^\rho \cup \partial\mathcal{B}_0^{T^\rho} \\ \mathbf{R}^\varphi &= \rho_0\,\mathrm{D}_t\boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 = \mathbf{0} \quad \text{in } \mathcal{B}_0 \quad \partial\mathcal{B}_0 = \partial\mathcal{B}_0^\varphi \cup \partial\mathcal{B}_0^{T^\varphi} \end{aligned}$$

- dirichlet / essential boundary conditions

$$\begin{aligned} \rho_0 - \bar{\rho}_0 &= 0 \quad \text{on } \partial\mathcal{B}_0^\rho \\ \boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}} &= \mathbf{0} \quad \text{on } \partial\mathcal{B}_0^\varphi \end{aligned}$$

- neumann / natural boundary conditions

$$\begin{aligned} \boldsymbol{R}\cdot\boldsymbol{N} - \bar{T}^\rho &= 0 \quad \text{on } \partial\mathcal{B}_0^{T^\rho} \\ \boldsymbol{P}\cdot\boldsymbol{N} - \bar{\boldsymbol{T}}^\varphi &= \mathbf{0} \quad \text{on } \partial\mathcal{B}_0^{T^\varphi} \end{aligned}$$

… and boundary conditions

from strong form…

$$\begin{aligned} \mathsf{R}^\rho &= \mathrm{D}_t\rho_0 - \mathrm{Div}(\boldsymbol{R}) - \mathcal{R}_0 = 0 \\ \mathbf{R}^\varphi &= \rho_0\,\mathrm{D}_t\boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 = \mathbf{0} \end{aligned}$$

- weak form

$$\begin{aligned} \mathsf{G}^\rho(\delta\rho;\rho_0,\boldsymbol{\varphi}) &= 0 \quad \forall\delta\rho \text{ in } \mathcal{H}_1^0(\mathcal{B}_0) \\ \mathsf{G}^\varphi(\delta\boldsymbol{\varphi};\rho_0,\boldsymbol{\varphi}) &= 0 \quad \forall\delta\boldsymbol{\varphi} \text{ in } \mathcal{H}_1^0(\mathcal{B}_0) \end{aligned}$$

- weak form expressions

no derivatives — second derivatives

$$\begin{aligned} \mathsf{G}^\rho &= \int_{\mathcal{B}_0}\delta\rho\,\mathrm{D}_t\rho_0\,\mathrm{d}V - \int_{\mathcal{B}_0}\delta\rho\,\mathrm{Div}(\boldsymbol{R})\,\mathrm{d}V - \int_{\mathcal{B}_0}\delta\rho\,\mathcal{R}_0\,\mathrm{d}V \\ \mathsf{G}^\varphi &= \int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\rho_0\mathrm{D}_t\boldsymbol{v}\,\mathrm{d}V - \int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\mathrm{Div}(\boldsymbol{P})\,\mathrm{d}V - \int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\boldsymbol{b}_0\,\mathrm{d}V \end{aligned}$$

… to nonsymmetric weak form

from nonsymmetric weak form…

- integration by parts

$$\int_{\mathcal{B}_0} \delta\rho \,\mathrm{Div}(\boldsymbol{R})\mathrm{d}V = \int_{\mathcal{B}_0} \mathrm{Div}(\delta\rho\boldsymbol{R})\ \mathrm{d}V - \int_{\mathcal{B}_0} \nabla\delta\rho \cdot \boldsymbol{R}\mathrm{d}V$$
$$\int_{\mathcal{B}_0} \delta\boldsymbol{\varphi}\cdot\mathrm{Div}(\boldsymbol{P})\mathrm{d}V = \int_{\mathcal{B}_0} \mathrm{Div}(\delta\boldsymbol{\varphi}\cdot\boldsymbol{P})\mathrm{d}V - \int_{\mathcal{B}_0} \nabla\delta\boldsymbol{\varphi} : \boldsymbol{P}\mathrm{d}V$$

- gauss theorem & boundary conditions

$$\int_{\mathcal{B}_0} \mathrm{Div}(\delta\rho\boldsymbol{R})\ \mathrm{d}V = \int_{\partial\mathcal{B}_0^{T^\rho}} \delta\rho\ \boldsymbol{R}\cdot\boldsymbol{N}\mathrm{d}A = \int_{\partial\mathcal{B}_0^{T^\rho}} \delta\rho\,\bar{T}^\rho\,\mathrm{d}A$$
$$\int_{\mathcal{B}_0} \mathrm{Div}(\delta\boldsymbol{\varphi}\cdot\boldsymbol{P})\mathrm{d}V = \int_{\partial\mathcal{B}_0^{T^\varphi}} \boldsymbol{\varphi}\cdot\boldsymbol{P}\cdot\boldsymbol{N}\mathrm{d}A = \int_{\partial\mathcal{B}_0^{T^\varphi}} \delta\boldsymbol{\varphi}\cdot\bar{\boldsymbol{T}}^\varphi\mathrm{d}A$$

- weak form

first derivatives first derivatives

$$\mathsf{G}^\rho = \int_{\mathcal{B}_0} \delta\rho\ \ \mathrm{D}_t\rho_0\mathrm{d}V + \int_{\mathcal{B}_0} \nabla\delta\rho\cdot\boldsymbol{R}\mathrm{d}V - \int_{\partial\mathcal{B}_0^{T^\rho}} \delta\rho\,\bar{T}^\rho\mathrm{d}A - \int_{\mathcal{B}_0} \delta\rho\,\mathcal{R}_0\,\mathrm{d}V$$
$$\mathsf{G}^\varphi = \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi}\cdot\rho_0\mathrm{D}_t\boldsymbol{v}\ \mathrm{d}V + \int_{\mathcal{B}_0} \nabla\delta\boldsymbol{\varphi} : \boldsymbol{P}\mathrm{d}V - \int_{\partial\mathcal{B}_0^{T^\varphi}} \delta\boldsymbol{\varphi}\cdot\bar{\boldsymbol{T}}^\varphi\mathrm{d}A - \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi}\cdot\boldsymbol{b}_0\,\mathrm{d}V$$

