# 04 - kinematic equations - kinematics of growth




| day | date | | topic |
|---|---|---|---|
| tue | jan | 08 | motivation - everything grows! |
| thu | jan | 10 | basics maths - notation and tensors |
| tue | jan | 15 | basic kinematics - large deformation and growth |
| thu | jan | 17 | kinematics - growing hearts |
| tue | jan | 22 | guest lecture - growing surfaces |
| thu | jan | 24 | kinematics - growing leaflets |
| tue | jan | 29 | basic balance equations - closed and open systems |
| thu | jan | 31 | basic constitutive equations - growing tumors |
| tue | feb | 05 | volume growth - finite elements for growth |
| thu | feb | 07 | volume growth - growing arteries |
| tue | feb | 12 | volume growth - growing skin |
| thu | feb | 14 | volume growth - growing hearts |
| tue | feb | 19 | basic constitutive equations - growing bones |
| thu | feb | 21 | density growth - finite elements for growth |
| tue | feb | 26 | density growth - growing bones |
| thu | feb | 28 | everything grows! - midterm summary |
| tue | mar | 05 | midterm |
| thu | mar | 07 | remodeling - remodeling arteries and tendons |
| tue | mar | 12 | class project - discussion, presentation, evaluation |
| thu | mar | 14 | class project - discussion, presentation, evaluation |
| thu | mar | 14 | written part of final projects due |



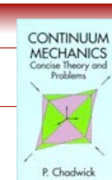
## kinematic equations

**kinematic equations** [kɪnəˈmætɪk ɪˈkweɪ.ʒəns] describe the motion of objects without the consideration of the masses or forces that bring about the motion. the basis of kinematics is the choice of coordinates. the 1st and 2nd time derivatives of the position coordinates give the velocities and accelerations. the difference in placement between the beginning and the final state of two points in a body expresses the numerical value of strain. strain expresses itself as a change in size and/or shape.

WIKIPEDIA The Free Encyclopedia



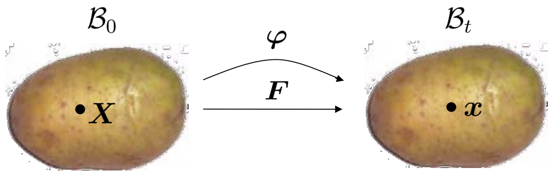
## kinematic equations




**kinematics** [kɪnəˈmætɪks] is the study of motion per se, regardless of the forces causing it. the primitive concepts concerned are position, time and body, the latter abstracting into mathematical terms intuitive ideas about aggregations of matter capable of motion and deformation.

chadwick "continuum mechanics" [1976]

## potato - kinematics




- nonlinear deformation map $\boldsymbol{\varphi}$

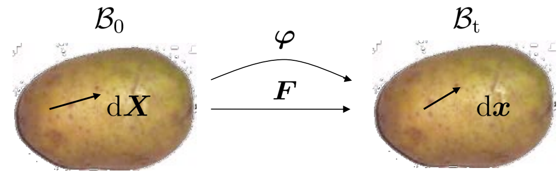
$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t) \quad \text{with} \quad \boldsymbol{\varphi} : \mathcal{B}_0 \times \mathbb{R} \rightarrow \mathcal{B}_t$$

- spatial derivative of $\boldsymbol{\varphi}$ - deformation gradient

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} \quad \text{with} \quad \boldsymbol{F} : T\mathcal{B}_0 \rightarrow T\mathcal{B}_t \qquad \boldsymbol{F} = \left.\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}}\right|_{t\ \text{fixed}}$$



## potato - kinematics




- transformation of line elements - deformation gradient $F_{ij}$

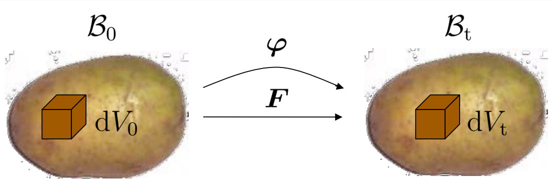
$$\mathrm{d}x_i = F_{ij}\, \mathrm{d}X_j \quad \text{with} \quad F_{ij} : T\mathcal{B}_0 \rightarrow T\mathcal{B}_t \qquad F_{ij} = \left.\frac{\partial \varphi_i}{\partial X_j}\right|_{t\ \text{fixed}}$$

- uniaxial tension (incompressible), simple shear, rotation

$$F_{ij}^{\text{uni}} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha^{-\frac{1}{2}} \end{bmatrix} \quad F_{ij}^{\text{shr}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F_{ij}^{\text{rot}} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



## potato - kinematics




- transformation of volume elements - determinant of $\boldsymbol{F}$

$$\mathrm{d}V_0 = \mathrm{d}\boldsymbol{X}_1 \cdot [\mathrm{d}\boldsymbol{X}_2 \times \mathrm{d}\boldsymbol{X}_3] \qquad \mathrm{d}V_t = \mathrm{d}\boldsymbol{x}_1 \cdot [\mathrm{d}\boldsymbol{x}_2 \times \mathrm{d}\boldsymbol{x}_3]$$

$$= \det([\mathrm{d}\boldsymbol{x}_1, \mathrm{d}\boldsymbol{x}_2, \mathrm{d}\boldsymbol{x}_3])$$

