15 - finite element method - density growth - alternative formulation

functional adaptation of proxima femur

$$\mathrm{D}_t \rho_0 = \mathcal{R}_0 \qquad \mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m} \psi_0 - \psi_0^*$$

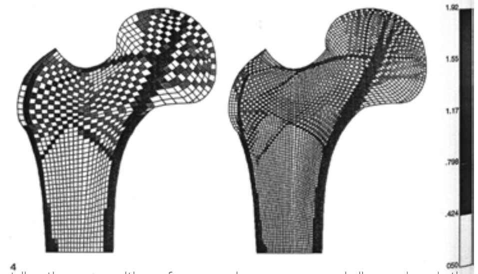
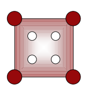


fig. 4. the density distribution resulting from a bone remodeling simulation carried out using the traditional element-based algorithm. this type of behavior is clearly nonbiological in nature and motivates the question: are the current strain-energy-based continuum formulations incapable of predicting the expected continuous results near bone ends or is this difficulty technical in nature to be overcome with appropriate numerical implementation?

jacobs, levenston, beaupre, simo, carter [1995]

from integral equation...



- integral equations cannot be evaluated analytically

$$\mathbf{R}_j^e = \int_\zeta \int_\eta \int_\xi \nabla N_\varphi^j(\xi,\eta,\zeta) \cdot \boldsymbol{P}_{n+1}(\xi,\eta,\zeta)\,\det(\boldsymbol{J}(\xi,\eta,\zeta))\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

$$\mathbf{K}_{jl}^e = \int_\zeta \int_\eta \int_\xi \nabla N_\varphi^j(\xi,\eta,\zeta) \cdot \mathrm{D}_F \boldsymbol{P}(\xi,\eta,\zeta) \cdot \nabla N_\varphi^l(\xi,\eta,\zeta)\,\det(\boldsymbol{J}(\xi,\eta,\zeta))\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

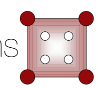
- idea - numerical interation / quadrature

$$\mathbf{R}_j^e \approx \sum_{i=0}^{n} \nabla N_\varphi^j(\xi_i,\eta_i,\zeta_i) \cdot \boldsymbol{P}_{n+1}(\xi_i,\eta_i,\zeta_i)\,\det(\boldsymbol{J}(\xi_i,\eta_i,\zeta_i)\,w_i$$

$$\mathbf{K}_{jl}^e \approx \sum_{i=0}^{n} \nabla N_\varphi^j(\xi_i,\eta_i,\zeta_i) \cdot \mathrm{D}_F \boldsymbol{P}(\xi_i,\eta_i,\zeta_i) \cdot \nabla N_\varphi^l(\xi_i,\eta_i,\zeta_i)\,\det(\boldsymbol{J}(\xi_i,\eta_i,\zeta_i)\,w_i$$

... to discrete sum

numerical integration

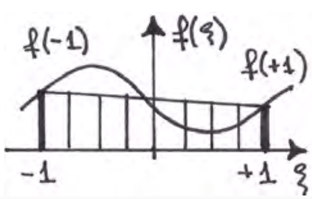


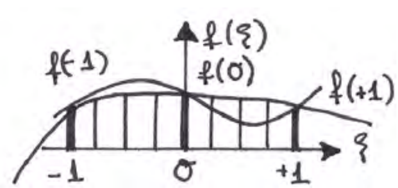
- integral equations are approximated by discrete sums

$$\int_a^b f(\xi)\mathrm{d}\xi \approx [b-a] \sum_{i=0}^{n} f(\xi_i)\, w_i$$

ξ_i ... quadrature point coordinates

w_i... quadrature point weights



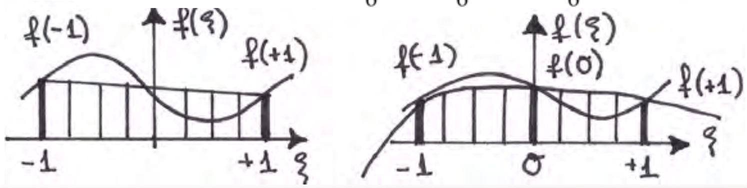


newton cotes quadrature - accuracy [n-1]

equidistant quadrature points @ $\xi_i = -1 + 2\,\frac{i}{n}$

n=2 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(\xi_0)\,w_0 + f(\xi_1)\,w_1\,]$
$= f(-1) + f(+1)$ trapezoidal rule

n=3 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(\xi_0)\,w_0 + f(\xi_1)\,w_1 + f(\xi_2)\,w_2\,]$
$= 2\,[\,f(-1)\frac{1}{6} + f(0)\frac{4}{6} + f(+1)\frac{1}{6}\,]$ simpson rule



gauss legendre quadrature - accuracy [2n-1]

optimized quadrature points

n=1 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(0)\,1\,]$

n=2 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(-\frac{1}{\sqrt{3}})\,\frac{1}{2} + f(+\frac{1}{\sqrt{3}})\,\frac{1}{2}\,]$

n=3 $\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(-\frac{3}{\sqrt{5}})\,\frac{5}{18} + f(0)\,\frac{8}{18} + f(+\frac{3}{\sqrt{5}})\,\frac{5}{18}\,]$

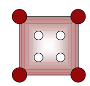
most fe programs prefer gauss over newton!