… to symmetric weak form

temporal discretization

- discretization $\mathcal{T} = \bigcup_{n=0}^{n_{\mathrm{step}}-1} [t_n, t_{n+1}]$ $\quad \Delta t = t_{n+1} - t_n$
- time discrete weak form

$$\mathsf{G}^\rho(\delta\rho; \rho_{0\,n+1}, \boldsymbol{\varphi}_{n+1}) = 0 \quad \forall\delta\rho \text{ in } \mathcal{H}_1^0(\mathcal{B}_0)$$
$$\mathsf{G}^\varphi(\delta\boldsymbol{\varphi}; \rho_{0\,n+1}, \boldsymbol{\varphi}_{n+1}) = 0 \quad \forall\delta\boldsymbol{\varphi} \text{ in } \mathcal{H}_1^0(\mathcal{B}_0)$$

- interpolation of material time derivatives euler backward

$$\mathrm{D}_t\rho_0 = \tfrac{1}{\Delta t}[\rho_{0\,n+1} - \rho_{0\,n}]$$
$$\mathrm{D}_t\boldsymbol{v} = \tfrac{1}{\Delta t}[\boldsymbol{v}_{n+1} - \boldsymbol{v}_n]$$

… to semidiscrete weak form

spatial discretization

- discretization $\mathcal{B}_0 = \bigcup_{e=1}^{n_{\mathrm{el}}} \mathcal{B}_0^e$
- interpolation of test functions

$$\delta\rho_0^h|_{\mathcal{B}_0^e} = \sum_{i=1}^{n_{\mathrm{en}}} N_\rho^i\,\delta\rho_i \in \mathcal{H}_1^0(\mathcal{B}_0) \qquad \nabla\delta\rho_0^h|_{\mathcal{B}_0^e} = \sum_{i=1}^{n_{\mathrm{en}}} \delta\rho_i\,\nabla N_\rho^i$$
$$\delta\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \sum_{j=1}^{n_{\mathrm{en}}} N_\varphi^j\delta\boldsymbol{\varphi}_j \in \mathcal{H}_1^0(\mathcal{B}_0) \qquad \nabla\delta\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \sum_{j=1}^{n_{\mathrm{en}}} \delta\boldsymbol{\varphi}_j \otimes \nabla N_\varphi^j$$

- interpolation of trial functions

$$\rho_0^h|_{\mathcal{B}_0^e} = \sum_{k=1}^{n_{\mathrm{en}}} N_\rho^k\,\rho_k \in \mathcal{H}_1(\mathcal{B}_0) \qquad \nabla\rho_0^h|_{\mathcal{B}_0^e} = \sum_{k=1}^{n_{\mathrm{en}}} \rho_k\,\nabla N_\rho^k$$
$$\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \sum_{l=1}^{n_{\mathrm{en}}} N_\varphi^l\,\boldsymbol{\varphi}_l \in \mathcal{H}_1(\mathcal{B}_0) \qquad \nabla\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \sum_{l=1}^{n_{\mathrm{en}}} \boldsymbol{\varphi}_l \otimes \nabla N_\varphi^l$$

… to discrete weak form

from discrete weak form…

- discrete residual format

$$\mathsf{R}_I^\rho(\rho_{0\,n+1}^h, \boldsymbol{\varphi}_{n+1}^h) = 0 \qquad \forall\, I = 1, \ldots, n_{\mathrm{np}}$$
$$\mathbf{R}_J^\varphi(\rho_{0\,n+1}^h, \boldsymbol{\varphi}_{n+1}^h) = \mathbf{0} \qquad \forall\, J = 1, \ldots, n_{\mathrm{np}}$$

- discrete residuals

$$\mathsf{R}_I^\rho = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} N_\rho^i \tfrac{\rho_{0\,n+1} - \rho_{0\,n}}{\Delta t}\mathrm{d}V + \int_{\mathcal{B}_0^e} \nabla N_\rho^i \cdot \boldsymbol{R}_{n+1}\mathrm{d}V - \int_{\partial\mathcal{B}_0^e} N_\rho^i\,\bar{T}_{n+1}^\rho\mathrm{d}A - \int_{\mathcal{B}_0^e} N_\rho^i\mathcal{R}_{0\,n+1}\mathrm{d}V$$
$$\mathbf{R}_J^\varphi = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} N_\varphi^j\rho_{0\,n+1}\tfrac{\boldsymbol{v}_{0\,n+1} - \boldsymbol{v}_{0\,n}}{\Delta t}\mathrm{d}V + \int_{\mathcal{B}_0^e} \nabla N_\varphi^j \cdot \boldsymbol{P}_{n+1}\mathrm{d}V - \int_{\partial\mathcal{B}_0^e} N_\varphi^j\,\bar{\boldsymbol{T}}_{n+1}^\varphi\mathrm{d}A - \int_{\mathcal{B}_0^e} N_\varphi^j\boldsymbol{b}_{0\,n+1}\mathrm{d}V$$

… to discrete residuals

from discreate weak form…

- linearization / newton raphson scheme

$$\mathsf{R}^{\rho k+1}_{In+1} = \mathsf{R}^{\rho k}_{In+1} + \mathrm{d}\mathsf{R}^{\rho}_{I} \doteq 0 \quad \forall\, I = 1, \ldots, n_{\mathrm{np}}$$

$$\mathbf{R}^{\varphi k+1}_{Jn+1} = \mathbf{R}^{\varphi k}_{Jn+1} + \mathrm{d}\mathbf{R}^{\varphi}_{J} \doteq 0 \quad \forall\, J = 1, \ldots, n_{\mathrm{np}}$$

