09 - finite element method - volume growth - theory

09 - finite element method

day	date		topic
tue	jan	10	motivation - everything grows!
thu	jan	12	basics maths - notation and tensors
tue	jan	17	basic kinematics - large deformation and growth
thu	jan	19	kinematics - growing hearts
tue	jan	24	guest lecture - growing skin
thu	jan	26	guest lecture - growing leaflets
tue	jan	31	basic balance equations - closed and open systems
thu	feb	02	basic constitutive equations - growing tumors
tue	feb	07	volume growth - finite elements for growth
thu	feb	09	volume growth - growing arteries
tue	feb	14	volume growth - growing skin
thu	feb	16	volume growth - growing hearts
tue	feb	21	basic constitutive equations - growing bones
thu	feb	23	density growth - finite elements for growth
tue	feb	28	density growth - growing bones
thu	mar	01	everything grows! - midterm summary
tue	mar	06	midterm
thu	mar	08	remodeling - remodeling arteries and tendons
tue	mar	13	class project - discussion, presentation, evaluation
thu	mar	15	class project - discussion, presentation, evaluation
thu	mar	15	written part of final projects due

where are we???

figure 1. surface growth of a narwhal tusk. photograph of a narwhal tusk, left, demonstrates the characteristic helical growth pattern. computational simulation of surface growth, right, with an outward pointing velocity of the growth surface, here characterized through the bottom ring, and a helically upward pointing velocity of material grown at the surface.. menzel & kuhl [2012]

example 01: surface growth of horns

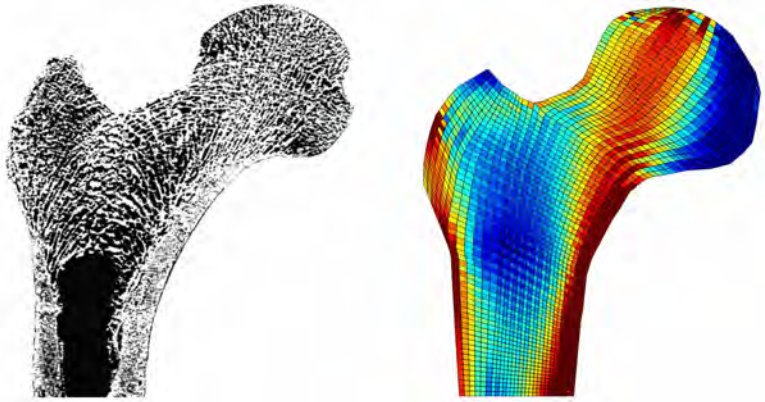


figure 2. density growth of the proximal femur for an energy-driven mass source. photograph of a thin section, left, demonstrates microstructural arrangement of trabeculae in the femur head aligned with the axis of maximum principal stress. computational simulation of density growth, right, predicts higher bone densities in regions of large mechanical stress and lower bone densities in unloaded regions. menzel & kuhl [2012]

example 02: density growth of bone

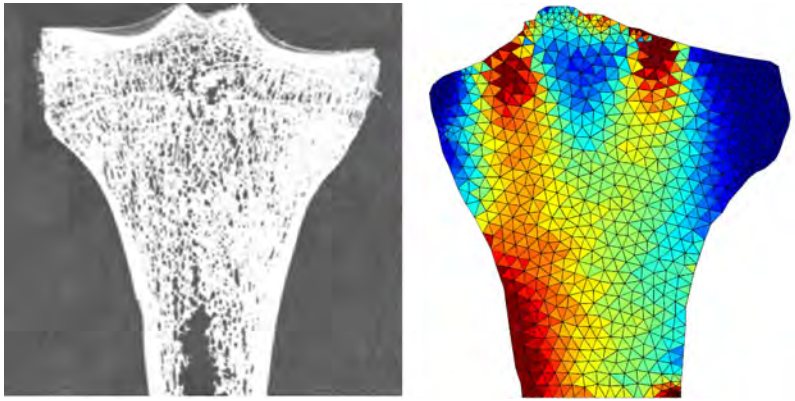


figure 3. density growth of the proximal tibia for an energy-driven mass source. photograph of a thin section, left, displays microstructural arrangement of trabeculae in the tibia head aligned with the axis of maximum principal stress. computational simulation of density growth, right, predicts a higher bone density in regions of large mechanical stress and lower bone density in unloaded regions. menzel & kuhl [2012]

example 03: density growth of bone

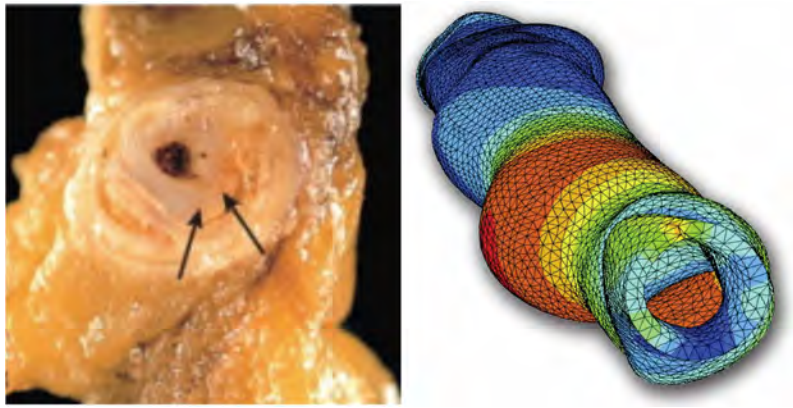


figure 4. volume growth of an artery for stress-driven isotropic growth. photograph of restenosis following balloon angioplasty, left, demonstrating residual atherosclerotic plaque and a new proliferative lesion caused by intimal thickening. computational simulation of isotropic volume growth, right, predicts wall thickening and re-narrowing of the lumen in response to changes in the mechanical environment. menzel & kuhl [2012]