@ integration point level

- constitutive equations - given $\boldsymbol{F} = \nabla\varphi$ calculate $\boldsymbol{P}$ 
- update density for current stress state
  from $\rho_{0\,n}$ and $\mathrm{D}_t\rho_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m}\psi_0(\boldsymbol{F}) - \psi_0^*$ calculate $\rho_{0\,n+1}$
- calculate first piola kirchhoff stress of solid material
  $\boldsymbol{P}^{\mathrm{neo}}(\boldsymbol{F}) = \mu_0\,\boldsymbol{F} + [\,\lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0\,]\,\boldsymbol{F}^{\text{-t}}$
- calculate first piola kirchhoff stress of porous material
  $\boldsymbol{P}\,(\boldsymbol{F}) = \left[\frac{\rho_0}{\rho_0^*}\right]^n \boldsymbol{P}^{\mathrm{neo}}$

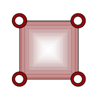
stress for righthand side vector

staggered solution - integration point based



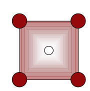
weinans, huiskes & grootenboer [1992], harrigan & hamilton [1992],, [1994], jacobs, levenston, beaupré,simo & carter [1995]

simultaneous solution - node point based

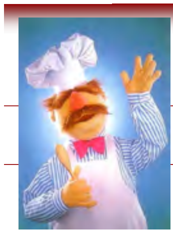


jacobs, levenston, beaupré,simo & carter [1995], fischer, jacobs, levenston & carter [1997], nackenhorst [1997], levenston [1997]]

sequential solution - element based



huiskes, weinans, grootenboer, dalstra, fudala & slooff [1987], carter, orr, fhyrie [1989], beaupré, orr & carter [1990], weinans, huiskes & grootenboer [1992], [1994], jacobs, levenston, beaupré, simo & carter [1995], huiskes [2000], carter & beaupré [2001]



recipe for finite element modeling

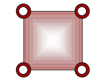
from continuous problem...

$$\begin{aligned} \mathrm{D}_t \rho_0 &= \mathrm{Div}(\boldsymbol{R}) + \mathcal{R}_0 \\ \rho_0 \, \mathrm{D}_t \boldsymbol{v} &= \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0 \end{aligned}$$

- temporal discretization — implicit euler backward
- spatial discretization — finite element method
- staggered/simultaneous — newton raphson iteration
- linearization — gateaux derivative

... to linearized discrete initial boundary value problem

from biological and mechanical equilibrium...



$$\begin{aligned} \mathrm{D}_t \rho_0 &= \mathrm{Div}(\boldsymbol{R}) + \mathcal{R}_0 \\ \rho_0 \, \mathrm{D}_t \boldsymbol{v} &= \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0 \end{aligned}$$

- strong form / residual format

$$\begin{aligned} \mathsf{R}^\rho(\rho_0, \boldsymbol{\varphi}) &= 0 \quad \text{in } \mathcal{B}_0 \\ \mathbf{R}^\varphi(\rho_0, \boldsymbol{\varphi}) &= \mathbf{0} \quad \text{in } \mathcal{B}_0 \end{aligned}$$

- residuals

$$\begin{aligned} \mathsf{R}^\rho &= \mathrm{D}_t \rho_0 - \mathrm{Div}(\boldsymbol{R}) - \mathcal{R}_0 \\ \mathbf{R}^\varphi &= \rho_0 \, \mathrm{D}_t \boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 \end{aligned}$$

... to residual format

residual equations...

$$\begin{aligned} \mathsf{R}^\rho &= \mathrm{D}_t \rho_0 - \mathrm{Div}(\boldsymbol{R}) - \mathcal{R}_0 = 0 \quad \text{in } \mathcal{B}_0 \quad \partial\mathcal{B}_0 = \partial\mathcal{B}_0^\rho \cup \partial\mathcal{B}_0^{T^\rho} \\ \mathbf{R}^\varphi &= \rho_0 \, \mathrm{D}_t \boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 = \mathbf{0} \quad \text{in } \mathcal{B}_0 \quad \partial\mathcal{B}_0 = \partial\mathcal{B}_0^\varphi \cup \partial\mathcal{B}_0^{T^\varphi} \end{aligned}$$

- dirichlet / essential boundary conditions

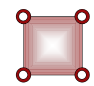
$$\begin{aligned} \rho_0 - \bar{\rho}_0 &= 0 \quad \text{on } \partial\mathcal{B}_0^\rho \\ \boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}} &= \mathbf{0} \quad \text{on } \partial\mathcal{B}_0^\varphi \end{aligned}$$

- neumann / natural boundary conditions

$$\begin{aligned} \boldsymbol{R} \cdot \boldsymbol{N} - \bar{T}^\rho &= 0 \quad \text{on } \partial\mathcal{B}_0^{T^\rho} \\ \boldsymbol{P} \cdot \boldsymbol{N} - \bar{\boldsymbol{T}}^\varphi &= \mathbf{0} \quad \text{on } \partial\mathcal{B}_0^{T^\varphi} \end{aligned}$$

... and boundary conditions

from strong form...



$$\begin{aligned} \mathsf{R}^\rho &= \mathrm{D}_t \rho_0 - \mathrm{Div}(\boldsymbol{R}) - \mathcal{R}_0 = 0 \\ \mathbf{R}^\varphi &= \rho_0 \, \mathrm{D}_t \boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 = \mathbf{0} \end{aligned}$$

- weak form

$$\begin{aligned} \mathsf{G}^\rho(\delta\rho; \rho_0, \boldsymbol{\varphi}) &= 0 \quad \forall \delta\rho \text{ in } \mathcal{H}_1^0(\mathcal{B}_0) \\ \mathsf{G}^\varphi(\delta\boldsymbol{\varphi}; \rho_0, \boldsymbol{\varphi}) &= 0 \quad \forall \delta\boldsymbol{\varphi} \text{ in } \mathcal{H}_1^0(\mathcal{B}_0) \end{aligned}$$

- weak form expressions

no derivatives ↙ second derivatives ↙

$$\begin{aligned} \mathsf{G}^\rho &= \int_{\mathcal{B}_0} \delta\rho \, \mathrm{D}_t \rho_0 \mathrm{d}V - \int_{\mathcal{B}_0} \delta\rho \, \mathrm{Div}(\boldsymbol{R}) \mathrm{d}V - \int_{\mathcal{B}_0} \delta\rho \, \mathcal{R}_0 \, \mathrm{d}V \\ \mathsf{G}^\varphi &= \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \rho_0 \mathrm{D}_t \boldsymbol{v} \, \mathrm{d}V - \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \mathrm{Div}(\boldsymbol{P}) \mathrm{d}V - \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \boldsymbol{b}_0 \, \mathrm{d}V \end{aligned}$$