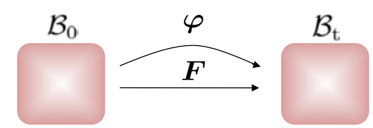
$$= \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3]) \qquad = \det(\boldsymbol{F})\ \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3])$$

- changes in volume - determinant of deformation tensor $J$

$$\mathrm{d}V_t = J\, \mathrm{d}V_0 \qquad J = \det(\boldsymbol{F})$$

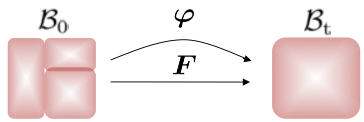


## kinematics of finite growth




[1] consider an elastic body $\mathcal{B}_0$ at time $t_0$, unloaded & stressfree

## kinematics of finite growth

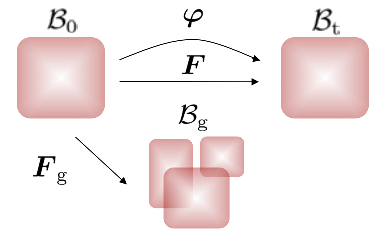



[1] consider an elastic body $\mathcal{B}_0$ at time $t_0$, unloaded & stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth



## kinematics of finite growth

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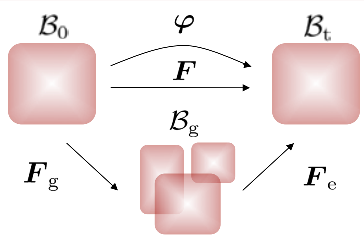
[3] after growing the elements, $\mathcal{B}_g$ may be incompatible



## kinematics of finite growth




[1] consider an elastic body $\mathcal{B}_0$ at time $t_0$, unloaded & stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

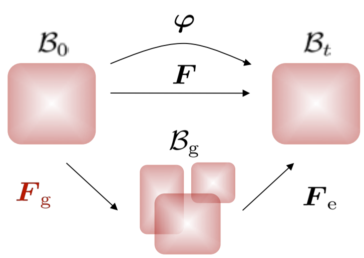
[4] loading generates compatible current configuration $\mathcal{B}_t$



## kinematics of finite growth

$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_g$$

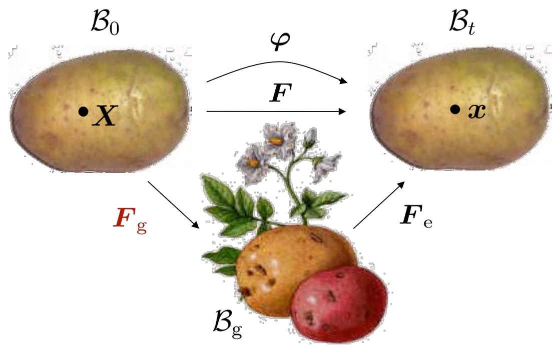
growth tensor




## multiplicative decomposition

Lee [1969], Simo [1992], Rodriguez, Hoger & Mc Culloch [1994], Epstein & Maugin [2000], Humphrey [2002], Ambrosi & Mollica [2002], Himpel, Kuhl, Menzel & Steinmann [2005]

## potato - kinematics of finite growth

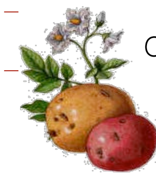



- incompatible growth configuration $\mathcal{B}_{\mathrm{g}}$ & growth tensor $\boldsymbol{F}_{\mathrm{g}}$

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{g}}$$

rodriguez, hoger & mc culloch [1994]



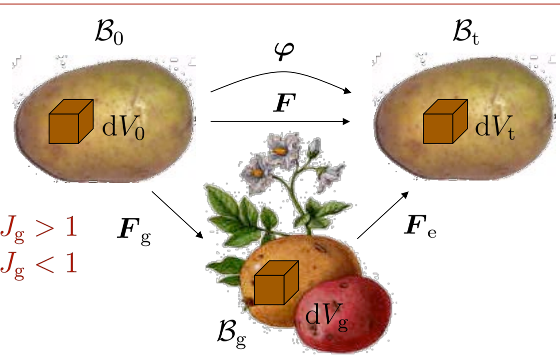


## concept of incompatible growth configuration

biologically, the notion of **incompatibility** implies that subelements of the grown configuration may overlap or have gaps. the implication of incompatibility is the existence of residual stresses necessary to `squeeze` these grown subelements back together. mathematically, the notion of **incompatibility** implies that unlike the deformation gradient, $\boldsymbol{F} = \left.\frac{\partial \varphi}{\partial \boldsymbol{X}}\right|_{t \text{ fixed}}$ the growth tensor cannot be derived as a gradient of a vector field. incompatible configurations are useful in finite strain inelasticity such as viscoelasticity, thermoelasticity, elastoplasticity and growth.