example 04: volume growth of arteries

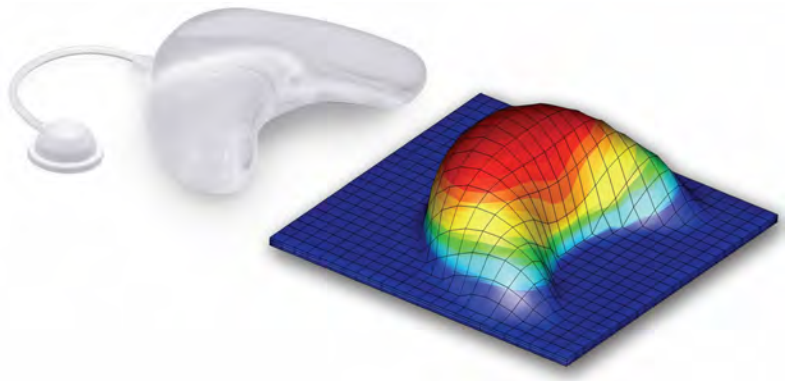


figure 5. area growth of skin for stretch-driven transversely isotropic growth. photograph of a tissue expander to induce controlled in situ skin growth for defect correction in reconstructive surgery, left, computational simulation of transversely isotropic area growth, right,predicts area growth in response to controlled mechanical overstretch during tissue expansion. menzel & kuhl [2012]

example 05: area growth of skin

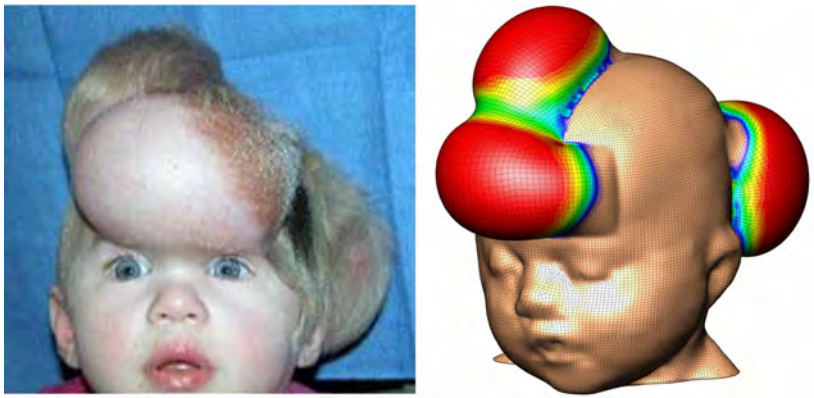


figure 6. area growth of skin for stretch-driven transversely isotropic growth. photograph of tissue expansion in pediatric forehead reconstruction, left, shows forehead, anterior and posterior scalp expansion to trigger skin growth in situ. computational simulation of transversely isotropic area growth, right, predicts area growth in response to controlled mechanical overstretch during tissue expansion menzel & kuhl [2012]

example 06: area growth of skin

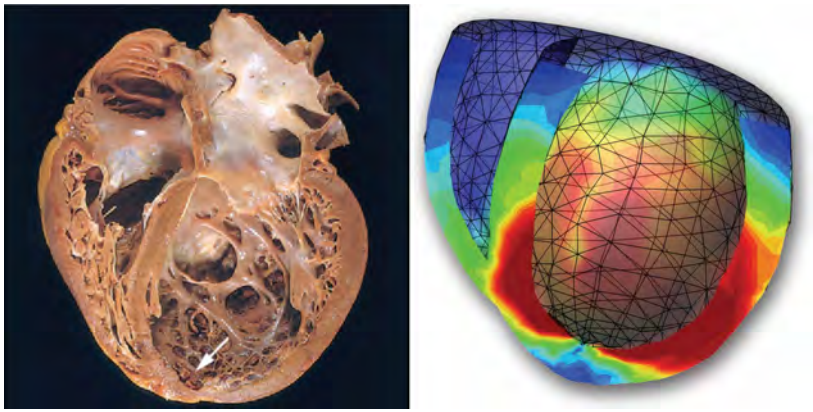


figure 7. fiber growth of the heart for strain-driven transversely isotropic growth. photograph of a heart in dilated cardiomyopathy, left, illustrates an increase in cavity size at a constant wall thickness associated with volume-overload induced eccentric growth. computational simulation of transversely isotropic fiber growth, right, predicts an enlargement of the left ventricular cavity in response to overstretch. menzel & kuhl [2012]

example 07: fiber growth of the heart

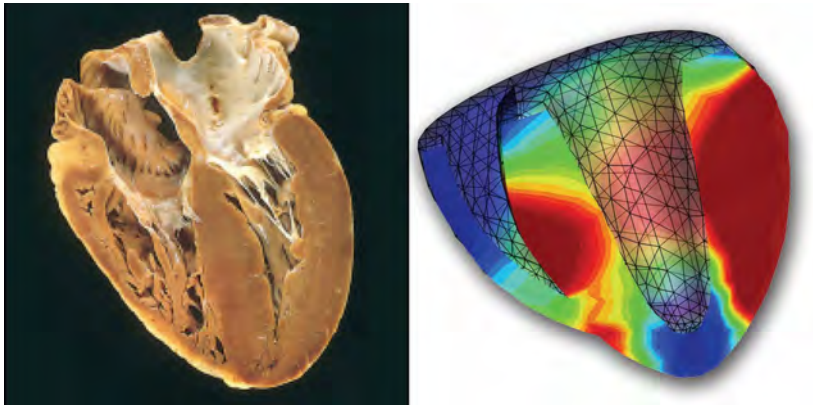


figure 8. cross-fiber growth of the heart for stress-driven wall thickening. photograph of a heart in hypertrophic cardiomyopathy, left, illustrates an increase in wall thickness at a constant cardiac size, typically associated with pressure-overload induced concentric growth. computational simulation of transversely isotropic cross-fiber growth, right, predicts a significant wall thickening in response to hypertension. menzel & kuhl [2012]