... to nonsymmetric weak form

from nonsymmetric weak form…

- integration by parts

$$\int_{\mathcal{B}_0} \delta\rho \,\mathrm{Div}(\boldsymbol{R})\mathrm{d}V = \int_{\mathcal{B}_0} \mathrm{Div}(\delta\rho \boldsymbol{R})\,\mathrm{d}V - \int_{\mathcal{B}_0} \nabla\delta\rho \cdot \boldsymbol{R}\mathrm{d}V$$
$$\int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \mathrm{Div}(\boldsymbol{P})\mathrm{d}V = \int_{\mathcal{B}_0} \mathrm{Div}(\delta\boldsymbol{\varphi} \cdot \boldsymbol{P})\mathrm{d}V - \int_{\mathcal{B}_0} \nabla\delta\boldsymbol{\varphi} : \boldsymbol{P}\mathrm{d}V$$

- gauss theorem & boundary conditions

$$\int_{\mathcal{B}_0} \mathrm{Div}(\delta\rho \boldsymbol{R})\,\mathrm{d}V = \int_{\partial\mathcal{B}_0^{T^\rho}} \delta\rho\, \boldsymbol{R} \cdot \boldsymbol{N}\mathrm{d}A = \int_{\partial\mathcal{B}_0^{T^\rho}} \delta\rho\, \bar{T}^\rho\, \mathrm{d}A$$
$$\int_{\mathcal{B}_0} \mathrm{Div}(\delta\boldsymbol{\varphi} \cdot \boldsymbol{P})\mathrm{d}V = \int_{\partial\mathcal{B}_0^{T^\varphi}} \boldsymbol{\varphi} \cdot \boldsymbol{P} \cdot \boldsymbol{N}\mathrm{d}A = \int_{\partial\mathcal{B}_0^{T^\varphi}} \delta\boldsymbol{\varphi} \cdot \bar{\boldsymbol{T}}^\varphi \mathrm{d}A$$

- weak form

first derivatives first derivatives

$$\mathsf{G}^\rho = \int_{\mathcal{B}_0} \delta\rho \quad \mathrm{D}_t\rho_0 \mathrm{d}V + \int_{\mathcal{B}_0} \nabla\delta\rho \cdot \boldsymbol{R}\mathrm{d}V - \int_{\partial\mathcal{B}_0^{T^\rho}} \delta\rho\, \bar{T}^\rho \mathrm{d}A - \int_{\mathcal{B}_0} \delta\rho\, \mathcal{R}_0\, \mathrm{d}V$$
$$\mathsf{G}^\varphi = \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \rho_0 \mathrm{D}_t \boldsymbol{v}\, \mathrm{d}V + \int_{\mathcal{B}_0} \nabla\delta\boldsymbol{\varphi} : \boldsymbol{P}\mathrm{d}V - \int_{\partial\mathcal{B}_0^{T^\varphi}} \delta\boldsymbol{\varphi} \cdot \bar{\boldsymbol{T}}^\varphi \mathrm{d}A - \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \boldsymbol{b}_0\, \mathrm{d}V$$

… to symmetric weak form

temporal discretization

- discretization $\mathcal{T} = \bigcup_{n=0}^{n_{\mathrm{step}}-1} [t_n, t_{n+1}]$ $\quad \Delta t = t_{n+1} - t_n$
- time discrete weak form

$$\mathsf{G}^\rho\,(\delta\rho; \rho_{0\,n+1}, \boldsymbol{\varphi}_{n+1}) = 0 \quad \forall\delta\rho \text{ in } \mathcal{H}_1^0(\mathcal{B}_0)$$
$$\mathsf{G}^\varphi(\delta\boldsymbol{\varphi}; \rho_{0\,n+1}, \boldsymbol{\varphi}_{n+1}) = 0 \quad \forall\delta\boldsymbol{\varphi} \text{ in } \mathcal{H}_1^0(\mathcal{B}_0)$$

- interpolation of material time derivatives euler backward

$$\mathrm{D}_t\rho_0 = \tfrac{1}{\Delta t}[\rho_{0\,n+1} - \rho_{0\,n}]$$
$$\mathrm{D}_t\boldsymbol{v} = \tfrac{1}{\Delta t}[\boldsymbol{v}_{n+1} - \boldsymbol{v}_n]$$

… to semidiscrete weak form

spatial discretization

- discretization $\quad \mathcal{B}_0 = \bigcup_{e=1}^{n_{\mathrm{el}}} \mathcal{B}_0^e$
- interpolation of test functions

$$\delta\rho_0^h|_{\mathcal{B}_0^e} = \textstyle\sum_{i=1}^{n_{\mathrm{en}}} N_\rho^i\, \delta\rho_i \in \mathcal{H}_1^0(\mathcal{B}_0) \qquad \nabla\delta\rho_0^h|_{\mathcal{B}_0^e} = \textstyle\sum_{i=1}^{n_{\mathrm{en}}} \delta\rho_i\, \nabla N_\rho^i$$
$$\delta\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \textstyle\sum_{j=1}^{n_{\mathrm{en}}} N_\varphi^j \delta\boldsymbol{\varphi}_j \in \mathcal{H}_1^0(\mathcal{B}_0) \qquad \nabla\delta\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \textstyle\sum_{j=1}^{n_{\mathrm{en}}} \delta\boldsymbol{\varphi}_j \otimes \nabla N_\varphi^j$$

- interpolation of trial functions

$$\rho_0^h|_{\mathcal{B}_0^e} = \textstyle\sum_{k=1}^{n_{\mathrm{en}}} N_\rho^k\, \rho_k \in \mathcal{H}_1(\mathcal{B}_0) \qquad \nabla\rho_0^h|_{\mathcal{B}_0^e} = \textstyle\sum_{k=1}^{n_{\mathrm{en}}} \rho_k\, \nabla N_\rho^k$$
$$\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \textstyle\sum_{l=1}^{n_{\mathrm{en}}} N_\varphi^l\, \boldsymbol{\varphi}_l \in \mathcal{H}_1(\mathcal{B}_0) \qquad \nabla\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e} = \textstyle\sum_{l=1}^{n_{\mathrm{en}}} \boldsymbol{\varphi}_l \otimes \nabla N_\varphi^l$$