## potato - kinematics of finite growth




growth $J_{\mathrm{g}} > 1$
resorption $J_{\mathrm{g}} < 1$

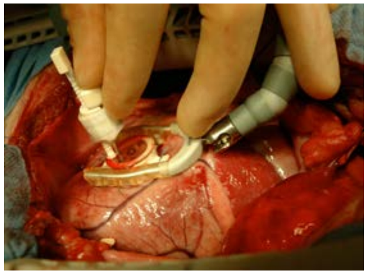
- changes in volume - determinant of growth tensor $J_{\mathrm{g}}$

$$\mathrm{d}V_{\mathrm{g}} = J_{\mathrm{g}}\, \mathrm{d}V_0 \qquad J_{\mathrm{g}} = \det(\boldsymbol{F}_{\mathrm{g}})$$



me338a - continuum mechanics - 2010

## kinematics of cardiac growth



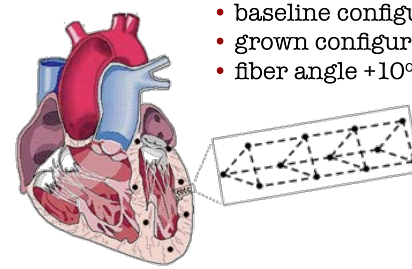
surgically implantation of 4x3 beads across the left ventricular wall

4d coordiantes from in vivo biplane videofluoroscopic marker images

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

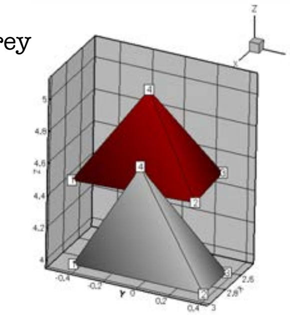


## kinematics of cardiac growth



given: coordinates of

- baseline configuration X, grey
- grown configuration x, red
- fiber angle +10°



X1 = [+2.80; -0.27; +3.75];
X2 = [+2.80; +0.53; +3.75];
X3 = [+2.50; +0.52; +3.75];
X4 = [+2.80; +0.13; +4.45];

x1 = [+2.62; -0.44; +4.34];
x2 = [+2.62; 0.44; +4.34];
x3 = [+2.32; 0.44; +4.35];
x4 = [+2.60; 0.07; +4.85];

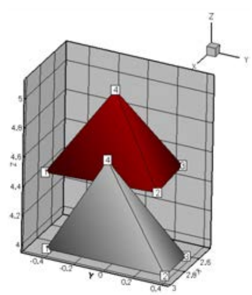


## kinematics of cardiac growth

**[1] Determine three vectors $\mathrm{d}\boldsymbol{X}_i$ that span the tetrahedron at baseline.** Take an arbitrary point of the tetrahedron as origin, e.g., $\boldsymbol{X}_4$, and calculate the three vectors $\mathrm{d}\boldsymbol{X}_1$, $\mathrm{d}\boldsymbol{X}_2$, and $\mathrm{d}\boldsymbol{X}_3$ from the origin to any other point using the coordinates $\boldsymbol{X}$ at baseline such that $\mathrm{d}\boldsymbol{X}_i = \boldsymbol{X}_i - \boldsymbol{X}_4$ for $i = 1,2,3$.

```
matlab
dX1 = X1 - X4
dX2 = X2 - X4
dX3 = X3 - X4
```

dX1 = [+0.00, -0.40, -0.70]
dX2 = [+0.00, +0.40, -0.70]
dX3 = [ -0.30, +0.39, -0.70]



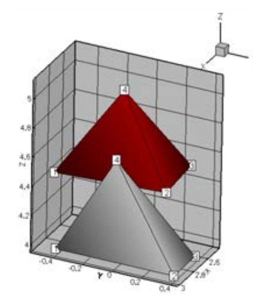


## kinematics of cardiac growth

**[2] Determine the same three vectors $\mathrm{d}\boldsymbol{x}_i$ that span the tetrahedron after growth.** Take the same point as origin, e.g., $\boldsymbol{x}_4$, and calculate the vectors $\mathrm{d}\boldsymbol{x}_1$, $\mathrm{d}\boldsymbol{x}_2$, and $\mathrm{d}\boldsymbol{x}_3$ from the origin to any other point using the coordinates $\boldsymbol{x}$ after growth such that $\mathrm{d}\boldsymbol{x}_i = \boldsymbol{x}_i - \boldsymbol{x}_4$ for $i = 1,2,3$.

```
matlab
dx1 = x1 - x4
dx2 = x2 - x4
dx3 = x3 - x4
```

dx1 = [+0.02, -0.51, -0.51]
dx2 = [+0.02, +0.37, -0.51]
dx3 = [ -0.28, +0.37, -0.50]



## kinematics of cardiac growth

**[3] Determine the growth tensor $\boldsymbol{F}^g$ that maps the baseline line elements $\mathrm{d}\boldsymbol{X}_i$ onto the grown line elements $\mathrm{d}\boldsymbol{x}_i$.**
The growth tensor maps line elements according to $\mathrm{d}\boldsymbol{x}_i = \boldsymbol{F}^g \cdot \mathrm{d}\boldsymbol{X}_i$. The application of this mapping to all three line elements $\mathrm{d}\boldsymbol{X}_i$ defines three vector valued equations, i.e., nine equations to solve for the nine components of $\boldsymbol{F}^g$. To obtain a more compact notation, rearrange all baseline line elements from [1] and all grown line elements from [2] in $3 \times 3$ matrices, i.e., $\boldsymbol{C} := [\ \mathrm{d}\boldsymbol{X}_1;\ \mathrm{d}\boldsymbol{X}_2;\ \mathrm{d}\boldsymbol{X}_3\ ]$ and $\boldsymbol{c} := [\ \mathrm{d}\boldsymbol{x}_1;\ \mathrm{d}\boldsymbol{x}_2;\ \mathrm{d}\boldsymbol{x}_3\ ]$. Now, determine the growth tensor $\boldsymbol{F}^g$ by using the equation $\boldsymbol{F}^g \cdot \boldsymbol{C} = \boldsymbol{c}$, thus $\boldsymbol{F}^g = \boldsymbol{c} \cdot \boldsymbol{C}^{-1}$.

```
matlab
C = [ dX1 dX2 dX3 ];        c = [ dx1 dx2 dx3 ];        F = c/C;
dx1_check = F * dX1;    dx2_check = F * dX2;    dx3_check = F * dX3;
```