example 08: cross-fiber growth of the heart

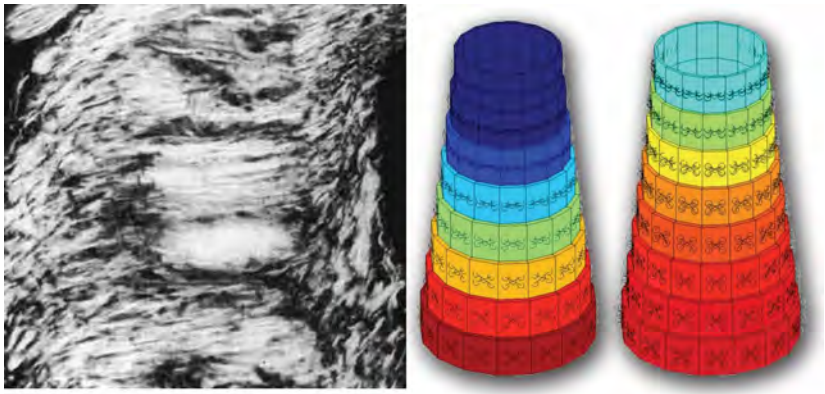


figure 9. collagen remodeling in arteries. polarized light micrograph of tangentially sectioned brain artery, left, showing variation of collagen fiber orientation from circumferential inner to helical outer layer. computational simulation of stress- and strain-driven fiber distribution, left and right, predicts a smooth variation of collagen fiber orientation from circumferential inner to helical outer layer. menzel & kuhl [2012]

example 09: collagen remodeling in arteries

recipe for finite element modeling

from continuous problem…

$$\rho_0 \, \mathrm{D}_t \boldsymbol{v} \quad = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0$$

- temporal discretization — implicit euler backward
- spatial discretization — finite element method
- staggered/simultaneous — newton raphson iteration
- linearization — gateaux derivative

… to linearized discrete initial boundary value problem

key transformation - from strong form to weak form (1d)

- strong / differential form

$$\sum f = f^{\text{int}} + f^{\text{ext}} \doteq 0 \qquad f^{\text{int}} = P'(\varphi)$$

- strong form / residual format

$$\mathsf{R}(\varphi) = P'(\varphi) + f^{\text{ext}} \doteq 0$$

- weak / integral form - nonsymmetric $\forall \delta\varphi$

$$\mathsf{G}(\delta\varphi;\varphi) = \int \delta\varphi \cdot [\, P'(\varphi) + f^{\text{ext}} \,]\mathrm{d}x \doteq 0$$

- integration by parts

$$\int \delta\varphi \cdot P' \mathrm{d}x = \int [\delta\varphi \cdot P]' \mathrm{d}x - \int \delta\varphi' \cdot P \mathrm{d}x$$

- integral theorem & neumann bc's

$$\int [\delta\varphi \cdot P]' \mathrm{d}x = \delta\varphi \cdot P|_{x=0}^{x=l}$$

- weak form / integral form - symmetric $\forall \delta\varphi$

$$\int \delta\varphi' \cdot P \mathrm{d}x - \delta\varphi \cdot P|_{x=0}^{x=l} - \int \delta\varphi \cdot f^{\text{ext}} \doteq 0$$

from equilibrium equation…

- start with nonlinear mechanical equilibrium equation

$$\rho_0 \underbrace{\mathrm{D}_t \boldsymbol{v}}_{\approx \mathbf{0} \text{ quasi-static}} = \mathrm{Div}(\boldsymbol{P}) + \underbrace{\boldsymbol{b}_0}_{\approx \mathbf{0} \text{ no gravity}}$$

- cast it into its residual format

$$\mathbf{R}^{\varphi}(\boldsymbol{\varphi}) = \mathbf{0} \qquad \text{in } \mathcal{B}_0$$

- with residual

$$\mathbf{R}^{\varphi} = \rho_0 \mathrm{D}_t \boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0$$

… to residual format

residual equation…

- strong / differential form

$$\mathbf{R}^{\varphi} = \rho_0 \mathrm{D}_t \boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 = \mathbf{0} \quad \text{in } \mathcal{B}_0$$

- dirichlet / essential boundary conditions (displacements)

$$\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}} = \mathbf{0} \quad \text{on } \partial\mathcal{B}_0^{\varphi} \quad \text{with } \partial\mathcal{B}_0^{\varphi} \cup \partial\mathcal{B}_0^{T\varphi} = \partial\mathcal{B}_0$$

- neumann / natural boundary conditions (tractions)

$$\boldsymbol{P} \cdot \boldsymbol{N} - \bar{\boldsymbol{T}}^{\varphi} = \mathbf{0} \quad \text{on } \partial\mathcal{B}_0^{T\varphi} \quad \text{and } \partial\mathcal{B}_0^{\varphi} \cap \partial\mathcal{B}_0^{T\varphi} = \emptyset$$