- incremental residual

$$\mathrm{d}\mathsf{R}^{\rho}_{I} = \sum_{K=1}^{n_{\mathrm{np}}} \mathsf{K}^{\rho\rho}_{IK}\mathrm{d}\rho_K + \sum_{L=1}^{n_{\mathrm{np}}} \mathbf{K}^{\rho\varphi}_{IL} \cdot \mathrm{d}\boldsymbol{\varphi}_L$$

$$\mathrm{d}\mathbf{R}^{\varphi}_{J} = \sum_{K=1}^{n_{\mathrm{en}}} \mathbf{K}^{\varphi\rho}_{JK}\mathrm{d}\rho_K + \sum_{L=1}^{n_{\mathrm{en}}} \mathbf{K}^{\varphi\varphi}_{JL} \cdot \mathrm{d}\boldsymbol{\varphi}_L$$

- system of equations

$$\begin{bmatrix} \mathsf{K}^{IK}_{\rho\rho} & \mathbf{K}^{IL}_{\rho\varphi} \\ \mathbf{K}^{\varphi\rho}_{JK} & \mathbf{K}^{\varphi\varphi}_{JL} \end{bmatrix} \begin{bmatrix} \mathrm{d}\rho_K \\ \mathrm{d}\boldsymbol{\varphi}_L \end{bmatrix} = - \begin{bmatrix} \mathsf{R}^{\rho k}_{I\,n+1} \\ \mathbf{R}^{\varphi k}_{J\,n+1} \end{bmatrix}$$

… to linearized weak form

linearization

- iteration matrices

$$\mathsf{K}^{\rho\rho}_{IK} = \frac{\partial \mathsf{R}^{\rho}_{I}}{\partial \rho_K} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}^e_0} N^i_\rho \frac{1}{\Delta t} N^k_\rho - N^i_\rho \mathrm{D}_{\rho_0} \mathcal{R} N^k_\rho \mathrm{d}V + \int_{\mathcal{B}^e_0} \nabla N^i_\rho \cdot \mathrm{D}_{\nabla\rho_0} \boldsymbol{R} \cdot \nabla N^k_\rho \mathrm{d}V$$

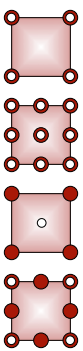
$$\mathbf{K}^{\rho\varphi}_{IL} = \frac{\partial \mathsf{R}^{\rho}_{I}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}^e_0} - N^i_\rho \mathrm{D}_F \mathcal{R}_0 \cdot \nabla N^l_\varphi \mathrm{d}V$$

$$\mathbf{K}^{\varphi\rho}_{JK} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \rho_K} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}^e_0} \nabla N^j_\varphi \cdot \mathrm{D}_{\rho_0} \boldsymbol{P} N^k_\rho \mathrm{d}V$$

$$\mathbf{K}^{\varphi\varphi}_{JL} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}^e_0} N^j_\varphi \rho \tfrac{1}{\Delta t} \boldsymbol{I} N^l_\varphi + \nabla N^j_\varphi \cdot \mathrm{D}_F \boldsymbol{P} \cdot \nabla N^l_\varphi \mathrm{d}V$$

… to linearized weak form

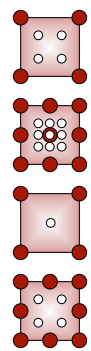
node point based



- loop over all time steps
  - global newton iteration
    - loop over all elements
      - loop over all quadrature points
        - evaulate balance of mass and momentum
      - determine element residuals & partial derivatives
    - determine global residuals and iterational matrices
  - determine $\rho_{0\,n+1}$ and $\boldsymbol{\varphi}_{n+1}$
- determine state of biological equilibrium

density ρ_0 as nodal degree of freedom

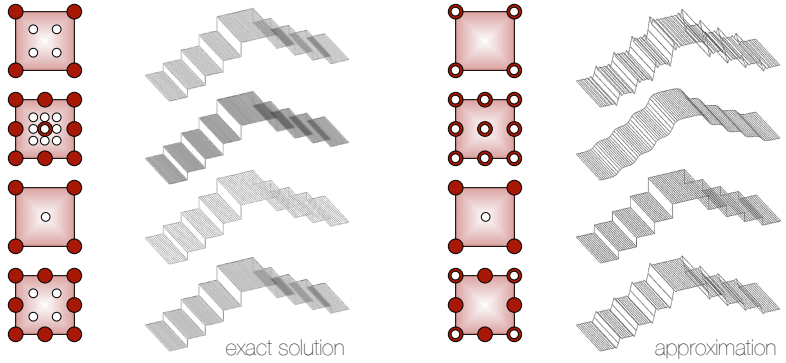
integration point based



- loop over all time steps
  - global newton iteration
    - loop over all elements
      - loop over all quadrature points
        - local newton iteration to determine $\rho_{0\,n+1}$
      - determine element residual & partial derivative
    - determine global residual and iterational matrix
  - determine $\boldsymbol{\varphi}_{n+1}$
- determine state of biological equilibrium