… to discrete weak form

from discrete weak form…

- discrete residual format

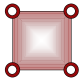
$$\mathsf{R}_I^\rho\,(\rho_0{}_{n+1}^h, \boldsymbol{\varphi}_{n+1}^h) = 0 \qquad \forall\, I = 1, ..., n_{\mathrm{np}}$$
$$\mathbf{R}_J^\varphi(\rho_0{}_{n+1}^h, \boldsymbol{\varphi}_{n+1}^h) = \mathbf{0} \qquad \forall\, J = 1, ..., n_{\mathrm{np}}$$

- discrete residuals

$$\mathsf{R}_I^\rho = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} N_\rho^i\, \tfrac{\rho_{0\,n+1} - \rho_{0\,n}}{\Delta t} \mathrm{d}V + \int_{\mathcal{B}_0^e} \nabla N_\rho^i \cdot \boldsymbol{R}_{n+1} \mathrm{d}V$$
$$- \int_{\partial\mathcal{B}_0^e} N_\rho^i\, \bar{T}_{n+1}^\rho \mathrm{d}A - \int_{\mathcal{B}_0^e} N_\rho^i \mathcal{R}_{0\,n+1} \mathrm{d}V$$
$$\mathbf{R}_J^\varphi = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} N_\varphi^j \rho_{0\,n+1}\, \tfrac{\boldsymbol{v}_{0\,n+1} - \boldsymbol{v}_{0\,n}}{\Delta t} \mathrm{d}V + \int_{\mathcal{B}_0^e} \nabla N_\varphi^j \cdot \boldsymbol{P}_{n+1} \mathrm{d}V$$
$$- \int_{\partial\mathcal{B}_0^e} N_\varphi^j\, \bar{\boldsymbol{T}}_{n+1}^\varphi \mathrm{d}A - \int_{\mathcal{B}_0^e} N_\varphi^j \boldsymbol{b}_{0\,n+1} \mathrm{d}V$$

… to discrete residuals

from discreate weak form…



- linearization / newton raphson scheme

$$\mathsf{R}^{\rho k+1}_{I\,n+1} = \mathsf{R}^{\rho k}_{I\,n+1} + \mathrm{d}\mathsf{R}^{\rho}_{I} \doteq 0 \quad \forall\, I = 1, ..., n_{\mathrm{np}}$$
$$\mathbf{R}^{\varphi k+1}_{J\,n+1} = \mathbf{R}^{\varphi k}_{J\,n+1} + \mathrm{d}\mathbf{R}^{\varphi}_{J} \doteq 0 \quad \forall\, J = 1, ..., n_{\mathrm{np}}$$

- incremental residual

$$\mathrm{d}\mathsf{R}^{\rho}_{I} = \textstyle\sum_{K=1}^{n_{\mathrm{np}}} \mathsf{K}^{\rho\rho}_{IK}\,\mathrm{d}\rho_K + \sum_{L=1}^{n_{\mathrm{np}}} \mathbf{K}^{\rho\varphi}_{IL} \cdot \mathrm{d}\boldsymbol{\varphi}_L$$
$$\mathrm{d}\mathbf{R}^{\varphi}_{J} = \textstyle\sum_{K=1}^{n_{\mathrm{en}}} \mathbf{K}^{\varphi\rho}_{JK}\,\mathrm{d}\rho_K + \sum_{L=1}^{n_{\mathrm{en}}} \mathbf{K}^{\varphi\varphi}_{JL} \cdot \mathrm{d}\boldsymbol{\varphi}_L$$

- system of equations

$$\begin{bmatrix} \mathsf{K}^{IK}_{\rho\rho} & \mathbf{K}^{IL}_{\rho\varphi} \\ \mathbf{K}^{JK}_{\varphi\rho} & \mathbf{K}^{JL}_{\varphi\varphi} \end{bmatrix} \begin{bmatrix} \mathrm{d}\rho_K \\ \mathrm{d}\boldsymbol{\varphi}_L \end{bmatrix} = - \begin{bmatrix} \mathsf{R}^{\rho\,k}_{I\,n+1} \\ \mathbf{R}^{\varphi\,k}_{J\,n+1} \end{bmatrix}$$

… to linearized weak form

finite elements - node point based

linearization

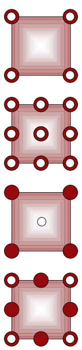
- iteration matrices

$$\mathsf{K}^{\rho\rho}_{IK} = \frac{\partial \mathsf{R}^{\rho}_{I}}{\partial \rho_K} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} N_\rho^i \frac{1}{\Delta t} N_\rho^k - N_\rho^i \mathrm{D}_{\rho_0} \mathcal{R} N_\rho^k \mathrm{d}V$$
$$+ \int_{\mathcal{B}_0^e} \nabla N_\rho^i \cdot \mathrm{D}_{\nabla\rho_0} \boldsymbol{R} \cdot \nabla N_\rho^k \mathrm{d}V$$
$$\mathbf{K}^{\rho\varphi}_{IL} = \frac{\partial \mathsf{R}^{\rho}_{I}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} - N_\rho^i \mathrm{D}_F \mathcal{R}_0 \cdot \nabla N_\varphi^l \mathrm{d}V$$
$$\mathbf{K}^{\varphi\rho}_{JK} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \rho_K} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} \nabla N_\varphi^j \cdot \mathrm{D}_{\rho_0} \boldsymbol{P} N_\rho^k \mathrm{d}V$$
$$\mathbf{K}^{\varphi\varphi}_{JL} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}_0^e} N_\varphi^j \rho \tfrac{1}{\Delta t} \boldsymbol{I} N_\varphi^l + \nabla N_\varphi^j \cdot \mathrm{D}_F \boldsymbol{P} \cdot \nabla N_\varphi^l \mathrm{d}V$$