$$F = \begin{matrix} +1.0000 & 0.0000 & -0.0286 \\ -0.0367 & +1.1000 & +0.1000 \\ -0.0333 & 0.0000 & +0.7286 \end{matrix}$$

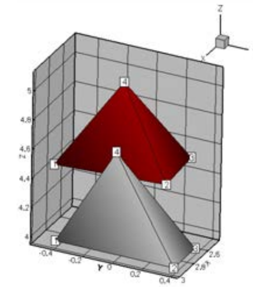


## kinematics of cardiac growth

**[4] Control your results by calculating $\mathrm{d}\boldsymbol{x}_i = \boldsymbol{F}^g \cdot \mathrm{d}\boldsymbol{X}_i$.**
Do the calculated grown line elements $\mathrm{d}\boldsymbol{x}_i$ match the ones you had calculated in [2]?

```
matlab
dx1_check = F * dX1;
dx2_check = F * dX2;
dx3_check = F * dX3;
```

```
dx1_check = [+0.02, -0.51, -0.51]
dx2_check = [+0.02, +0.37, -0.51]
dx3_check = [ -0.28, +0.37, -0.50]
```





## kinematics of cardiac growth

**[5] Determine the grown fiber direction $\boldsymbol{n}^{\mathrm{fib}} = \boldsymbol{F}^g \cdot \boldsymbol{N}^{\mathrm{fib}}$.**
The growth tensor can be used to map the measured baseline fiber direction $\boldsymbol{N}^{\mathrm{fib}}$ onto the grown fiber direction $\boldsymbol{n}^{\mathrm{fib}}$. Determine $\boldsymbol{n}^{\mathrm{fib}}$ and comment on how $\boldsymbol{N}^{\mathrm{fib}}$ and $\boldsymbol{n}^{\mathrm{fib}}$ deviate.

```
matlab
alpha = 10.0;
N_fib = [0.0; -cosd(alpha); sind(alpha)]
n_fib = F * N_fib;
theta = acosd((n_fib'*N_fib)/(norm(n_fib)*norm(N_fib)))
```

```
N_fib = [ 0.0000, -0.9848, +0.1736]
n_fib  = [-0.0050, -1.0659, +0.1265]
theta = 3.2420
```

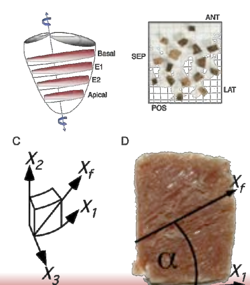


## kinematics of cardiac growth

**[6] Determine the fiber stretch upon growth $\lambda^g = \sqrt{\boldsymbol{n}^{\mathrm{fib}} \cdot \boldsymbol{n}^{\mathrm{fib}}}$.**
Since the fiber orientation $\boldsymbol{N}^{\mathrm{fib}}$ was given as a unit vector, the length of the grown vector $\boldsymbol{n}^{\mathrm{fib}} = \boldsymbol{F} \cdot \boldsymbol{N}^{\mathrm{fib}}$ corresponds to the relative change in fiber length, i.e., the amount of growth along the fiber direction, $\lambda^g = \sqrt{\boldsymbol{n}^{\mathrm{fib}} \cdot \boldsymbol{n}^{\mathrm{fib}}} = \sqrt{\boldsymbol{N}^{\mathrm{fib}} \cdot \boldsymbol{F}^{g\,t} \cdot \boldsymbol{F}^g \cdot \boldsymbol{N}^{\mathrm{fib}}}$.

```
matlab
lambda = sqrt(n_fib'*n_fib)
```

```
lambda = 1.0734
```




ennis, nguyen, riboh, wigström, harri
ton, daughters, ingels, miller [2010]

## kinematics of cardiac growth

**[7] Determine the second order Green Lagrange strain tensor $\boldsymbol{E} = \frac{1}{2}\,[\,\boldsymbol{F}^t \cdot \boldsymbol{F} - \boldsymbol{I}\,]$.**
$\boldsymbol{E}$ is called the Green Lagrange strain tensor and it is used to characterize strains with respect to the reference configuration in a finite strain setting.