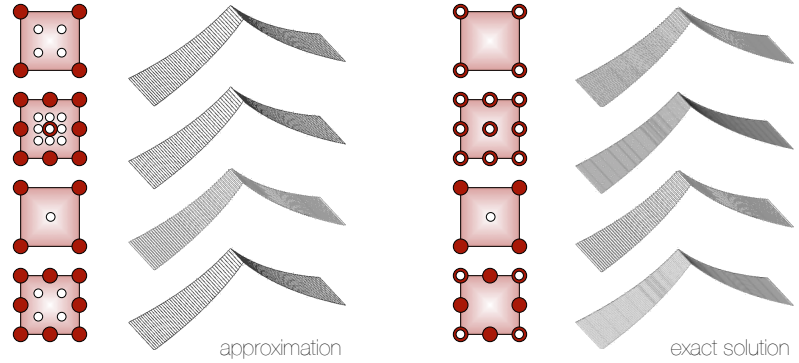
density ρ_0 as internal variable

integration vs node point based



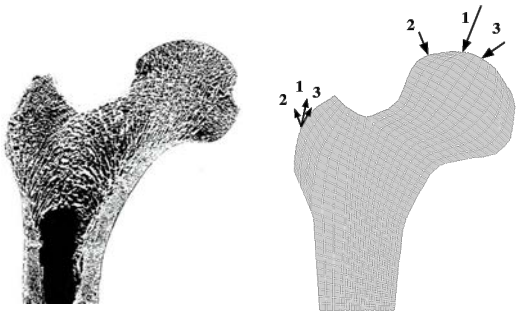
discontinuous model problem

integration vs node point based



continuous model problem

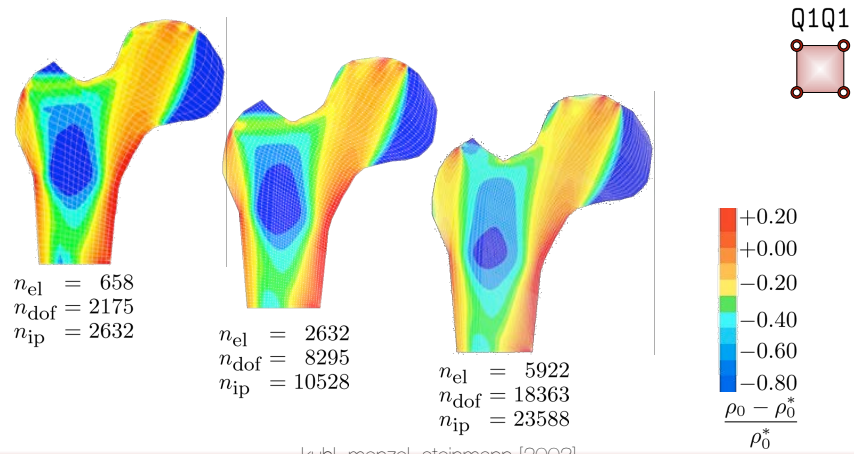
different load cases



[1]	midstance phase of gait	2317 N	24°	703 N	28°
[2]	extreme range of abduction	1158 N	-15°	351 N	-8 °
[3]	extreme range of adduction	1548 N	56°	468 N	35°

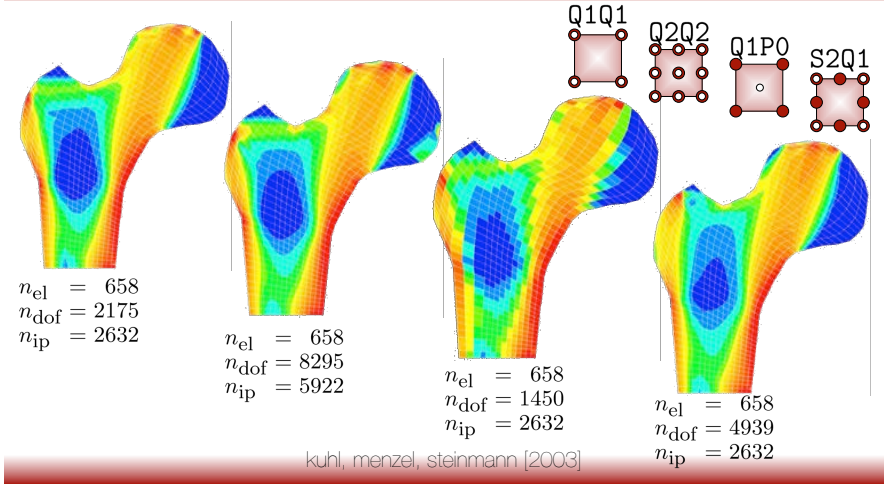
carter & beaupré [2001]

node point based - h-refinement



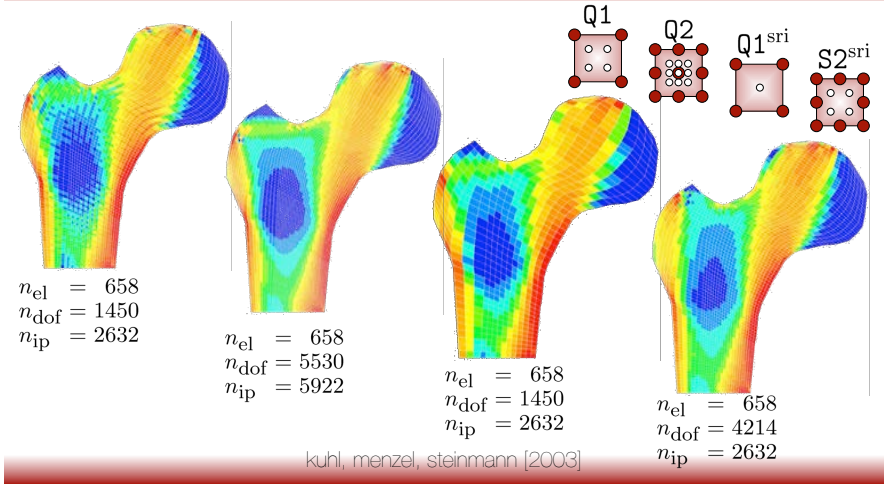
kuhl, menzel, steinmann [2003]

node point based - p-refinement



kuhl, menzel, steinmann [2003]

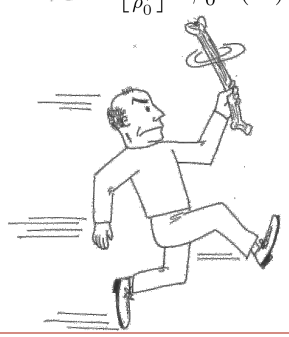
integration point based - p-refinement

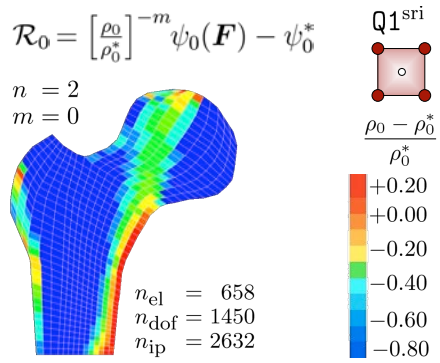


kuhl, menzel, steinmann [2003]

parameter sensitivity - instabilities

$$\psi_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^n \psi_0^{\text{neo}}(\boldsymbol{F}) \qquad \mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m} \psi_0(\boldsymbol{F}) - \psi_0^*$$





certain parameters induce checkerboard modes

harrigon & hamilton [1992],[1994]