… to linearized weak form

finite elements - node point based

node point based

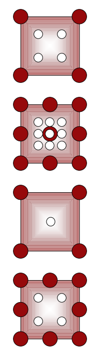


- loop over all time steps
  - global newton iteration
    - loop over all elements
      - loop over all quadrature points
        - evaulate balance of mass and momentum
      - determine element residuals & partial derivatives
    - determine global residuals and iterational matrices
  - determine $\rho_{0\,n+1}$ and $\boldsymbol{\varphi}_{n+1}$
- determine state of biological equilibrium

density ρ_0 as nodal degree of freedom

finite elements - node point based

integration point based

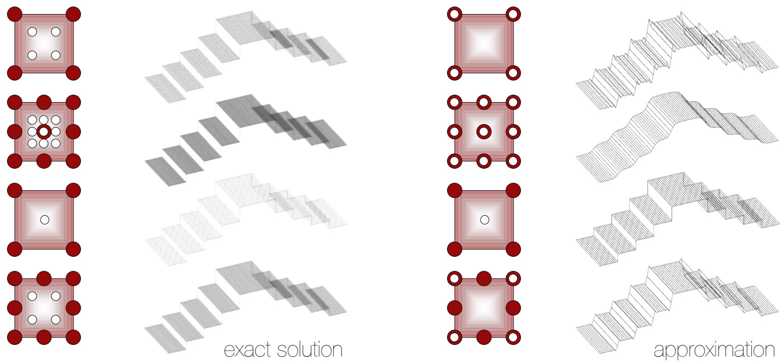


- loop over all time steps
  - global newton iteration
    - loop over all elements
      - loop over all quadrature points
        - local newton iteration to determine $\rho_{0\,n+1}$
      - determine element residual & partial derivative
    - determine global residual and iterational matrix
  - determine $\boldsymbol{\varphi}_{n+1}$
- determine state of biological equilibrium

density ρ_0 as internal variable

finite elements - integration point based

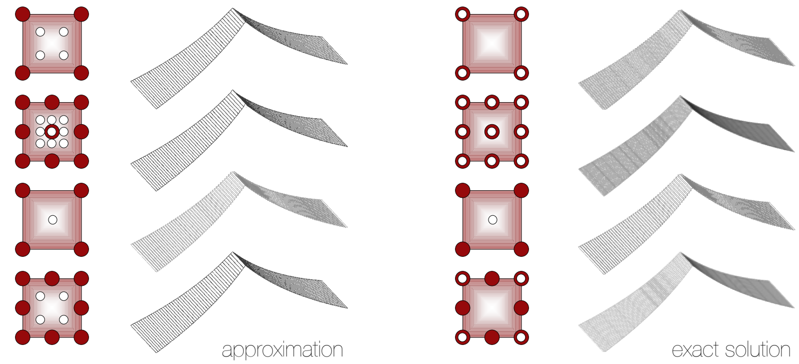
integration vs node point based



discontinuous model problem

finite element method

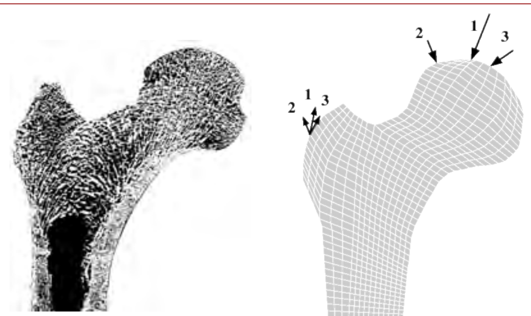
integration vs node point based



continuous model problem

finite element method

different load cases

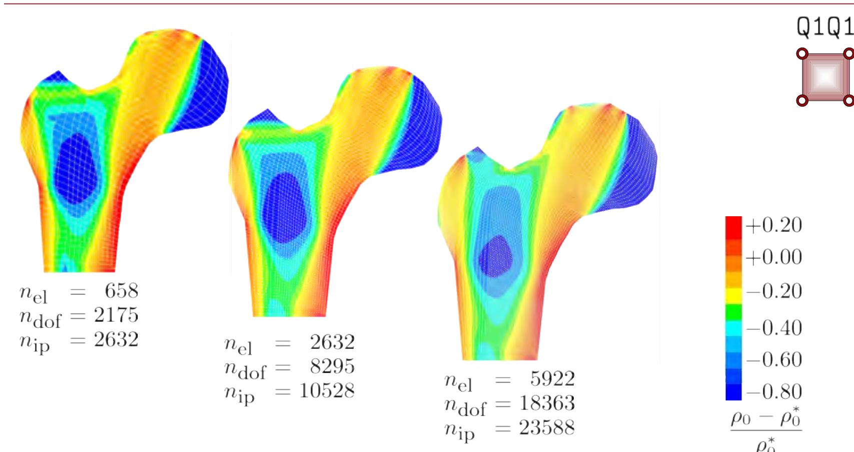


[1]	midstance phase of gait	2317 N	24°	703 N	28°
[2]	extreme range of abduction	1158 N	-15°	351 N	-8 °
[3]	extreme range of adduction	1548 N	56°	468 N	35°

carter & beaupré [2001]

example - adaptation in bone

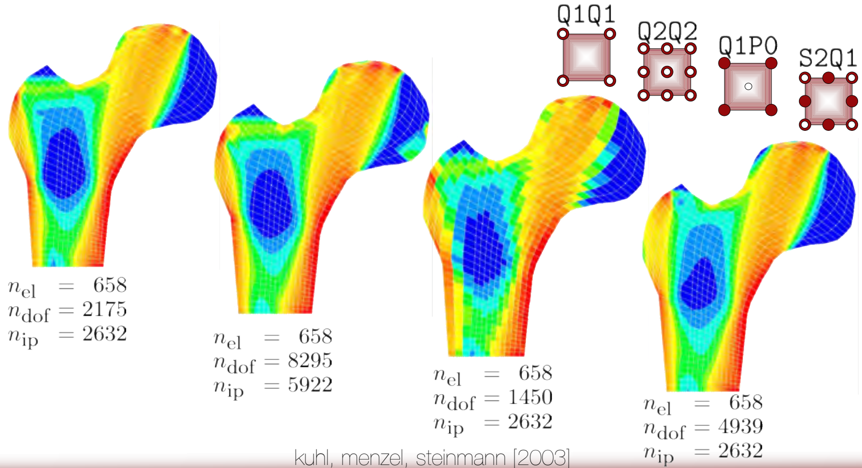
node point based - h-refinement



kuhl, menzel, steinmann [2003]