… and boundary conditions

from strong form…

- strong / differential form

$$\mathbf{R}^{\varphi} = \rho_0 \mathrm{D}_t \boldsymbol{v} - \mathrm{Div}(\boldsymbol{P}) - \boldsymbol{b}_0 = \mathbf{0} \qquad \text{in } \mathcal{B}_0$$

- mulitplication with test function & integration

$$\mathsf{G}^{\varphi}(\delta\boldsymbol{\varphi};\boldsymbol{\varphi}) = \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \mathbf{R}^{\varphi} \mathrm{d}V = 0 \qquad \forall \delta\boldsymbol{\varphi} \text{ in } \mathcal{H}_1^0(\mathcal{B}_0)$$

- weak form / nonsymmetric

$$\mathsf{G}^{\varphi} = \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \rho_0 \mathrm{D}_t \boldsymbol{v} \,\mathrm{d}V - \int_{\mathcal{B}_0} \underbrace{\delta\boldsymbol{\varphi}}_{\text{no derivative}} \cdot \underbrace{\mathrm{Div}(\boldsymbol{P})}_{\text{second derivative}} \mathrm{d}V - \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \boldsymbol{b}_0 \mathrm{d}V$$

… to nonsymmetric weak form

from non-symmetric weak form…

- integration by parts

$$\int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\mathrm{Div}(\boldsymbol{P})\mathrm{d}V=\int_{\mathcal{B}_0}\mathrm{Div}(\delta\boldsymbol{\varphi}\cdot\boldsymbol{P})\mathrm{d}V-\int_{\mathcal{B}_0}\nabla\delta\boldsymbol{\varphi}:\boldsymbol{P}\mathrm{d}V$$

- gauss theorem & boundary conditions

$$\int_{\mathcal{B}_0}\mathrm{Div}(\delta\boldsymbol{\varphi}\cdot\boldsymbol{P})\mathrm{d}V=\int_{\partial\mathcal{B}_0^{T^\varphi}}\boldsymbol{\varphi}\cdot\boldsymbol{P}\cdot\boldsymbol{N}\mathrm{d}A=\int_{\partial\mathcal{B}_0^{T^\varphi}}\delta\boldsymbol{\varphi}\cdot\bar{\boldsymbol{T}}^\varphi\mathrm{d}A$$

- weak form / symmetric

first derivative, first derivative

$$\mathsf{G}^\varphi=\int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\rho_0\,\mathrm{D}_t\boldsymbol{v}\,\mathrm{d}V+\int_{\mathcal{B}_0}\nabla\delta\boldsymbol{\varphi}:\boldsymbol{P}\mathrm{d}V-\int_{\partial\mathcal{B}_0^{T^\varphi}}\delta\boldsymbol{\varphi}\bar{\boldsymbol{T}}^\varphi\mathrm{d}A-\int_{\mathcal{B}_0}\delta\boldsymbol{\varphi}\cdot\boldsymbol{b}_0\mathrm{d}V$$

… to symmetric weak form

spatial discretization

- discretization $\mathcal{B}_0=\bigcup_{e=1}^{n_{\mathrm{el}}}\mathcal{B}_0^e$

- interpolation of test functions

$$\delta\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e}=\sum_{j=1}^{n_{\mathrm{en}}}N_\varphi^j\delta\boldsymbol{\varphi}_j\in\mathcal{H}_1^0(\mathcal{B}_0)\quad\nabla\delta\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e}=\sum_{j=1}^{n_{\mathrm{en}}}\delta\boldsymbol{\varphi}_j\otimes\nabla N_\varphi^j$$

- interpolation of trial functions

$$\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e}=\sum_{l=1}^{n_{\mathrm{en}}}N_\varphi^l\boldsymbol{\varphi}_l\in\mathcal{H}_1(\mathcal{B}_0)\qquad\nabla\boldsymbol{\varphi}^h|_{\mathcal{B}_0^e}=\sum_{l=1}^{n_{\mathrm{en}}}\boldsymbol{\varphi}_l\otimes\nabla N_\varphi^l$$

… to discrete weak form

from discrete weak form…

- discrete weak form

$$\mathsf{G}^\varphi=\delta\boldsymbol{\varphi}_J\cdot\mathbf{R}_j^\varphi(\boldsymbol{\varphi}_{n+1}^h)=0\qquad\forall\delta\boldsymbol{\varphi}_J$$

- discrete residual format

$$\mathbf{R}_J^\varphi(\boldsymbol{\varphi}_{n+1}^h)=\mathbf{0}\qquad\forall\ J=1,\ldots,n_{\mathrm{np}}$$

- discrete residual

$$\mathbf{R}_J^\varphi=\mathbf{A}_{e=1}^{n_{\mathrm{el}}}\int_{\mathcal{B}_0^e}N_\varphi^j\mathrm{D}_t\boldsymbol{v}_{n+1}\mathrm{d}V+\int_{\mathcal{B}_0^e}\nabla N_\varphi^j\cdot\boldsymbol{P}_{n+1}\mathrm{d}V-\int_{\partial\mathcal{B}_0^e}N_\varphi^j\bar{\boldsymbol{T}}_{n+1}^\varphi\mathrm{d}A-\int_{\mathcal{B}_0^e}N_\varphi^j\boldsymbol{b}_{0\,n+1}\mathrm{d}V$$

… to discrete residual

discrete residual

- discrete residual

check in matlab!

$$\mathbf{R}_J^\varphi=\mathbf{A}_{e=1}^{n_{\mathrm{el}}}\int_{\mathcal{B}_0^e}\nabla N_\varphi^j\cdot\boldsymbol{P}_{n+1}\mathrm{d}V$$

- residual of mechanical equilibrium/balance of momentum

righthand side vector for global system of equations

from discrete residual ...

