kinematic equations [kɪˈmætɪk ɪˈkwɛər.ʃənz] describe the motion of objects without the consideration of the masses or forces that bring about the motion. The basis of kinematics is the choice of coordinates. The 1st and 2nd time derivatives of the position coordinates give the velocities and accelerations. The difference in placement between the beginning and the final state of two points in a body expresses the numerical value of strain. Strain expresses itself as a change in size and/or shape.

kinematics [kɪˈmætɪks] is the study of motion per se, regardless of the forces causing it. The primitive concepts concerned are position, time and body, the latter abstracting into mathematical terms intuitive ideas about aggregations of matter capable of motion and deformation.
potato - kinematics

- nonlinear deformation map \( \varphi \)
  \[ x = \varphi(X, t) \] with \( \varphi : B_0 \times \mathbb{R} \rightarrow B_t \)

- spatial derivative of \( \varphi \) - deformation gradient
  \[ \frac{dx}{dt} = F \cdot dX \] with \( F : TB_0 \rightarrow TB_t \)
  \[ F = \frac{\partial \varphi}{\partial X} \] \( t \) fixed

kinematic equations

- transformation of line elements - deformation gradient \( F_{ij} \)
  \[ dx_i = F_{ij} \cdot dX_j \] with \( F_{ij} : TB_0 \rightarrow TB_t \)

- uniaxial tension (incompressible), simple shear, rotation
  \[ F_{\text{uni}} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha^{-\frac{1}{2}} \end{bmatrix} \]
  \[ F_{\text{shr}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
  \[ F_{\text{rot}} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

potato - kinematics

- transformation of volume elements - determinant of \( F \)
  \[ dV_0 = dX_1 \cdot [dX_2 \times dX_3] \]
  \[ dV_t = \det((dX_1, dX_2, dX_3)) \]

- changes in volume - determinant of deformation tensor \( J \)
  \[ dV_t = J \cdot dV_0 \]
  \[ J = \det(F') \]

kinematic equations

kinematics of growth

[1] consider an elastic body \( B_0 \) at time \( t_0 \), unloaded & stressfree
[1] Consider an elastic body $B_0$ at time $t_0$, unloaded & stress free.

[2] Imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth.

[3] After growing the elements, $B_g$ may be incompatible.

[4] Loading generates compatible current configuration $B_t$.

**Kinematics of finite growth**

$$F = F_c \cdot F_g$$

growth tensor

**Multiplicative decomposition**

biologically, the notion of incompatibility implies that subelements of the grown configuration may overlap or have gaps. The implication of incompatibility is the existence of residual stresses necessary to "squeeze" these grown subelements back together. Mathematically, the notion of incompatibility implies that unlike the deformation gradient, \( F = \frac{\partial \varphi}{\partial \mathbf{X}} \), the growth tensor cannot be derived as a gradient of a vector field. Incompatible configurations are useful in finite strain inelasticity such as viscoelasticity, thermoelasticity, elastoplasticity and growth.

\[ F = F_c \cdot F_g \]

changes in volume - determinant of growth tensor \( J_g = \det(F_g) \)

\[ dV_g = J_g dV_0 \]

potato - kinematics of finite growth

incompatible growth configuration \( B_g \) & growth tensor \( F_g \)

growth \( J_g > 1 \)
resorption \( J_g < 1 \)
**kinematics of cardiac growth**

Surgically implantation of 4x3 beads across the left ventricular wall.

4d coordinates from in vivo biplane videofluoroscopic marker images.

Tsamis, Cheng, Nguyen, Langer, Miller, Kuhl (2012)

**example - growth of the heart**

1. Determine three vectors $\mathbf{dX}$, that span the tetrahedron at baseline.
   Take an arbitrary point of the tetrahedron as origin, e.g., $X_4$, and calculate the three vectors $\mathbf{dX}_1$, $\mathbf{dX}_2$, and $\mathbf{dX}_3$ from the origin to any other point using the coordinates $X$ at baseline such that $\mathbf{dX}_i = X_i - X_4$ for $i = 1, 2, 3$.