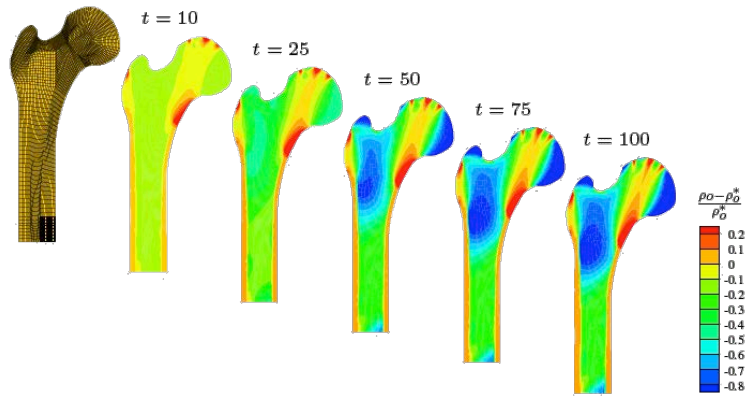
total hip replacement vs hip resurfacing

- about 120,000 artificial hip replacements in us per year
- aseptic loosening caused by adaptive bone remodeling
- goal prediction of dredisctribution of bone density

total hip replacement

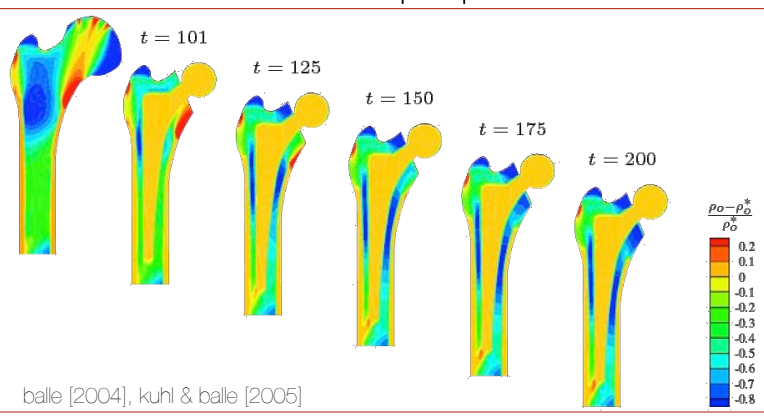
total hip replacement is a surgical procedure in which the hip joint is replaced by a prostetic implant. a total hip replacement consists of replacing both the acetabulum and the femoral head. hip replacement is currently the most successful and reliable orthopaedic operation. risks and complications include aseptic loosening, dislocation, and pain.in the long term, many problems relate to **bone resorption and subsequent loosening** or fracture often requiring revision surgery.

convential total hip replacement



ward 's triangle • trabeculae • dense cortical shaft

convential total hip replacement



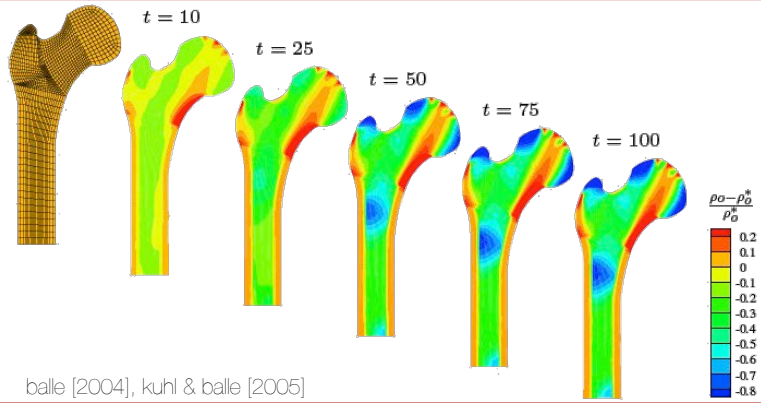
balle [2004], kuhl & balle [2005]

stress shielding • bone resorption • implant loosening

hip resurfacing

hip resurfacing is a surgical procedure which has been developed as an intervention alternative to total hip replacement. the potential advantages of hip resurfacing include **less bone removal**, a potentially lower number of hip dislocations due to a relatively larger femoral head size, and possibly easier revision surgery for a subsequent total hip replacement device. the potential disadvantages are femoral neck fractures, aspectic loosening, and metal wear.

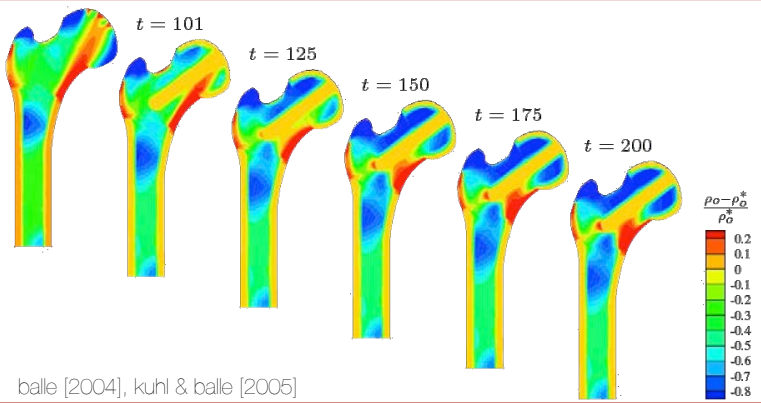
new birmingham hip resurfacing



balle [2004], kuhl & balle [2005]

ward ‘s triangle • trabeculae • dense cortical shaft

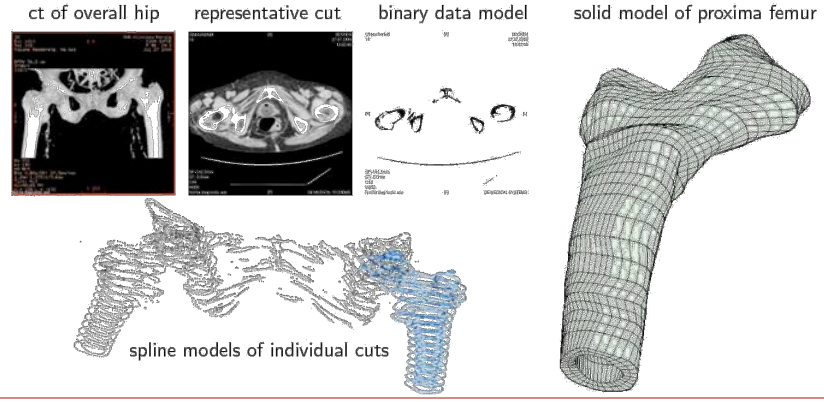
new birmingham hip resurfacing



balle [2004], kuhl & balle [2005]