- linearization / newton raphson scheme

$$\mathbf{R}^{\varphi k+1}_{Jn+1} = \mathbf{R}^{\varphi k}_{Jn+1} + \mathrm{d}\mathbf{R}^{\varphi}_{J} \doteq 0 \quad \forall\, J = 1, ..., n_{\mathrm{np}}$$

- incremental residual

$$\mathrm{d}\mathbf{R}^{\varphi}_{J} = \sum_{L=1}^{n_{\mathrm{en}}} \mathbf{K}^{\varphi\varphi}_{JL} \cdot \mathrm{d}\boldsymbol{\varphi}_L \qquad \mathbf{K}^{\varphi\varphi}_{JL} = \frac{\mathrm{d}\mathbf{R}^{\varphi}_{J}}{\mathrm{d}\boldsymbol{\varphi}_L}$$

- system of equations

$$\mathbf{K}^{\varphi\varphi}_{JL}\, \mathrm{d}\boldsymbol{\varphi}_L = -\mathbf{R}^{\varphi\, k}_{J\, n+1}$$

- incremental iterative update

$$\Delta\boldsymbol{\varphi}_L = \Delta\boldsymbol{\varphi}_L + \mathrm{d}\boldsymbol{\varphi}_L \qquad \forall\, L = 1, ..., n_{\mathrm{np}}$$

... to linearized residual

linearized residual

- stiffness matrix / iteration matrix

$$\mathbf{K}^{\varphi\varphi}_{JL} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}^e_0} N^j_\varphi \rho \mathrm{D}_\varphi(\mathrm{D}_t \boldsymbol{v}) N^l_\varphi \mathrm{d}V + \int_{\mathcal{B}^e_0} \nabla N^j_\varphi \cdot \mathrm{D}_F \boldsymbol{P} \cdot \nabla N^l_\varphi \mathrm{d}V$$

4th order tensor - derivatives of 2nd order tensors wrt 2nd order tensor

- linearization of residual wrt nodal dofs

iteration matrix for global system of equations

linearized residual

- stiffness matrix / iteration matrix

check in matlab!

$$\mathbf{K}^{\varphi\varphi}_{JL} = \frac{\partial \mathbf{R}^{\varphi}_{J}}{\partial \boldsymbol{\varphi}_L} = \mathbf{A}_{e=1}^{n_{\mathrm{el}}} \int_{\mathcal{B}^e_0} \nabla N^j_\varphi \cdot \mathrm{D}_F \boldsymbol{P} \cdot \nabla N^l_\varphi \mathrm{d}V$$

- linearization of residual wrt nodal dofs

iteration matrix for global system of equations

from integral equation...

- integral equations cannot be evaluated analytically

$$\mathbf{R}^{\mathrm{e}}_{j} = \int_\zeta \int_\eta \int_\xi \nabla N^j_\varphi(\xi,\eta,\zeta) \cdot \boldsymbol{P}_{n+1}(\xi,\eta,\zeta)\, \det(\boldsymbol{J}(\xi,\eta,\zeta))\, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$$

$$\mathbf{K}^{\mathrm{e}}_{jl} = \int_\zeta \int_\eta \int_\xi \nabla N^j_\varphi(\xi,\eta,\zeta) \cdot \mathrm{D}_F \boldsymbol{P}(\xi,\eta,\zeta) \cdot \nabla N^l_\varphi(\xi,\eta,\zeta)\, \det(\boldsymbol{J}(\xi,\eta,\zeta))\, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$$

- idea - numerical interation / quadrature

$$\mathbf{R}^{\mathrm{e}}_{j} \approx \sum_{i=0}^{n} \nabla N^j_\varphi(\xi_i,\eta_i,\zeta_i) \cdot \boldsymbol{P}_{n+1}(\xi_i,\eta_i,\zeta_i)\, \det(\boldsymbol{J}(\xi_i,\eta_i,\zeta_i)\, w_i$$

$$\mathbf{K}^{\mathrm{e}}_{jl} \approx \sum_{i=0}^{n} \nabla N^j_\varphi(\xi_i,\eta_i,\zeta_i) \cdot \mathrm{D}_F \boldsymbol{P}(\xi_i,\eta_i,\zeta_i) \cdot \nabla N^l_\varphi(\xi_i,\eta_i,\zeta_i)\, \det(\boldsymbol{J}(\xi_i,\eta_i,\zeta_i)\, w_i$$

... to discrete sum

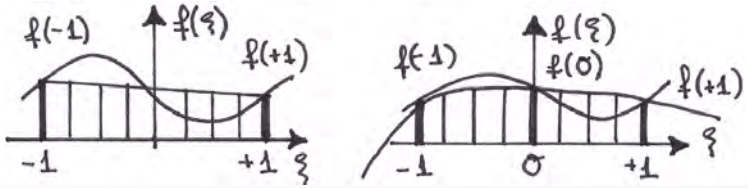
numerical integration

- integral equations are approximated by discrete sums

$$\int_a^b f(\xi)\mathrm{d}\xi \approx [\,b-a\,]\sum_{i=0}^{n} f(\xi_i)\,w_i$$

ξ_i ... quadrature point coordinates

w_i... quadrature point weights



newton cotes quadrature - accuracy [n-1]