example - adaptation in bone

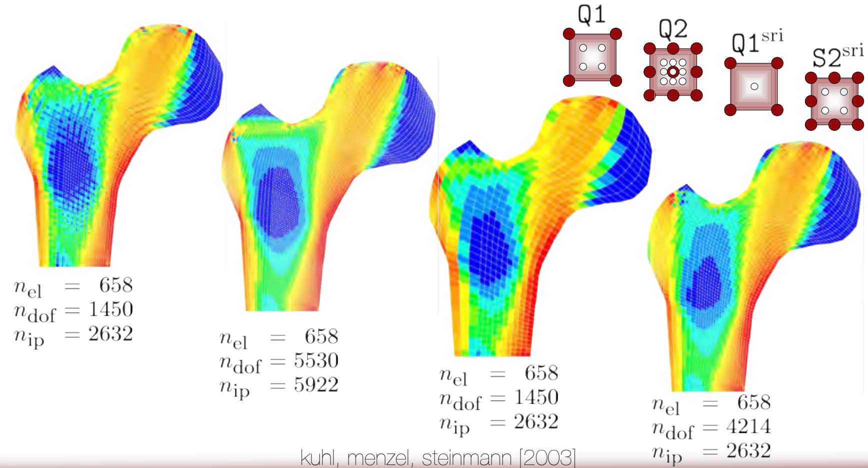
node point based - p-refinement



kuhl, menzel, steinmann [2003]

example - adaptation in bone

integration point based - p-refinement

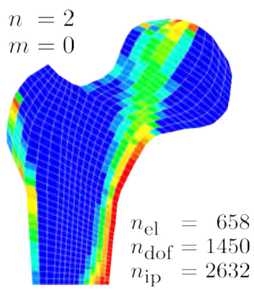


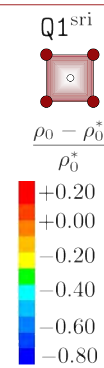
kuhl, menzel, steinmann [2003]

example - adaptation in bone

parameter sensitivity - instabilities

$$\psi_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^n \psi_0^{\text{neo}}(\boldsymbol{F}) \qquad \mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m} \psi_0(\boldsymbol{F}) - \psi_0^*$$





certain parameters induce checkerboard modes

harrigon & hamilton [1992],[1994]

example - adaptation in bone

total hip replacement vs hip resurfacing

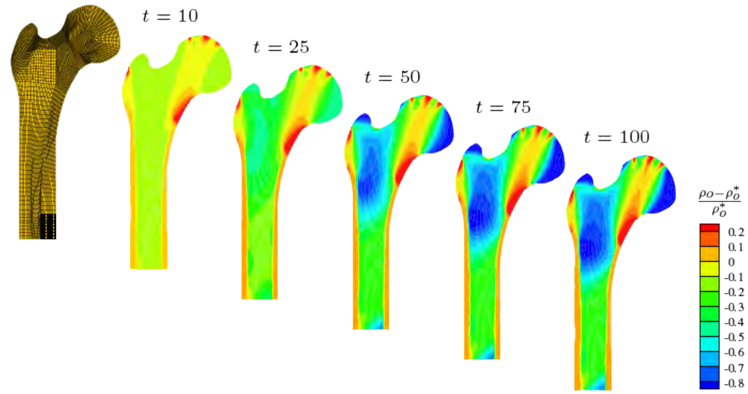
- about 120,000 artificial hip replacements in us per year
- aseptic loosening caused by adaptive bone remodeling
- goal prediction of dredisctribution of bone density

example - hip replacement

total hip replacement

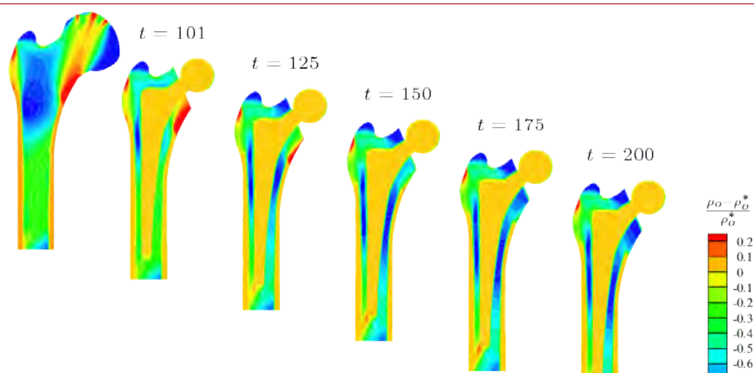
total hip replacement is a surgical procedure in which the hip joint is replaced by a prostetic implant. a total hip replacement consists of replacing both the acetabulum and the femoral head. hip replacement is currently the most successful and reliable orthopaedic operation. risks and complications include aseptic loosening, dislocation, and pain.in the long term, many problems relate to **bone resorption and subsequent loosening** or fracture often requiring revision surgery.

convential total hip replacement



ward's triangle • trabeculae • dense cortical shaft

convential total hip replacement



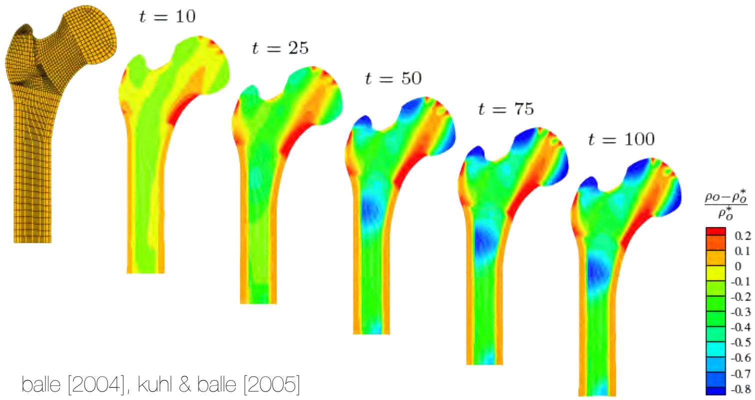
balle [2004], kuhl & balle [2005]

stress shielding • bone resorption • implant loosening

hip resurfacing

hip resurfacing is a surgical procedure which has been developed as an intervention alternative to total hip replacement. the potential advantages of hip resurfacing include **less bone removal**, a potentially lower number of hip dislocations due to a relatively larger femoral head size, and possibly easier revision surgery for a subsequent total hip replacement device. the potential disadvantages are femoral neck fractures, aspectic loosening, and metal wear.

