13 - basic constitutive equations
- growing bones

osteoporosis

osteoporosis is a disease of bones that leads to an increased risk of fracture. Osteoporosis literally means porous bones. In osteoporosis the bone mineral density is reduced, bone microarchitecture is disrupted, and the amount of variety of proteins in bone is altered. The diagnosis of osteoporosis can be made using conventional radiography. Bone mineral density can be measured by dual energy X-ray absorptiometry, DXA or DEXA. Osteoporosis can be prevented with lifestyle changes and sometimes medication. Lifestyle changes include exercise and preventing falls as well as reducing protein intake which may cause calcium to be taken from the bones.

motivation - growing bone

where are we???

history - 19th century

motivation - growing bone
“...es ist demnach unter dem gesetz der transformation der knochen dasjenige gesetz zu verstehen, nach welchem im folge primärer abänderungen der form und inanspruchnahme bestimmte umwandlungen der inneren architektur und umwandlungen der äusseren form sich vollziehen...”

Wolff "gesetz der transformation der knochen" [1892]

**history - 19th century**

---

**different load cases**

<table>
<thead>
<tr>
<th>Case Description</th>
<th>Force (N)</th>
<th>Angle</th>
<th>Force (N)</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 midstance phase of gait</td>
<td>2317</td>
<td>24°</td>
<td>703</td>
<td>28°</td>
</tr>
<tr>
<td>2 extreme range of abduction</td>
<td>1158</td>
<td>-15°</td>
<td>351</td>
<td>-8°</td>
</tr>
<tr>
<td>3 extreme range of adduction</td>
<td>1548</td>
<td>55°</td>
<td>468</td>
<td>35°</td>
</tr>
</tbody>
</table>

carter & beaupré [2001]

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**motivation - growing bone**

**experiment vs simulation**

- dense system of compressive trabaculae carrying stress into calcar region
- secondary arcuate system, medial joint surface to lateral metaphyseal region
- ward’s triangle, low density region contrasting dense cortical shaft

carter & beaupré [2001]
class project - me337 - mechanics of growth

rebecca e. taylor

neuromuscular biomechanics lab

scott delp

kate holzbaur

joseph c. doll

human performance lab

amir shamloo

ryan p. jackson

chun hua zheng

nathaniel a. benz

thor bezier

inter-arm asymmetry in high performance tennis players

example - twisted tennis arm density

example - twisted tennis arm density
Critical muscle forces during serve

- Deltoid
- Latissimus dorsi
- Pectoralis major
- Triceps

Phase (I): Max ext rotation

Phase (III): Ball impact

Example - Twisted tennis arm density

Inter-arm asymmetry in bone density

Thor Besier - Human Performance Lab - Stanford

Example - Twisted tennis arm density

Inter-arm asymmetry in bone density

Example - Twisted tennis arm density

Phase (II) Maximum external shoulder rotation

$t = 10 \Delta t$
$t = 24 \Delta t$
$t = 48 \Delta t$
$t = 64 \Delta t$

BMD left 1.107g/mm²
BMD right 1.369g/mm²

Example - Twisted tennis arm density
A physical-conditioning program for pitchers is geared to striking a balance between muscle strength and endurance, tendon/ligament strength and flexibility, and optimal cartilage and **bone density**. Bone hypertrophy occurs in response to physical activity. The bones in the throwing arm of a baseball pitcher are **denser and thicker** than those of the other arm. Bone hypertrophy is **stimulated by the magnitude of loading** rather than by the frequency.

**The real secret** to Tim Lincecum’s overpowering velocity is all **stored within his pitching mechanics**. It has little to do with his size and strength. Six things in Tim Lincecum’s pitching delivery create his amazing arm speed:

- move fast from back leg to front leg
- use back leg to move out very low to ground
- get throwing arm up very late in delivery
- stride length of over 100% of pitcher’s height
- brace front leg to increase upper body speed
- land in a straight line toward plate

Tim Lincecum’s pitching mechanics, because he moves fast into a long stride and stays low, forces his body to put as many muscles on stretch as quickly as possible, which helps develop maximum elastic energy so that his body acts like a huge rubber band stretching to its maximum length ready to be let go and whip the arm through.
pitcher’s arm

maximal external shoulder rotation stimulates twisted density growth

taylor, Zheng, Jackson, Doll, Chen, Holzbaur, Besier, Kuhl [2009]

example - pitcher’s arm

neo hooke’ian elasticity of solid materials

undeformed potato

• free energy
  $\psi_{\text{neo}} = \frac{1}{2} \lambda_0 \ln^2(\det(F)) + \frac{1}{2} \mu_0 [F^t \cdot F : I - n^\text{dim} - 2 \ln(\det(F))]$