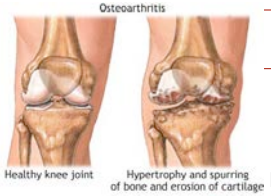
improved ingrowth • anatomic situation • less resorption

computer tomography of human femur



patient specific medical treatment

osteoarthritis



osteoarthritis, also known as degenerative joint disease or osteoarthrosis, is a group of mechanical abnormalities involving degradation of joints, including articular cartilage and subchondral bone. it affects about 27 million people in the united states alone. symptoms may include joint pain, tenderness, stiffness, locking, and sometimes an effusion. a variety of causes, hereditary, developmental, metabolic, and mechanical, may initiate processes leading to loss of cartilage. when bone surfaces become less well protected by cartilage, bone may be exposed and damaged. treatment generally involves a combination of exercise, lifestyle modification, or, in severe cases,surgical joint replacement.

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anisotropy of bone mineral density

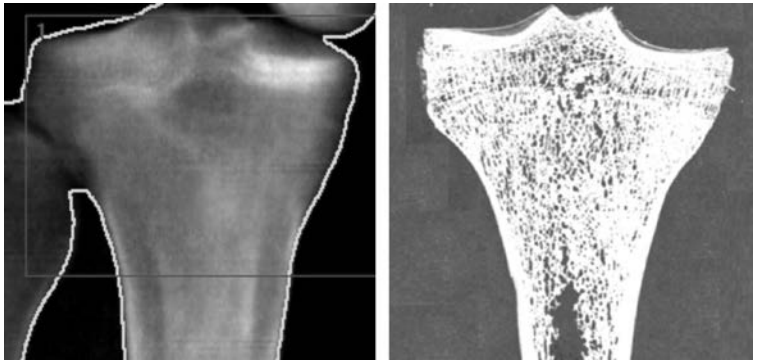


figure 1. healthy proxima tibia. isometric desity distribution, left, visualized through DEXA scan, which displays the characteristic heterogeneous density distriubtion with a dense region in the medial plateau, a lower density region in the lateral plateau, and the lowest density in the central region, pang et al. [2012]. anisotropic density distriubtion, right, visualized through a thin section. Trabeculae are aligned with axis of maximum principal loading, wolff [1870].

anisotropy of bone mineral density

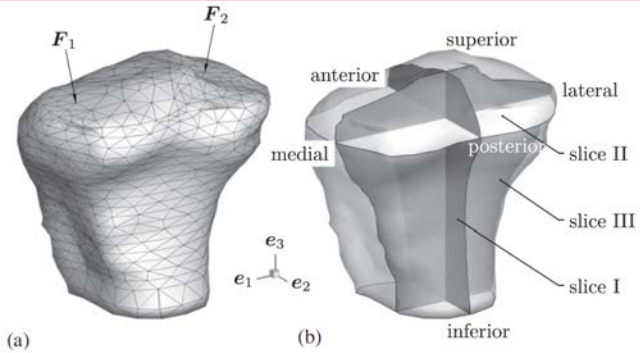
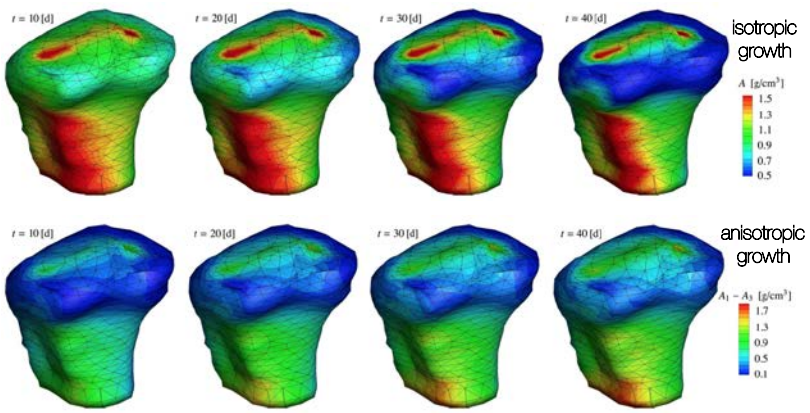


Fig. 6. Geometry of the upper part of the proximal tibia. (a) Finite element mesh with applied concentrated forces F_1, F_2 and (b) location of slices I (sagittal slice), II (axial slice) and III (coronal slice) used in Figs. 9–13.

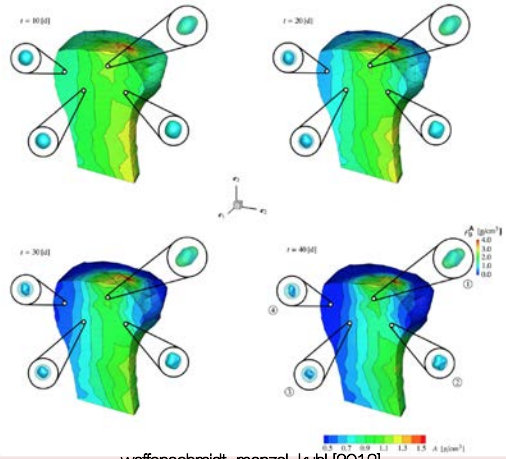
waffenschmidt, menzel, kuhl [2012]