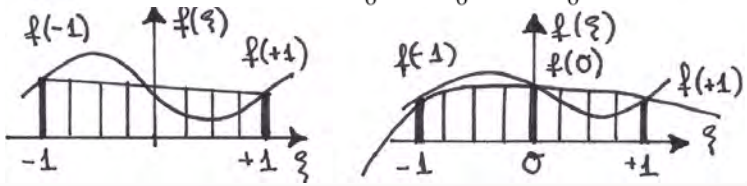
equidistant quadrature points @ $\xi_i = -1 + 2\,\frac{i}{n}$

n=2 $\displaystyle\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(\xi_0)\,w_0 + f(\xi_1)\,w_1\,]$

$= f(-1) + f(+1)$ trapezoidal rule

n=3 $\displaystyle\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(\xi_0)\,w_0 + f(\xi_1)\,w_1 + f(\xi_2)\,w_2\,]$

$= 2\,[\,f(-1)\frac{1}{6} + f(0)\frac{4}{6} + f(+1)\frac{1}{6}\,]$ simpson rule



gauss legendre quadrature - accuracy [2n-1]

optimized quadrature points

n=1 $\displaystyle\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(0)\,1\,]$

n=2 $\displaystyle\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(-\tfrac{1}{\sqrt{3}})\,\tfrac{1}{2} + f(+\tfrac{1}{\sqrt{3}})\,\tfrac{1}{2}\,]$

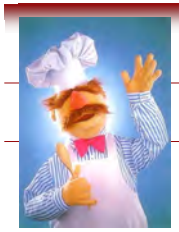
n=3 $\displaystyle\int_{\xi=-1}^{+1} f(\xi)\mathrm{d}\xi \approx 2\,[\,f(-\tfrac{3}{\sqrt{5}})\,\tfrac{5}{18} + f(0)\,\tfrac{8}{18} + f(+\tfrac{3}{\sqrt{5}})\,\tfrac{5}{18}\,]$

most fe programs prefer gauss over newton!

@ integration point level

- constitutive equations - given $\boldsymbol{F} = \nabla\boldsymbol{\varphi}$ calculate $\boldsymbol{P}$
- update growth multiplier for current stress state from ϑ_{n} and $\mathrm{D}_t\vartheta = k_\vartheta(\vartheta)\,\mathrm{tr}(\boldsymbol{C}_{\mathrm{e}}\cdot\boldsymbol{S}_{\mathrm{e}})$ calculate $\vartheta_{\mathrm{n}+1}$
- update growth tensor
  $\boldsymbol{F}^{\mathrm{g}} = \vartheta\,\boldsymbol{I}$ and $\boldsymbol{F}^{\mathrm{g}-1} = \frac{1}{\vartheta}\,\boldsymbol{I}$
- calculate elastic tensor
  $\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}}\cdot\boldsymbol{F}^{\mathrm{g}}$ $\quad\boldsymbol{F}^{\mathrm{e}} = \boldsymbol{F}\cdot\boldsymbol{F}^{\mathrm{g}-1} = \boldsymbol{F}/\vartheta$
- calculate stress
  $\boldsymbol{P}(\boldsymbol{F}^{\mathrm{e}}) = \mu\,\boldsymbol{F}^{\mathrm{e}} + [\,\lambda\,\ln(\det(\boldsymbol{F}^{\mathrm{e}})) - \mu\,]\,\boldsymbol{F}^{\mathrm{e}-\mathrm{t}}$

stress for righthand side vector



recipe for temporal discretization

explicit euler forward

- evolution of growth multiplier

$$\mathrm{D}_t\vartheta = \tfrac{1}{\Delta t}[\,\vartheta_{\mathrm{n}+1} - \vartheta_{\mathrm{n}}\,] = \dot{\vartheta} \qquad \text{finite difference approximation}$$

$$\mathrm{D}_t\vartheta = k(\vartheta_{\mathrm{n}})\,\mathrm{tr}(\boldsymbol{C}^{\mathrm{e}}\cdot\boldsymbol{S}^{\mathrm{e}}) \qquad \text{euler forward}$$

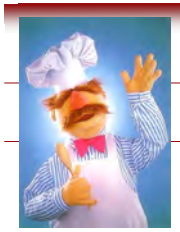
- direct update of growth multiplier

$$\vartheta_{\mathrm{n}+1} = \vartheta_{\mathrm{n}} + k(\vartheta_{\mathrm{n}})\,\mathrm{tr}(\boldsymbol{C}^{\mathrm{e}}\cdot\boldsymbol{S}^{\mathrm{e}})\Delta t \doteq 0$$

$$k(\vartheta_{\mathrm{n}}) = k_{\vartheta}^{+}\left[\frac{\vartheta^{\max}-\vartheta_{\mathrm{n}}}{\vartheta^{\max}-1}\right] \qquad \text{if} \qquad \mathrm{tr}(\boldsymbol{C}^{\mathrm{e}}\cdot\boldsymbol{S}^{\mathrm{e}}) > 0$$

$$k(\vartheta_{\mathrm{n}}) = k_{\vartheta}^{+}\left[\frac{\vartheta_{\mathrm{n}}-\vartheta^{\min}}{\vartheta^{\min}-1}\right] \qquad \text{if} \qquad \mathrm{tr}(\boldsymbol{C}^{\mathrm{e}}\cdot\boldsymbol{S}^{\mathrm{e}}) < 0$$

conditionally stable - limited to small time steps



recipe for temporal discretization

implicit euler backward

- evolution of growth multiplier

$$\mathrm{D}_t\vartheta = \tfrac{1}{\Delta t}[\,\vartheta_{\mathrm{n}+1} - \vartheta_{\mathrm{n}}\,] = \dot{\vartheta} \qquad \text{finite difference approximation}$$

$$\mathrm{D}_t\vartheta = k(\vartheta_{\mathrm{n}+1})\,\mathrm{tr}(\boldsymbol{C}^{\mathrm{e}}\cdot\boldsymbol{S}^{\mathrm{e}}) \qquad \text{euler backward}$$