**[8] Determine the displacement gradient tensor $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}$.**
$\boldsymbol{H} = \nabla \boldsymbol{u}$ is the nonsymmetric displacement gradient tensor which can also be expressed as $\boldsymbol{H} = \partial \boldsymbol{u} \,/\, \partial \boldsymbol{X} = \partial[\boldsymbol{x} - \boldsymbol{X}] \,/\, \partial \boldsymbol{X} = \boldsymbol{F} - \boldsymbol{I}$.

```
matlab
E = 1/2 * (F'*F - eye(3))
```

```
matlab
H = F - eye(3)
```

```
     0.0012  -0.0202  -0.0283
E=  -0.0202   0.1050   0.0550
    -0.0283   0.0550  -0.2292
```

```
     0.0000   0        -0.0286
H=  -0.0367   0.1000    0.1000
    -0.0333   0        -0.2714
```



## kinematics of cardiac growth

**[9] Determine the small strain tensor $\boldsymbol{\epsilon} = \frac{1}{2}\,(\boldsymbol{H} + \boldsymbol{H}^t)$.**
Compare the small strain approximation $\boldsymbol{\epsilon}$ with the large strain Green Lagrange tensor $\boldsymbol{E}$ and comment on your results.

**[10] Determine the normal strain $\epsilon_n = \boldsymbol{N}^{\text{fib}} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{N}^{\text{fib}}$.**
Compare the small strain approximation of the normal strain $\epsilon_n$ with the large strain fiber stretch $\lambda^g$.

```
matlab
epsilon =  1/2 * (H+H')     eps_n   = N_fib'*epsilon*N_fib;
```

```
            0.0000  -0.0183  -0.0310
epsilon =  -0.0183   0.1000   0.0500     eps_n   = 0.0717
           -0.0310   0.0500  -0.2714
```



## kinematics of cardiac growth

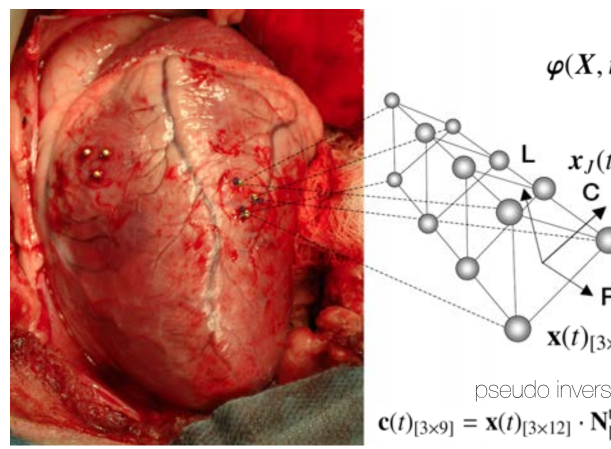
**[11] Determine the volume change $J^g = \det(\boldsymbol{F}^g)$ and compare it with the small strain volume dilation $e = \text{tr}(\boldsymbol{\epsilon})$.**
What does this imply in terms of tissue growth?

```
matlab
J = det(F)
dV = dot(dX3,cross(dX2,dX1))
dv = dot(dx3,cross(dx2,dx1))
J_check = dv / dV;
e = trace(epsilon)
```

```
J = 0.8004
J_check =  0.1345 / 0.1608 = 0.8004
e = -0.1714
```



## kinematics of cardiac growth




deformation

$$\boldsymbol{\varphi}(\boldsymbol{X},t) = \sum_{I=1}^{n_{\text{apx}}} \boldsymbol{c}_I(t)\, N_I(\boldsymbol{X})$$

valid for all data points

$$\boldsymbol{x}_J(t) = \sum_{I=1}^{n_{\text{apx}}} \boldsymbol{c}_I(t)\, N_I(\boldsymbol{X}_J)$$

with coordinates

$$\boldsymbol{x} = [x_c, x_l, x_r]^t$$

system for 12 markers

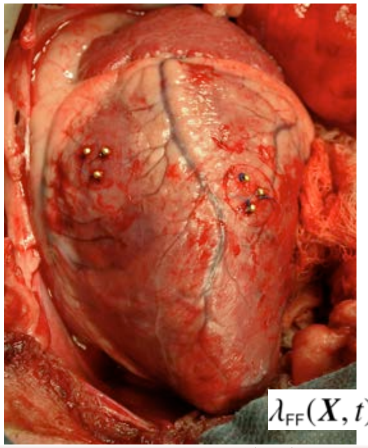
$$\mathbf{x}(t)_{[3\times12]} = \mathbf{c}(t)_{[3\times9]} \cdot \mathbf{N}_{[9\times12]}$$

pseudo inverse to determine coefficients

$$\mathbf{c}(t)_{[3\times9]} = \mathbf{x}(t)_{[3\times12]} \cdot \mathbf{N}^t_{[12\times9]} \cdot [\,\mathbf{N}_{[9\times12]} \cdot \mathbf{N}^t_{[12\times9]}\,]^{-1}$$

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

## kinematics of cardiac growth



deformation

$$\boldsymbol{\varphi}(\boldsymbol{X},t) = \sum_{I=1}^{n_{\mathrm{apx}}} \boldsymbol{c}_I(t)\, N_I(\boldsymbol{X})$$

deformation gradient

$$\boldsymbol{F}(\boldsymbol{X},t) = \sum_{I=1}^{n_{\mathrm{apx}}} \boldsymbol{c}_I(t) \otimes \nabla N_I(\boldsymbol{X})$$

spatial gradient

$$\nabla(\circ) = [\partial_c(\circ), \partial_l(\circ), \partial_r(\circ)]^{\mathrm{t}}$$

volume changes

$$J(\boldsymbol{X},t) = \det(\boldsymbol{F}(\boldsymbol{X},t))$$

fiber stretch

$$\lambda_{\mathrm{FF}}(\boldsymbol{X},t) = [\,\boldsymbol{f}(\boldsymbol{X}) \cdot \boldsymbol{F}^{\mathrm{t}}(\boldsymbol{X},t) \cdot \boldsymbol{F}(\boldsymbol{X},t) \cdot \boldsymbol{f}(\boldsymbol{X})\,]^{1/2}$$

tsamis, cheng, nguyen, langer, miller, kuhl [2012]