• definition of stress
  $P_{\text{neo}} = D_F \psi_{\text{neo}}$

  $\mu_0 F + [\lambda_0 \ln(\det(F)) - \mu_0]F^{-1}$

undeformed potato

• definition of tangent operator
  $A_{\text{neo}} = D_F F^{-1} P_{\text{neo}}$

  $\lambda_0 F^{-1} \otimes F^{-1} + \mu_0 \mathbb{I} \otimes \mathbb{I}$

  $+[\mu_0 - \lambda_0 \ln(\det(F))] F^{-1} \otimes F^{-1}$

deformed potato

constitutive equations

undeformed potato

constitutive equations

undeformed potato

constitutive equations

undeformed potato

constitutive equations

undeformed potato

constitutive equations

undeformed potato

constitutive equations

undeformed potato
consitutive equations

- free energy
  \[ \psi_0^{\text{neo}} = \frac{1}{2} \lambda_0 \ln^2(\det(F_{ij})) + \frac{1}{2} \mu_0 \ln \left( F_{ij} F_{ij}^T - I \right) + 2 \ln(\det(F_{ij})) \]
- definition of stress
  \[ F_{ij}^{\text{neo}} = D_{ij,kl}^F \psi_0^{\text{neo}} = \frac{1}{2} \lambda_0 \ln^2(\det(F_{ij})) + \frac{1}{2} \mu_0 \ln(\det(F_{ij})) - \mu_0 F_{ij}^2 \]
- definition of tangent operator
  \[ A_{ijkl}^{\text{neo}} = D_{ijkl}^F \psi_0^{\text{neo}} = D_{ijkl} F_{ij}^{\text{neo}} \]

open systems - balance of mass

- mass flux \( \rho_0 = \text{Div}(R) + R_0 \)
  - cell movement (migration)
- mass source \( R_0 \)
  - cell growth (proliferation)
  - cell division (hyperplasia)
  - cell enlargement (hypertrophy)

biological equilibrium


balance equations

mechanical equilibrium

- volume specific version
  \[ \rho_0 \text{Div}(v) = \text{Div}(P + v \otimes R) + [b_0 + v R_0 - \nabla_x v \cdot R] \]
- subraction of weighted balance of mass
  \[ v \text{Div}(\rho_0) = \text{Div}(v \otimes R) + v R_0 - \nabla_x v \cdot R \]
- mass specific version
  \[ \rho_0 \text{Div}(v) = \text{Div}(P) + b_0 \]
density growth at constant volume

- free energy: \( \psi_c = \left( \frac{\rho_0}{\rho_0^*} \right)^m \psi_0^{\text{neo}}(\mathbf{F}) \)
- stress: \( P = \left( \frac{\rho_0}{\rho_0^*} \right)^m P^{\text{neo}}(\mathbf{F}) \)
- mass flux: \( R = R_0 \nabla X \rho_0 \)
- mass source: \( \mathcal{R}_0 = \left[ \frac{\rho_0}{\rho_0^*} \right]^{-m} \psi_0(\mathbf{F}) - \psi_0^* \)

constitutive coupling of growth and deformation

Gibson & Ashby [1999]

constitutive equations

Density growth - mass source

\[ D_t \rho_0 = \mathcal{R}_0 \]

1d model problem: \( \mathcal{R}_0 = \left[ \frac{\rho_0}{\rho_0^*} \right]^{-m} \psi_0 - \psi_0^* \)

\( f, u \) \( \uparrow \)

\( E = 1 \) \( \nu = 0 \) \( \rho_0^* = 1 \) \( \psi_0^* = 1 \)

Gradually increased workout

Density growth - mass source

\[ D_t \rho_0 = \mathcal{R}_0 \]

How does the mass source control growth?