- discrete residual

$$\mathsf{R}^{\vartheta}_{\mathrm{n}+1} = \vartheta_{\mathrm{n}+1} - \vartheta_{\mathrm{n}} - k(\vartheta_{\mathrm{n}+1})\,\mathrm{tr}(\boldsymbol{C}^{\mathrm{e}}\cdot\boldsymbol{S}^{\mathrm{e}})\Delta t \doteq 0$$

- local newton iteration

$$\mathsf{R}^{\vartheta\,\mathrm{k}+1}_{\mathrm{n}+1} = \mathsf{R}^{\vartheta\,\mathrm{k}}_{\mathrm{n}+1} + \mathrm{d}\mathsf{R}^{\vartheta} \doteq 0 \qquad \mathrm{d}\mathsf{R}^{\vartheta} = \frac{\mathrm{d}\mathsf{R}^{\vartheta}}{\mathrm{d}\vartheta}\,\mathrm{d}\vartheta$$

$$\vartheta_{\mathrm{n}+1} \leftarrow \vartheta_{\mathrm{n}} + \mathrm{d}\vartheta \qquad \mathrm{d}\vartheta = \left[\frac{\mathrm{d}\mathsf{R}^{\vartheta}}{\mathrm{d}\vartheta}\right]^{-1}\mathsf{R}^{\vartheta\,\mathrm{k}}_{\mathrm{n}+1} \qquad \text{iterative update}$$

unconditionally stable - larger time steps

@ integration point level

- discrete residual of growth multiplier

check in matlab!

$$\mathsf{R}^{\vartheta}_{\mathrm{n}+1} = \vartheta_{\mathrm{n}+1} - \vartheta_{\mathrm{n}} - k\;\mathrm{tr}(\boldsymbol{M}^{\mathrm{e}})\,\Delta t$$

- residual of biological equilibrium

local newton iteration

local newton interation to determine growth multiplier

given $\boldsymbol{F}$ and $\vartheta^{\mathrm{g}}_{\mathrm{n}}$
initialize $\vartheta^{\mathrm{g}} \leftarrow \vartheta^{\mathrm{g}}_{\mathrm{n}}$

local Newton iteration
- calculate elastic tensor $\boldsymbol{F}^{\mathrm{e}} = \boldsymbol{F} \,/\, \vartheta^{\mathrm{g}}$
- calculate elastic right Cauchy Green tensor $\boldsymbol{C}^{\mathrm{e}} = \boldsymbol{F}^{\mathrm{e}\,\mathrm{t}}\cdot\boldsymbol{F}^{\mathrm{e}}$
- calculate second Piola Kirchhoff stress $\boldsymbol{S}^{\mathrm{e}} = 2\,\partial\psi \,/\, \partial\boldsymbol{C}^{\mathrm{e}}$
- check growth criterion $\phi^{\mathrm{g}} = \mathrm{tr}(\boldsymbol{C}^{\mathrm{e}}\cdot\boldsymbol{S}^{\mathrm{e}}) > 0$?
  - calculate growth function $k^{\mathrm{g}} = [[\vartheta^{\max} - \vartheta^{\mathrm{g}}]/[\vartheta^{\max} - 1]]^{\gamma} \,/\, \tau$
  - calculate residual $\mathsf{R} = \vartheta^{\mathrm{g}} - \vartheta^{\mathrm{g}}_{\mathrm{n}} - k^{\mathrm{g}}\,\phi^{\mathrm{g}}\Delta t$
  - calculate tangent $\mathsf{K} = \partial\mathsf{R} \,/\, \partial\vartheta^{\mathrm{g}}$
- update growth multiplier $\vartheta^{\mathrm{g}} \leftarrow \vartheta^{\mathrm{g}} - \mathsf{R} \,/\, \mathsf{K}$

check convergence $\mathsf{R} \leq \mathrm{tol}$?

probing the material @the integration point

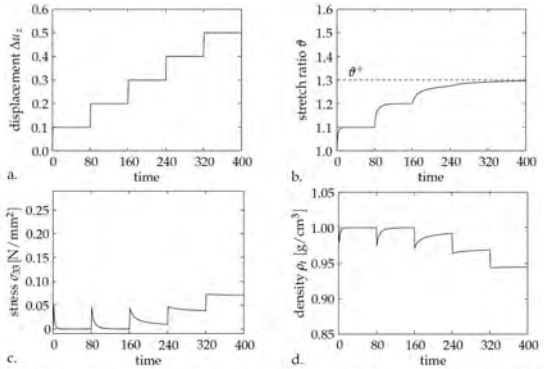


Figure 4.2: Isotropic simple tension test on a growing cube. (a) An incrementally increasing stretch is applied. (b) The stretch ratio converges time-dependently to the biological equilibrium. (c) The stresses vanish in the biological equilibrium state as long as $\vartheta < \vartheta^+$. (d) The density in the biological equilibrium state does not change as long as $\vartheta < \vartheta^+$.