   **matlab**

   
   $\begin{align*}
   \mathbf{dX}_1 &= \mathbf{X}_1 - \mathbf{X}_4 \\
   \mathbf{dX}_2 &= \mathbf{X}_2 - \mathbf{X}_4 \\
   \mathbf{dX}_3 &= \mathbf{X}_3 - \mathbf{X}_4 \\
   \mathbf{dX}_1 &= [+0.00, -0.40, -0.70] \\
   \mathbf{dX}_2 &= [+0.00, +0.40, -0.70] \\
   \mathbf{dX}_3 &= [-0.30, +0.39, -0.70]
   \end{align*}$

2. Determine the same three vectors $\mathbf{dx}$, that span the tetrahedron after growth.
   Take the same point as origin, e.g., $x_4$, and calculate the vectors $\mathbf{dx}_1$, $\mathbf{dx}_2$, and $\mathbf{dx}_3$ from the origin to any other point using the coordinates $x$ after growth such that $\mathbf{dx}_i = x_i - x_4$ for $i = 1, 2, 3$.

   **matlab**

   
   $\begin{align*}
   \mathbf{dx}_1 &= \mathbf{x}_1 - \mathbf{x}_4 \\
   \mathbf{dx}_2 &= \mathbf{x}_2 - \mathbf{x}_4 \\
   \mathbf{dx}_3 &= \mathbf{x}_3 - \mathbf{x}_4 \\
   \mathbf{dx}_1 &= [+0.02, -0.51, -0.51] \\
   \mathbf{dx}_2 &= [+0.02, +0.37, -0.51] \\
   \mathbf{dx}_3 &= [-0.28, +0.37, -0.50]
   \end{align*}$
kinematics of cardiac growth

[3] Determine the growth tensor $F^g$ that maps the baseline line elements $dX_i$ onto the grown line elements $dx_i$.

The growth tensor maps line elements according to $dx_i = F^g \cdot dX_i$. The application of this mapping to all three line elements $dX_i$ defines three vector valued equations, i.e., nine equations to solve for the nine components of $F^g$. To obtain a more compact notation, rearrange all baseline line elements from [1] and all grown line elements from [2] in $3 \times 3$ matrices, i.e., $C := [ dX_1 \; dX_2 \; dX_3 ]$ and $c := [ dx_1 \; dx_2 \; dx_3 ]$. Now, determine the growth tensor $F^g$ by using the equation $F^g \cdot C = c$, thus $F^g = c \cdot C^{-1}$.

matlab

\[
C = \begin{bmatrix} dX_1 & dX_2 & dX_3 \end{bmatrix}; \quad c = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}; \quad F = c/C;
\]

\[
dx_1_{\text{check}} = F \cdot dX_1; \quad dx_2_{\text{check}} = F \cdot dX_2; \quad dx_3_{\text{check}} = F \cdot dX_3;
\]

\[
+1.0000 \quad 0.0000 \quad -0.0286
\]

\[
F = \begin{bmatrix} -0.0367 & +1.1000 & +0.1000 \\ -0.0333 & 0.0000 & +0.7286 \end{bmatrix}
\]

kinematics of cardiac growth

[4] Control your results by calculating $dx_i = F^g \cdot dX_i$.

Do the calculated grown line elements $dx_i$ match the ones you had calculated in [2]?

matlab

\[
dx_1_{\text{check}} = F \cdot dX_1; \quad dx_2_{\text{check}} = F \cdot dX_2; \quad dx_3_{\text{check}} = F \cdot dX_3;
\]

\[
dx_1_{\text{check}} = [ +0.02, -0.51, -0.51 ]
\]

\[
dx_2_{\text{check}} = [ +0.02, +0.37, -0.51 ]
\]

\[
dx_3_{\text{check}} = [ -0.28, +0.37, -0.50 ]
\]

kinematics of cardiac growth

[5] Determine the grown fiber direction $n^{gb} = F^g \cdot N^{ib}$.