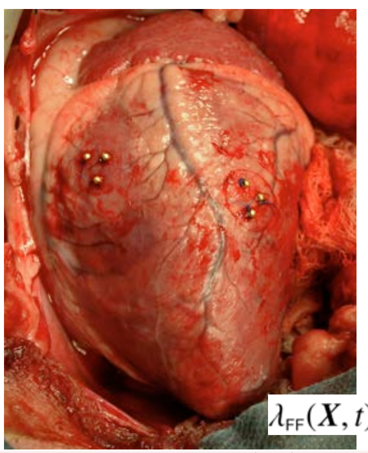
## kinematics of cardiac growth

| | epi 20% depth | p | mid 50% depth | p | endo 80% depth | p |
|---|---|---|---|---|---|---|
| $F^g_{CC}$ | 1.00±0.12 | 0.96 | 1.03±0.14 | 0.46 | 1.02±0.10 | 0.44 |
| $F^g_{CL}$ | 0.04±0.14 | 0.42 | 0.01±0.10 | 0.77 | 0.01±0.09 | 0.61 |
| $F^g_{CR}$ | −0.07±0.29 | 0.46 | −0.03±0.16 | 0.61 | 0.05±0.14 | 0.29 |
| $F^g_{LC}$ | −0.02±0.17 | 0.75 | −0.04±0.13 | 0.33 | −0.04±0.11 | 0.24 |
| $F^g_{LL}$ | 1.10±0.15 | 0.06 | 1.10±0.13 | 0.03 | 1.11±0.11 | 0.01 |
| $F^g_{LR}$ | 0.02±0.16 | 0.71 | 0.10±0.20 | 0.11 | 0.18±0.34 | 0.12 |
| $F^g_{RC}$ | −0.01±0.09 | 0.64 | −0.03±0.17 | 0.54 | −0.05±0.19 | 0.41 |
| $F^g_{RL}$ | 0.00±0.05 | 0.86 | −0.00±0.09 | 0.96 | −0.01±0.11 | 0.67 |
| $F^g_{RR}$ | 0.68±0.15 | 0.00 | 0.73±0.15 | 0.00 | 0.77±0.22 | 0.01 |
| $J^g$ | 0.74±0.19 | 0.00 | 0.82±0.19 | 0.01 | 0.89±0.21 | 0.10 |
| $\lambda^g_{FF}$ | 1.03±0.12 | 0.49 | 1.04±0.16 | 0.36 | 1.08±0.11 | 0.04 |

tsamis, cheng, nguyen, langer, miller, kuhl [2012]



## kinematics of cardiac growth



deformation

$$\boldsymbol{\varphi}(\boldsymbol{X},t) = \sum_{I=1}^{n_{\mathrm{apx}}} \boldsymbol{c}_I(t)\, N_I(\boldsymbol{X})$$

green lagrange strains

$$\boldsymbol{E}(\boldsymbol{X},t) = \frac{1}{2}\,[\,\boldsymbol{F}^{\mathrm{t}} \cdot \boldsymbol{F} - \boldsymbol{I}\,]$$

fiber strain

$$\mathrm{E}_{\mathrm{FF}}(\boldsymbol{X},t) = \boldsymbol{f}(\boldsymbol{X}) \cdot \boldsymbol{E}(\boldsymbol{X},t) \cdot \boldsymbol{f}(\boldsymbol{X})$$

relation of fiber strain to fiber stretch

$$\mathrm{E}_{\mathrm{FF}} = 1/2\ [\ \lambda_{\mathrm{FF}}^2 - 1\ ]$$

fiber stretch

$$\lambda_{\mathrm{FF}}(\boldsymbol{X},t) = [\,\boldsymbol{f}(\boldsymbol{X}) \cdot \boldsymbol{F}^{\mathrm{t}}(\boldsymbol{X},t) \cdot \boldsymbol{F}(\boldsymbol{X},t) \cdot \boldsymbol{f}(\boldsymbol{X})\,]^{1/2}$$