anisotropy of bone mineral density



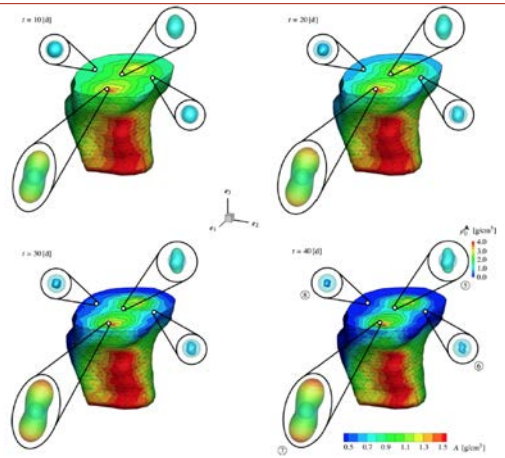
waffenschmidt, menzel, kuhl [2012]

anisotropy of bone mineral density



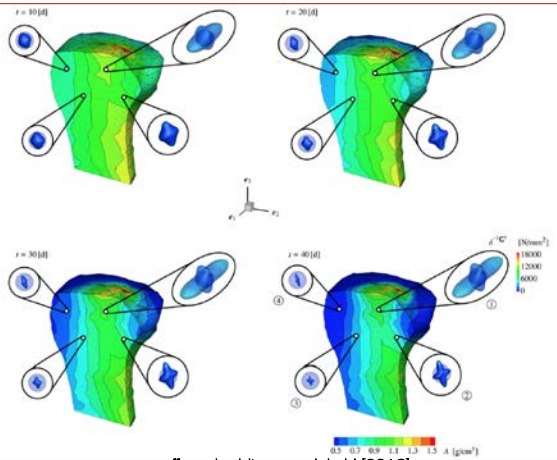
waffenschmidt, menzel, kuhl [2012]

anisotropy of bone mineral density



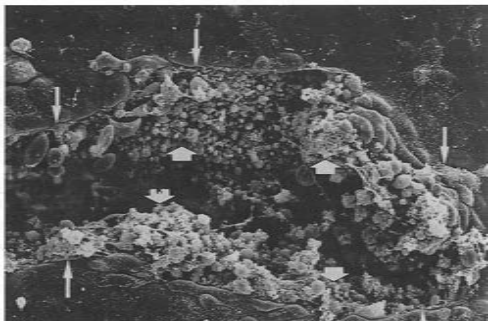
waffenschmidt, menzel, kuhl [2012]

anisotropy of bone mineral density



waffenschmidt, menzel, kuhl [2012]

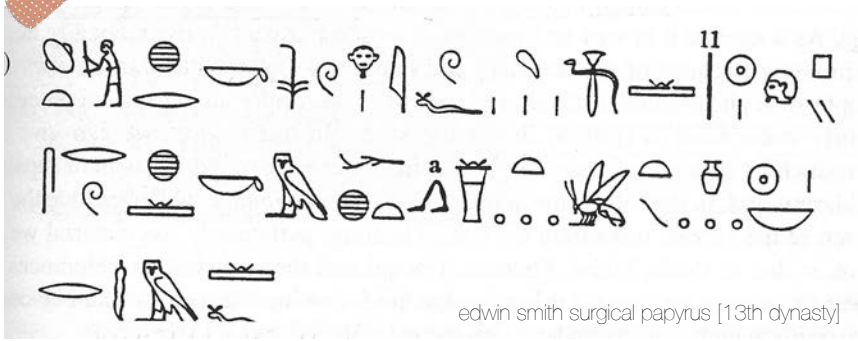
wound healing



murray [2003]

- epidermal migration / spreading of existing cells R
- increase of mitotic activity of about15 times in 1mm wide zone @wound edge $\mathcal{R}_0$

wound healing

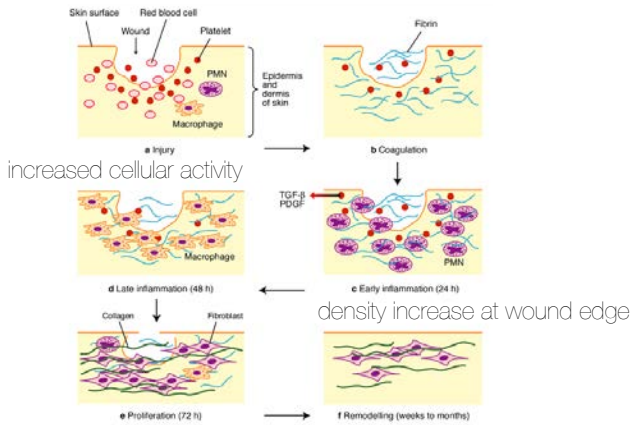


edwin smith surgical papyrus [13th dynasty]

…thou shouldst bind it with fresh meet the first day, and thou shouldst treat afterword with grease and honey every day until he recovers…

breadsted [1930]

wound healing

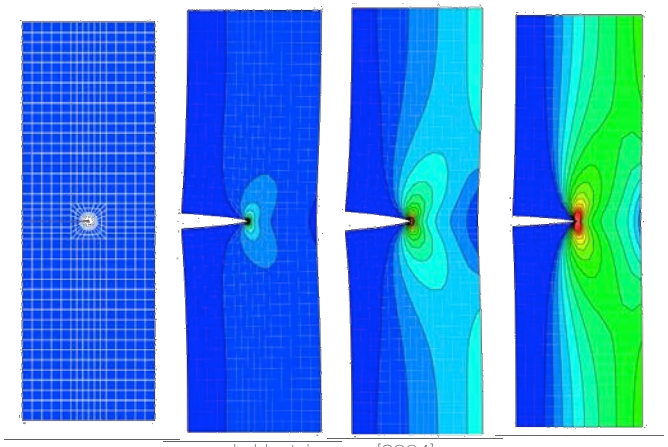


increased cellular activity

density increase at wound edge

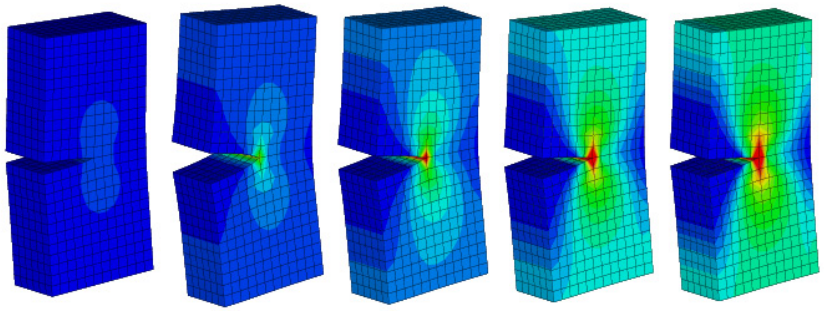
the phases of cutaneous wound healing. beanes, dang, soo, ting [2003]

tension - single edge notched specimen



kuhl, steinmann [2004]