The growth tensor can be used to map the measured baseline fiber direction $N^{ib}$ onto the grown fiber direction $n^{gb}$. Determine $n^{gb}$ and comment on how $N^{ib}$ and $n^{gb}$ deviate.

matlab

\[
alpha = 10.0;
\]

\[
N_{\text{fib}} = [0.0; -\cos(\alpha); \sin(\alpha)]
\]

\[
n_{\text{fib}} = F \cdot N_{\text{fib}};
\]

\[
theta = \arccos((n_{\text{fib}} \cdot N_{\text{fib}})/(\|n_{\text{fib}}\| \cdot \|N_{\text{fib}}\|))
\]

\[
N_{\text{fib}} = [0.0000, -0.9848, +0.1736]
\]

\[
n_{\text{fib}} = [-0.0050, -1.0659, +0.1265]
\]

\[
theta = 3.2420
\]

kinematics of cardiac growth

[6] Determine the fiber stretch upon growth $\lambda = \sqrt{N^{ib} \cdot N^{ib}}$.

Since the fiber orientation $N^{ib}$ was given as a unit vector, the length of the grown vector $n^{gb} = F \cdot N^{ib}$ corresponds to the relative change in fiber length, i.e., the amount of growth along the fiber direction, $\lambda = \sqrt{n^{gb} \cdot n^{gb}} = \sqrt{N^{ib} \cdot F^{g} \cdot F^{T} \cdot N^{ib}}$.

matlab

\[
lambda = \sqrt{(n_{\text{fib}} \cdot n_{\text{fib}})}
\]

\[
lambda = 1.0734
\]
Determine the second order Green Lagrange strain tensor $E = \frac{1}{2} [F^* F - I]$. $E$ is called the Green Lagrange strain tensor and it is used to characterize strains with respect to the reference configuration in a finite strain setting.

Determine the displacement gradient tensor $H = F - I$. $H = \nabla u$ is the nonsymmetric displacement gradient tensor which can also be expressed as $H = \frac{\partial u}{\partial X} = \frac{\partial |X - X|}{\partial X} = F - I$.

\[
E = \frac{1}{2} (F^* F - \text{eye}(3)) = \begin{bmatrix} 0.0012 & -0.0202 & -0.0283 \\ -0.0202 & 0.1050 & 0.0550 \\ -0.0283 & 0.0550 & -0.2292 \end{bmatrix}
\]

\[
H = F - \text{eye}(3) = \begin{bmatrix} 0.0000 & 0 & -0.0286 \\ -0.0367 & 0.1000 & 0.1000 \\ -0.0333 & 0 & -0.2714 \end{bmatrix}
\]

\[
\text{epsilon} = \frac{1}{2} (H + H^*) = \begin{bmatrix} 0.0000 & -0.0183 & -0.0310 \\ -0.0183 & 0.1000 & 0.0500 \\ -0.0310 & 0.0500 & -0.2714 \end{bmatrix}
\]

\[
\epsilon_n = N_{fib}^T \text{epsilon} N_{fib};
\]

\[
\epsilon_n = 0.0717 
\]

\[
J = \det(F) = 0.8004 
\]

\[
J_{check} = 0.8003 
\]

\[
e = -0.1714 
\]
kinematics of cardiac growth

Deformation

\[ \varphi(X, t) = \sum_{l=1}^{n_{apx}} c_l(t) N_l(X) \]

Spatial gradient

\[ F(X, t) = \sum_{l=1}^{n_{apx}} c_l(t) \nabla N_l(X) \]

Volume changes

\[ J(X, t) = \det(F(X, t)) \]

Fiber stretch

\[ \lambda_{FF}(X, t) = \sqrt{[f(X) \cdot (F^t(X, t) \cdot F(X, t) \cdot f(X))]}^{1/2} \]


comas, cheng, nguyen, langer, miller, kuhl [2012]

Example - growth of the heart

• Longitudinal growth by more than 10%
• Radial thinning by more than 20%
• Fiber lengthening by more than 5%
• Volume decrease by more than 15%