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

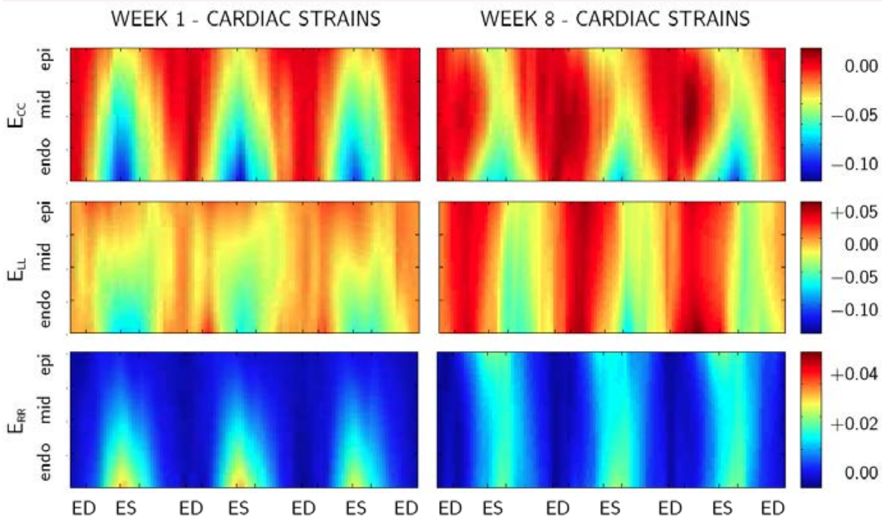


## kinematics of cardiac growth

- longitudinal growth by more than 10%
- radial thinning by more than 20%
- fiber lengthening by more than 5%
- volume decrease by more than 15%

| | epi 20% depth | p | mid 50% depth | p | endo 80% depth | p |
|---|---|---|---|---|---|---|
| $E^g_{CC}$ | 0.03±0.15 | 0.56 | 0.06±0.18 | 0.27 | 0.05±0.13 | 0.20 |
| $E^g_{LL}$ | 0.12±0.17 | 0.04 | 0.12±0.15 | 0.03 | 0.13±0.12 | 0.00 |
| $E^g_{RR}$ | −0.21±0.12 | 0.00 | −0.19±0.09 | 0.00 | −0.10±0.15 | 0.05 |
| $E^g_{CL}$ | 0.01±0.15 | 0.79 | −0.01±0.09 | 0.63 | −0.01±0.06 | 0.46 |
| $E^g_{LR}$ | 0.00±0.08 | 0.86 | 0.06±0.11 | 0.10 | 0.11±0.19 | 0.10 |
| $E^g_{CR}$ | −0.04±0.17 | 0.51 | −0.03±0.11 | 0.39 | 0.00±0.10 | 0.88 |
| $E^g_{FF}$ | 0.03±0.13 | 0.42 | 0.06±0.18 | 0.31 | 0.09±0.12 | 0.03 |

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

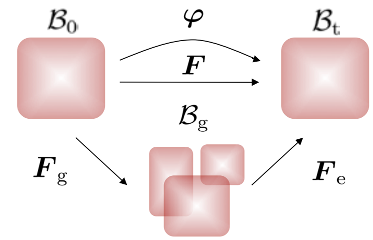



tsamis, cheng, nguyen, langer, miller, kuhl [2012]

# example - growth of the heart



## kinematics of finite growth



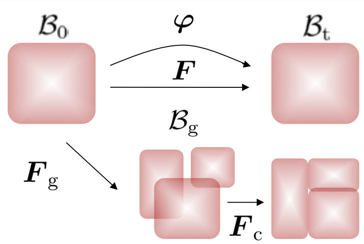

[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[4] loading generates compatible current configuration $\mathcal{B}_t$

# concept of residual stress



## kinematics of finite growth




[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

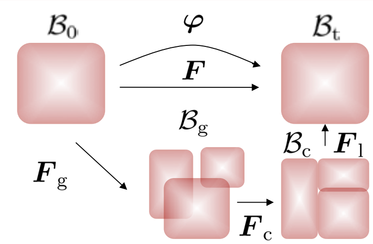
[3a] we then first apply a deformation $\boldsymbol{F}_c$ to squeeze the elements back together to the compatible configuration $\mathcal{B}_c$

[4] to generate the compatible current configuration $\mathcal{B}_t$

# concept of residual stress



## kinematics of finite growth




[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[3a] we then first apply a deformation $\boldsymbol{F}_c$ to squeeze the elements back together to the compatible configuration $\mathcal{B}_c$

[3b] and then load the compatible configuration $\mathcal{B}_c$ by $\boldsymbol{F}_l$

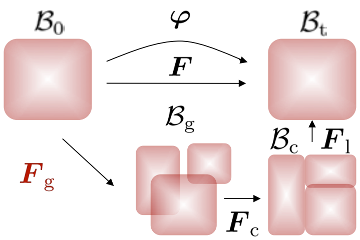
[4] to generate the compatible current configuration $\mathcal{B}_t$

# concept of residual stress

## kinematics of finite growth

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{l}} \cdot \boldsymbol{F}_{\mathrm{c}} \cdot \boldsymbol{F}_{\mathrm{g}}$$

growth tensor




$$\boldsymbol{F}_{\mathrm{e}} = \boldsymbol{F}_{\mathrm{l}} \cdot \boldsymbol{F}_{\mathrm{c}}$$

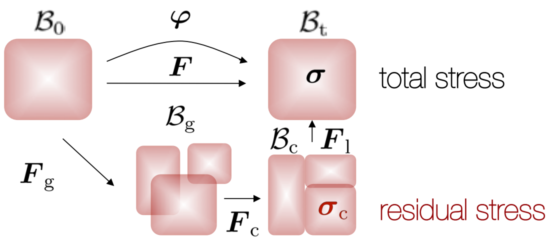
## multiplicative decomposition

lee [1969], simo [1992], rodriguez, hoger & mc culloch [1994], epstein & maugin [2000], humphrey [2002], ambrosi & mollica [2002], himpel, kuhl, menzel & steinmann [2005]