new birmingham hip resurfacing

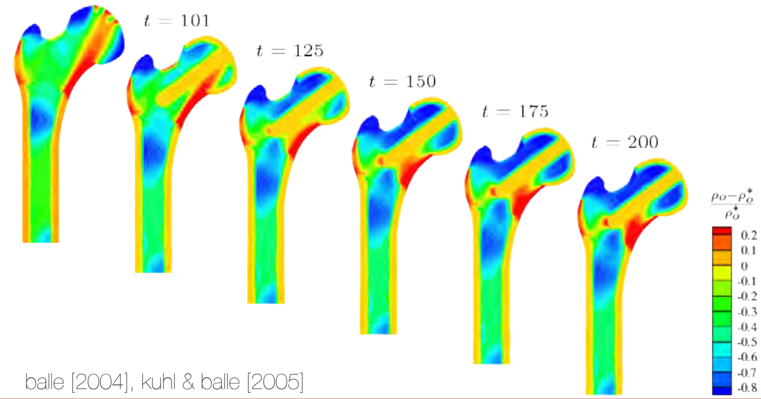


balle [2004], kuhl & balle [2005]

ward's triangle • trabeculae • dense cortical shaft

example - hip replacement

new birmingham hip resurfacing

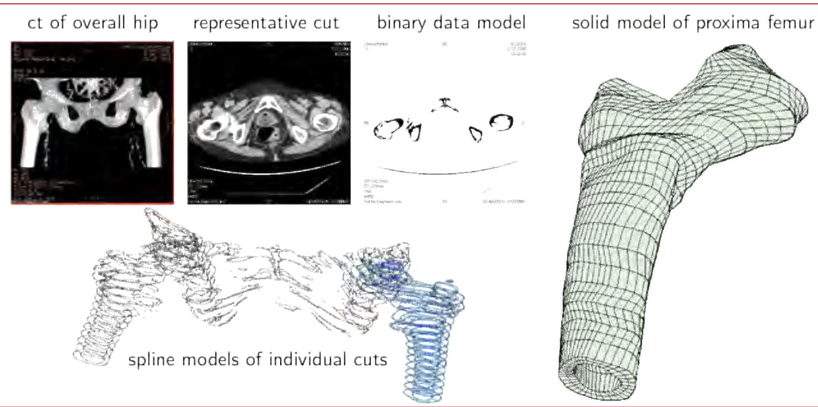


balle [2004], kuhl & balle [2005]

improved ingrowth • anatomic situation • less resorption

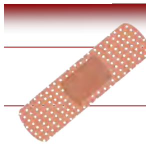
example - hip replacement

computer tomography of human femur

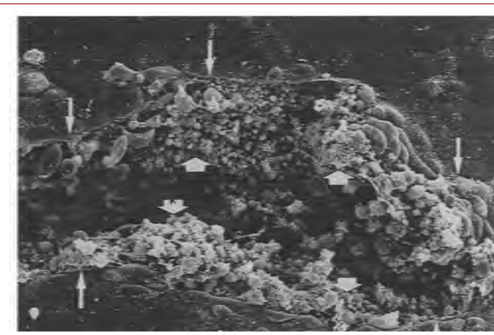


patient specific medical treatment

example - hip replacement



wound healing



murray [2003]

- epidermal migration / spreading of existing cells $\boldsymbol{R}$
- increase of mitotic activity of about15 times in 1mm wide zone @wound edge $\mathcal{R}_0$

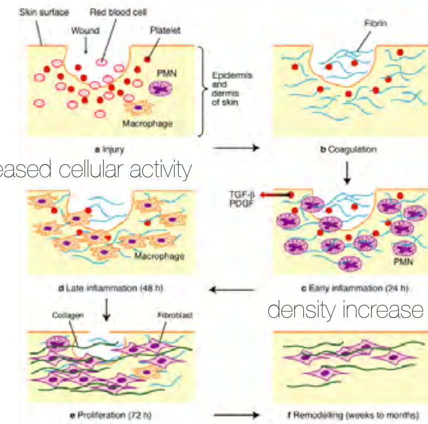
example - wound healing

wound healing

edwin smith surgical papyrus [13th dynasty]

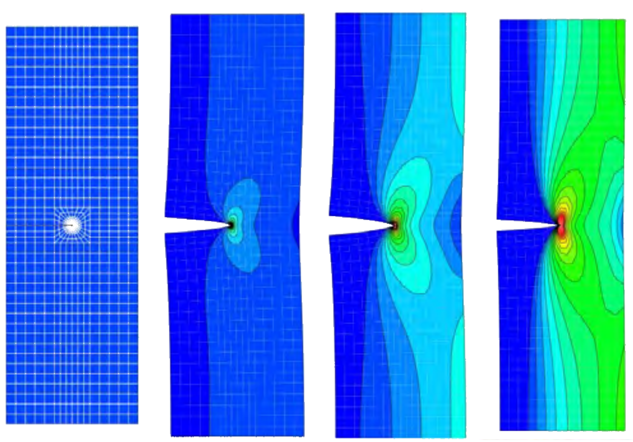
```
…thou shouldst bind it with fresh meet the first day,
and thou shouldst treat afterword with grease and honey
every day until he recovers…
```
breadsted [1930]

wound healing



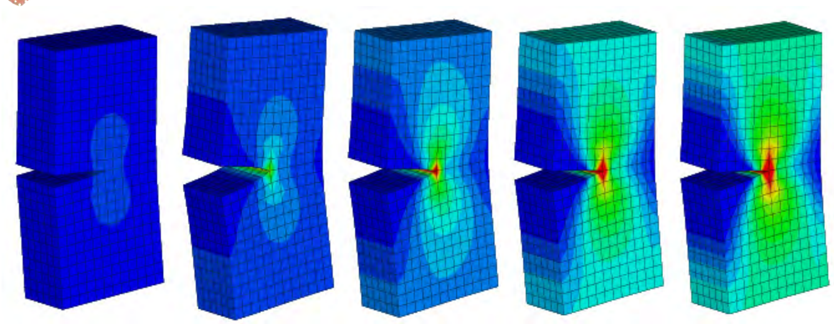
the phases of cutaneous wound healing. beanes, dang, soo, ting [2003]

tension - single edge notched specimen



kuhl, steinmann [2004]