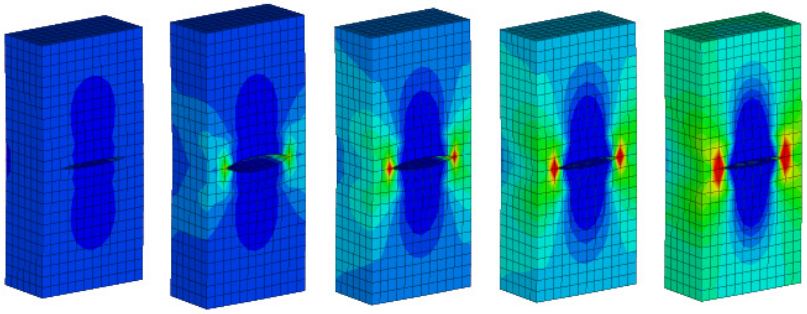
tension - single edge notched specimen



increased cell activity @wound edge

kuhl, steinmann [2004]

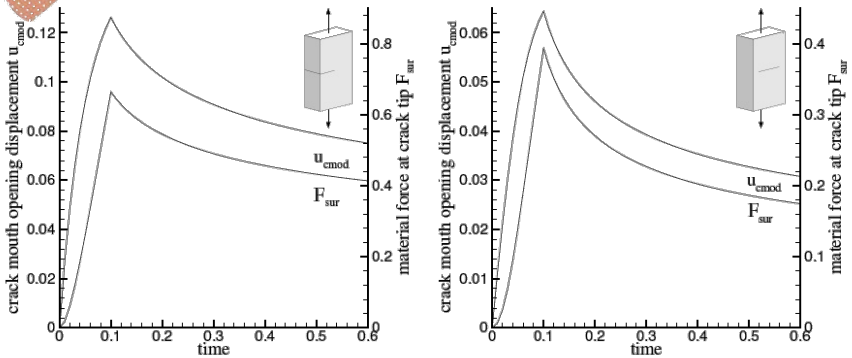
tension - surface cut specimen



increased cell activity @wound edge

kuhl, steinmann [2004]

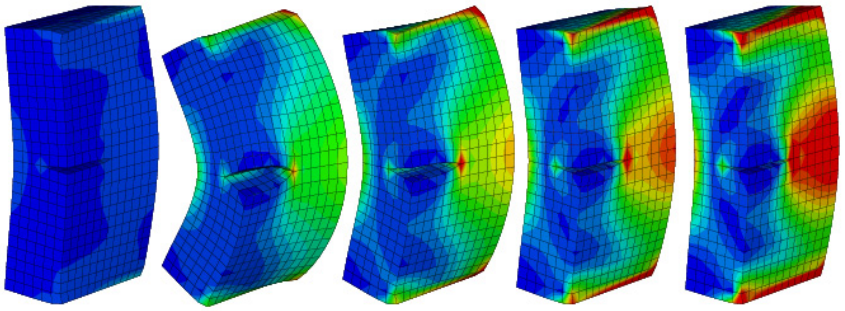
tension - crack mouth opening displacement



increased cell activity - wound healing & closure

kuhl, steinmann [2004]

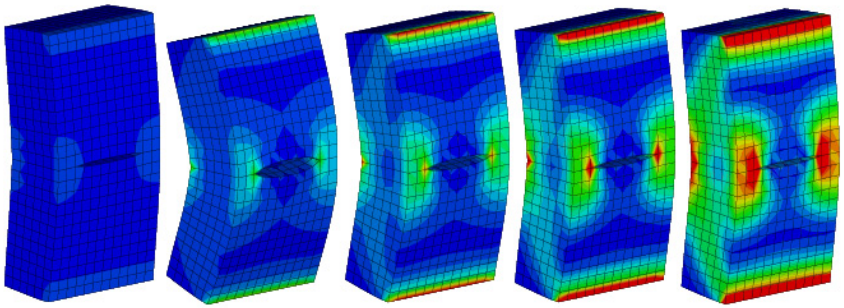
bending - single edge notched specimen



increased cell activity @wound edge

kuhl, steinmann [2004]

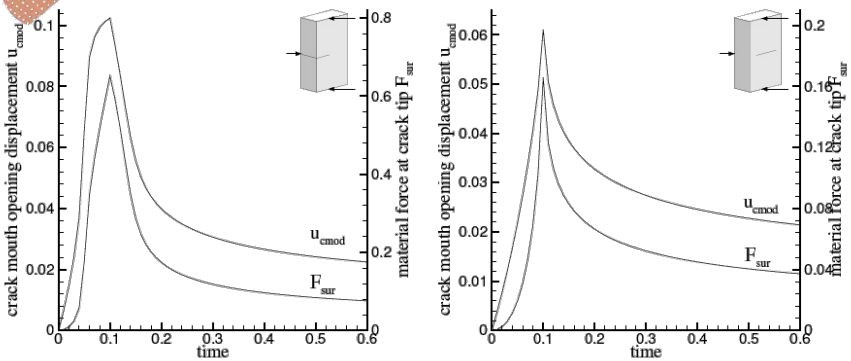
bending - surface cut specimen



increased cell activity @wound edge

kuhl, steinmann [2004]

bending - crack mouth opening displacement



increased cell activity - wound healing & closure

kuhl, steinmann [2004]