himpel, kuhl, menzel & steinmann [2005]

probing the material @the integration point

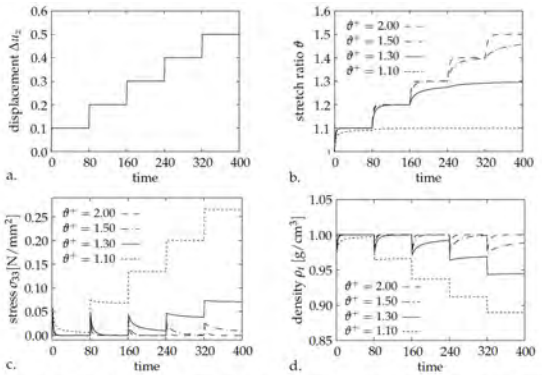


Figure 4.3: Variation of the limiting stretch ratio ϑ^+ in the simple tension test. The stretch ratio increases until the limiting value is reached. If the limiting value of the stretch ratio is reached the material behavior is purely elastic.

himpel, kuhl, menzel & steinmann [2005]

probing the material @the integration point

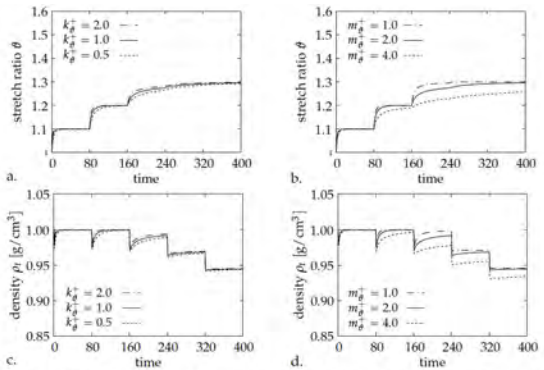


Figure 4.4: Variation of the material parameters k_ϑ^+ and m_ϑ^+ in the simple tension test. They influence the relaxation time, but not the final state at biological equilibrium.

himpel, kuhl, menzel & steinmann [2005]

probing the material @the integration point

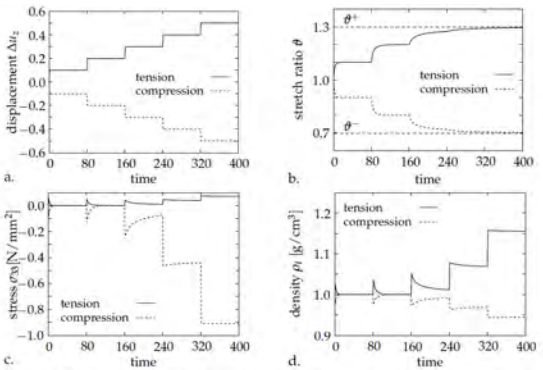


Figure 4.5: The material distinguishes between tension and compression. In case of tension the material grows, and in case of compression the material decreases.

himpel, kuhl, menzel & steinmann [2005]

@ integration point level

- constitutive equations - given $\boldsymbol{F}$ calculate $\boldsymbol{P}$

check in matlab!

$$\boldsymbol{P}(\boldsymbol{F}^{\mathrm{e}}) = \mu\,\boldsymbol{F}^{\mathrm{e}} + [\,\lambda\,\ln(\det(\boldsymbol{F}^{\mathrm{e}})) - \mu\,]\,\boldsymbol{F}^{\mathrm{e}-\mathrm{t}}$$

- stress calculation @ integration point level

stress for righthand side vector

@ integration point level

- tangent operator / constitutive moduli

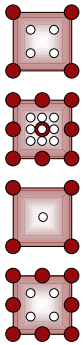
check in matlab!

$$\mathbf{A} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\boldsymbol{F}} = \left.\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}}\right|_{\boldsymbol{F}^{\mathrm{g}}} + \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}^{\mathrm{g}}} : \frac{\partial \boldsymbol{F}^{\mathrm{g}}}{\partial \vartheta} \otimes \left.\frac{\vartheta}{\boldsymbol{F}}\right|_{\boldsymbol{F}}$$

- linearization of stress wrt deformation gradient

tangent for iteration matrix

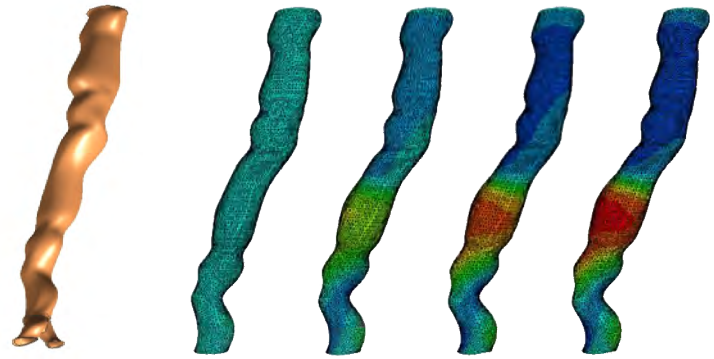
integration point based solution of growth equation



loop over all time steps	`nlin_fem`
global newton iteration	`nlin_fem`
loop over all elements	`brick_3d`
loop over all quadrature points	`cnst_vol`
local newton iteration ϑ_{n+1}	`upd_vol`
determine element residual & tangent	`cnst_vol`
determine global residual and tangent	`brick_3d`
determine φ_{n+1}	`nlin_fem`
determine state of biological equilibrium	`nlin_fem`

growth multiplier ϑ as internal variable

virtual stent implantation - patient specific model



tissue growth - response to virtual stent implantation

kuhl, maas, himpel & menzel [2007]