Gradually increased workout

Constative equations

Driving force for density growth - why does bone grow?

\[ R_0 = \left[ \frac{\rho_0}{\rho_0^*} \right]^{-m} \psi_0(\mathbf{F}) - \psi_0^* \]

Energy driven

Mechanotransduction

Bending of primary cilia

Shear stress driven

Malone, Anderson, Tummala, Kwon, Johnston, Stearns, Jacobs [2007]

Constative equations

Increasing forces causes density increase

Constative equations
density growth - mass source

\[ \frac{D_t \rho_0}{\rho_0} = \mathcal{R}_0 \]

\[ f \xrightarrow{\mathcal{R}_0} \left[ \frac{\rho_0}{\rho_0} \right]^{\frac{m}{2}} \psi_0 - \psi_0^* \]

first time interval \( f = 0.5 \text{N} \)

- \( u = 1.5910 \text{i} \)
- \( \varphi = X + u = 2.5910 \text{i} \)
- \( F = \frac{\partial \varphi}{\partial X} = \frac{2.5910 \text{i}}{2} = 2.5910 \text{homog.} \)
- \( E = 1 \)
- \( \lambda_0 = \frac{E}{\nu} = 1 \)
- \( \nu = 0 \)
- \( \mu_0 = \frac{E}{2(1+\nu)} = 0.5 \)

\[ \psi_0 = \left[ \frac{\rho_0}{\rho_0} \right]^{\frac{m}{4}} \mu_0 \text{ for } F^2 - 1 - 2 \ln(F) \]

increasing force causes energy increase

convergence towards biological equilibrium \( \frac{D_t \rho_0}{\rho_0} = 0 \)

constitutive equations

parameter sensitivity wrt \( n, m, \psi_0^* \)

parameter insensitivity wrt \( \rho_0, \Delta t \)

constitutive equations
the density develops such that the tissue can just support the given mechanical load.

**constitutive equations**

\[ D_t \rho_0 = \nabla \cdot R \]

\[ R = R_0 \nabla_X \rho_0 \]

Initial hat type density distribution

**density growth - mass source**

\[ D_t \rho_0 = R_0 \]

\[ R_0 = \left[ \frac{\rho_0}{\rho_b} \right]^{-m} \psi_0 - \psi^* \]

**density growth - mass flux**

\[ D_t \rho_0 = \nabla \cdot R \]

\[ R = R_0 \nabla_X \rho_0 \]

Initial hat type density distribution

whatz this mass flux good for in the end?

**equilibration of concentrations**

\[ R = R_0 \nabla_X \rho_0 \]
density growth - mass flux & source

\[ D_t \rho_0 = \text{Div}(R) + R_c \]

\[ R = 1.0000 \]

\[ R = 0.1000 \]

\[ R = 0.0100 \]

\[ R = 0.0010 \]

\[ R = 0.0001 \]

\[ R = 0.0000 \]

\[ R = R_0 \nabla_x \rho_0 \quad R_c = \left( \frac{\rho_0}{\rho_0} \right)^m \psi_0 - \psi_0^* \]

smoothing influence of mass flux

constitutive equations

human spaceflight to Mars could become a reality within the next 25 years, but not until some physiological problems are solved, including an alarming loss of bone mass, fitness and muscle strength. The rate at which we lose bone in space is 10-15 times greater than that of a post-menopausal woman and there is no evidence that bone loss ever slows in space. Further, it is not clear that space travelers will regain that bone on returning to gravity. During a trip to Mars, lasting between 13 and 30 months, unchecked bone loss could make an astronaut's skeleton the equivalent of a 100-year-old person.

http://www.acsm.org

deck growth - bone loss in space

nasa has collected data that humans in space lose bone mass at a rate of \( c = 1.5\% / \text{month} \). so far, no astronauts have been in space for more than 14 months but the predicted rate of bone loss seems constant in time. this could be a severe problem if we want to send astronauts on a 3 year trip to Mars and back. How long could an astronaut survive in a zero-g environment if we assume the critical bone density to be \( \rho_{0}^{\text{crit}} = 1.00 \text{ g/cm}^3 \)? you can assume an initial density of \( \rho_0 = 1.79 \text{ g/cm}^3 \).

example - bone loss in space

\[ D_t \rho_0 = c \rho_0^m \left[ \frac{\psi_0}{\psi_0^*} - 1 \right] \quad D_t \rho_0 = \frac{1}{\Delta t} \left[ \rho_{0}^{n+1} - \rho_0^n \right] \]

\[ \rho_{0}^{n+1} = \rho_0^n + c \rho_0^n \left[ \frac{\psi_0}{\psi_0^*} - 1 \right] \Delta t \quad \rho_0(t_0) = 1.79 \text{ g/cm}^3 \]

\[ \psi_0 = \psi_{0}^\text{mars} = 0.38 \rho_{0}^* \quad \rho_0(36) = 1.0098 \]

\[ \rho_0(37) = 0.9947 \]

\[ \psi_0 = \psi_{0}^\text{space} = 0.00 \rho_{0}^* \]

example - bone loss in space
E = 3.790 \rho_0^3 \text{ MPa} \quad \text{with } \rho_0 \text{ in g/cm}^3

carter & hayes [1977]