Example - growth of the heart

<table>
<thead>
<tr>
<th>epi</th>
<th>mid</th>
<th>endo</th>
</tr>
</thead>
<tbody>
<tr>
<td>20% depth</td>
<td>50% depth</td>
<td>80% depth</td>
</tr>
<tr>
<td>p</td>
<td>p</td>
<td>p</td>
</tr>
<tr>
<td>1.00±0.12</td>
<td>0.96</td>
<td>1.03±0.14</td>
</tr>
<tr>
<td>0.04±0.14</td>
<td>0.42</td>
<td>0.01±0.10</td>
</tr>
<tr>
<td>−0.07±0.29</td>
<td>0.46</td>
<td>−0.03±0.16</td>
</tr>
<tr>
<td>−0.02±0.17</td>
<td>0.75</td>
<td>−0.04±0.13</td>
</tr>
<tr>
<td>1.10±0.15</td>
<td>0.06</td>
<td>1.10±0.13</td>
</tr>
<tr>
<td>0.02±0.16</td>
<td>0.71</td>
<td>0.10±0.20</td>
</tr>
<tr>
<td>−0.01±0.09</td>
<td>0.64</td>
<td>−0.03±0.17</td>
</tr>
<tr>
<td>0.00±0.05</td>
<td>0.86</td>
<td>−0.00±0.09</td>
</tr>
</tbody>
</table>

\[ J^E | 0.74±0.19 | 0.00 | 0.82±0.19 | 0.01 | 0.89±0.21 | 0.10 \]

\[ J^F | 1.03±0.12 | 0.49 | 1.04±0.16 | 0.36 | 1.08±0.11 | 0.04 \]

comas, cheng, nguyen, langer, miller, kuhl [2012]
example - growth of the heart

concept of residual stress

kinematics of finite growth

[3] after growing the elements, $B_g$ may be incompatible

[3a] we then first apply a deformation $F_c$ to squeeze the elements back together to the compatible configuration $B_c$

[4] to generate the compatible current configuration $B_t$

kinematics of finite growth

[3] after growing the elements, $B_g$ may be incompatible

[3a] we then first apply a deformation $F_c$ to squeeze the elements back together to the compatible configuration $B_c$

[3b] and then load the compatible configuration $B_c$ by $F_1$

[4] to generate the compatible current configuration $B_t$
The additional deformation of squeezing the grown parts back to a compatible configuration gives rise to residual stresses (see thermal stresses).

An existence of residual strains in human arteries is well known. It can be observed as an opening up of a circular arterial segment after a radial cut. An opening angle of the arterial segment is used as a measure of the residual strains generally.
the classical opening angle experiment

photographs showing specimens obtained from different locations in the intestine in the no-load state (left, closed rings) and the zero-stress state (right, open sectors). the rings of jejunum (site 5 to site 8) turned inside out when cut open

zhao, sha, zhuang, gregersen [2002]

concept of residual stress

convince yourself - residual stresses in rhubarb

residual stresses can be easily visualized in a stalk of rhubarb made up of an outer layer, consisting of epidermal tissue and the collenchyma layers, and an inner layer consisting of parenchyma. when peeled, the outer strip shortens by -1% while the inner layer extends by +4%. the inner tissue grows faster than the outer tissue creating residual stresses resulting from axial tension in the outer wall and axial compression in the inner layer.

atkinson (1900), vanleever & gorely (2009), holland, kosmata, gorely, kuhl [2013]

concept of residual stress

convince yourself - residual stresses in rhubarb

118. Differential Growth.—Not all the tissues of a stem or other part grow at the same rate.1 On this account, and since adjacent tissues are closely united, those which elongate or grow more slowly are stretched by those which grow more rapidly. As a result either a state of tension exists, or the organ is distorted, or both.

Fig. 74.—Longitudinal tissue-tension in leaf-stalk of rhubarb. In C the strip of outer tissue, entirely removed from the main piece, is seen to have shortened, showing that, before being removed, it was in a state of longitudinal tissue-tension.

charles stuart gager "fundamentals of botany" [1916], holland, kosmata, gorely, kuhl [2013]