## kinematics of finite growth

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{l}} \cdot \boldsymbol{F}_{\mathrm{c}} \cdot \boldsymbol{F}_{\mathrm{g}}$$

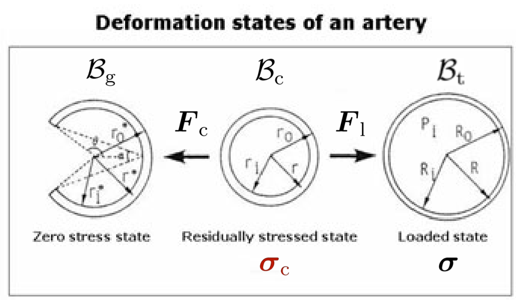



## residual stress

the additional deformation of squeezing the grown parts back to a compatible configuration gives rise to residual stresses (see thermal stresses)



## kinematics of finite growth




## residual stress

fung [1990], horný, chlup, zitný, mackov [2006]

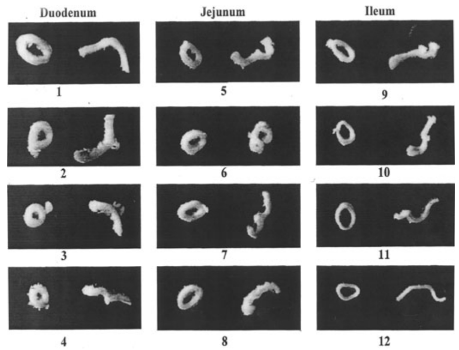


## the classical opening angle experiment


an existence of residual strains in human arteries is well known. it can be observed as an opening up of a circular arterial segment after a radial cut. an opening angle of the arterial segment is used as a measure of the residual strains generally.

fung [1990], horný, chlup, zitný, mackov [2006]

## the classical opening angle experiment

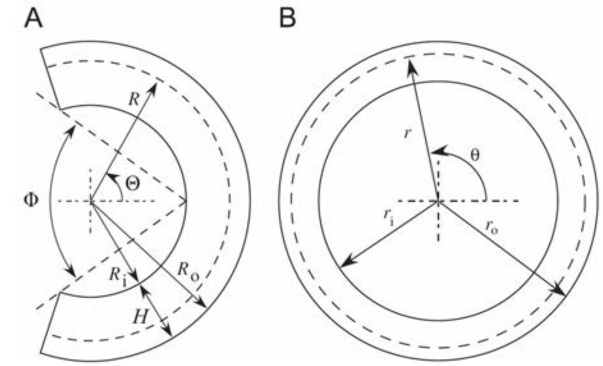



photographs showing specimens obtained from different locations in the intestine in the no-load state (left, closed rings) and the zero-stress state (right, open sectors). the rings of jejunum (site 5 to site 8) turned inside out when cut open

zhao, sha, zhuang, gregersen [2002]



## the classical opening angle experiment

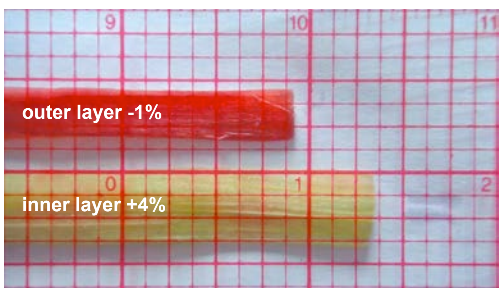



global geometrical adaptation - schematic diagram of an arterial cross section in the zero-stress state (A) and in the loaded state (B)

tsamis & stergiopulos [2009]



## convince yourself - residual stresses in rhubarb




residual stresses can be easily visualzed in a stalk of rhubarb made up of an outer layer, consisting of epidermal tissue and the collenchyma layers, and an inner layer consisting of parenchyma. when peeled, the **outer strip shortens by -1%** while the **inner layer extends by +4%**. the **inner tissue grows faster** than the outer tissue creating residual stresses resulting from axial tension in the outer wall and axial compression in the inner layer.

atkinson [1900], vandiever & goriely [2009], holland, kosmata, goriely, kuhl [2013]



## convince yourself - residual stresses in rhubarb

**118. Differential Growth.**—Not all the tissues of a stem or other part grow at the same rate.[1] On this account, and since adjacent tissues are closely united, those which elongate or grow more slowly are stretched by those which grow more rapidly. As a result either a state of tension exists, or the organ is distorted, or both.

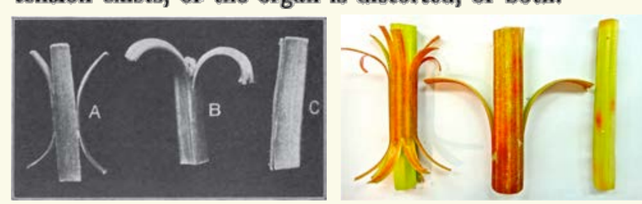



FIG. 74.—Longitudinal tissue-tension in leaf-stalk of rhubarb. In *C* the strip of outer tissues, entirely removed from the main piece, is seen to have shortened, showing that, before being removed, it was in a state of *longitudinal tissue-tension*.

charles stuart gager "fundamentals of botany" [1916], holland, kosmata, goriely, kuhl [2013]