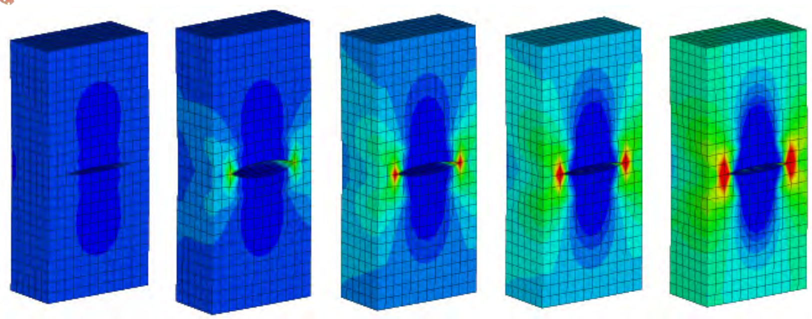
tension - single edge notched specimen



increased cell activity @wound edge

kuhl, steinmann [2004]

tension - surface cut specimen

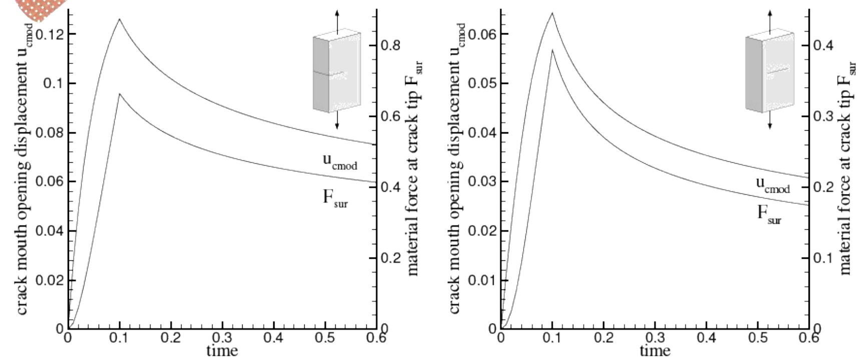


increased cell activity @wound edge

kuhl, steinmann [2004]

example - wound healing

tension - crack mouth opening displacement

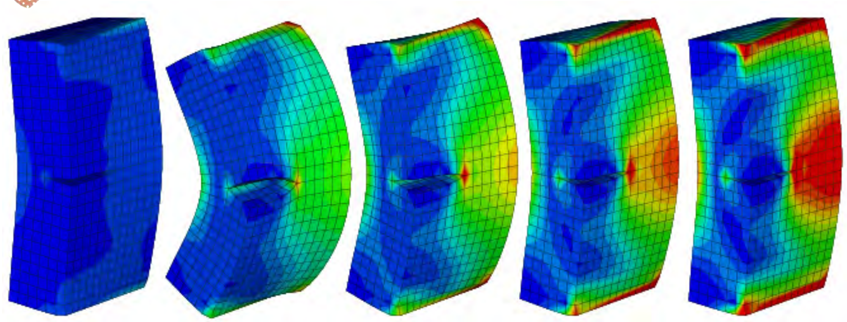


increased cell activity - wound healing & closure

kuhl, steinmann [2004]

example - wound healing

bending - single edge notched specimen

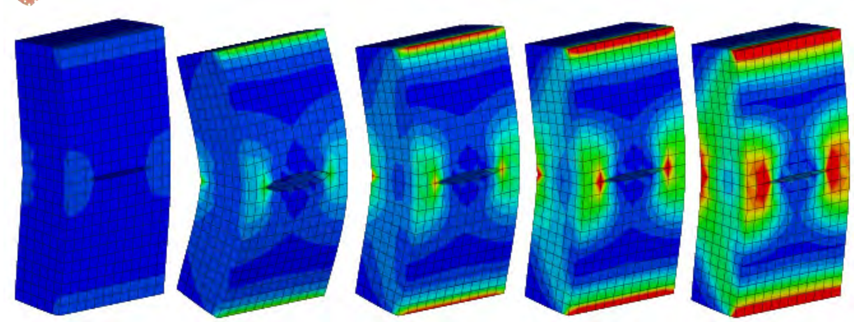


increased cell activity @wound edge

kuhl, steinmann [2004]

example - wound healing

bending - surface cut specimen

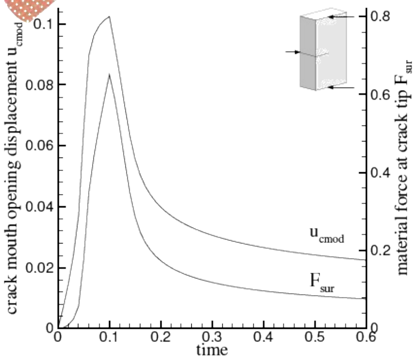


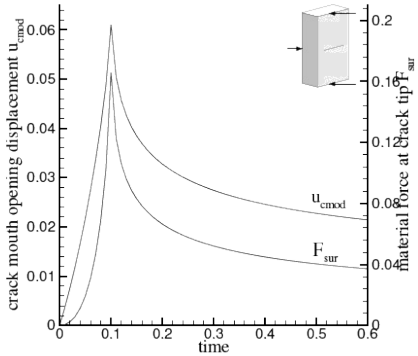
increased cell activity @wound edge

kuhl, steinmann [2004]

example - wound healing

bending - crack mouth opening displacement





increased cell activity - wound healing & closure

kuhl, steinmann [2004]

example - wound healing